The High School Identities

There are eleven basic identities of the **positive** integers N with the operations $+, \times, \uparrow$ which one learns in high school:

$$
\text{HSI}\left\{\begin{array}{l}
\widehat{\text{HSI}}\left\{\begin{array}{lc}
(1) & x + y \approx y + x \\
(2) & x + (y + z) \approx (x + y) + z \\
(3) & x \cdot 1 \approx x \\
(4) & x \cdot y \approx y \cdot x \\
(5) & x \cdot (y \cdot z) \approx (x \cdot y) \cdot z \\
(6) & x \cdot (y + z) \approx (x \cdot y) + (x \cdot z)
\end{array}\right. \\
\hline
\begin{array}{lc}
(7) & 1^{x} \approx 1 \\
(8) & x^{1} \approx x \\
(9) & x^{y+z} \approx x^{y} \cdot x^{z} \\
(10) & (x \cdot y)^{z} \approx x^{z} \cdot y^{z} \\
(11) & (x^{y})^{z} \approx x^{y \cdot z}.
\end{array}
\end{array}\right.
$$

These can be found in Dedekind's 1888 monograph *Was Sind Und Was Sollen Die Zahlen?*

Let

$$\mathbf{N} = (N, +, \times, \uparrow, 1) \qquad \widehat{\mathbf{N}} = (N, +, \times, 1)$$

There are two themes we want to consider in this talk:

(a) **Axioms** for the equational theory of $\mathbf{N}$.

(b) **Decidability** of the equational theory of $\mathbf{N}$.

First let us examine the same questions for $\widehat{\mathbf{N}}$ and $\widehat{\text{HSI}}$.

In this case the situation is straightforward since each term $t(\vec{x})$ in the language $\{+, \times, 1\}$ has a **normal form**, namely it is equivalent, modulo $\widehat{\text{HSI}}$, to a **polynomial** $p(\vec{x})$.

(a) $\widehat{\mathsf{HSI}}$ axiomatizes the equational theory of $\widehat{\mathbf{N}}$.

Clearly $\widehat{\mathsf{HSI}}$ holds in $\widehat{\mathbf{N}}$.

For the converse let $s(\vec{x}) \approx t(\vec{x})$ be an identity true of $\widehat{\mathbf{N}}$. Let $p(\vec{x})$ be the polynomial that each side can be reduced to using $\widehat{\mathsf{HSI}}$. By piecing together the two reductions one has a derivation of $s \approx t$ from $\widehat{\mathsf{HSI}}$.

(b) The equational theory of $\widehat{\mathbf{N}}$ is decidable.

To determine if an equation $s \approx t$ holds in $\widehat{\mathbf{N}}$ one computes the normal forms of the two sides and checks if they are the same polynomial.

When working with HSI and $\mathbf{N}$ we no longer have the benefit of such normal forms. The subject becomes much more difficult.

Models of HSI

Of course HSI holds in the **positive reals**.

G. Birkhoff showed that HSI holds for the algebra of **posets** where the operations are given by:

$+$ is disjoint union
$\times$ is cartesian product
$\uparrow$ is order preserving maps

Consequently they also hold for the algebra of **cardinal numbers**.

In a footnote Birkhoff says that Tukey pointed out that HSI holds more generally for **preorders** (= reflexive + transitive).

Aside from some examples in topos theory I know of few other natural models of HSI that have been studied.

Every model of HSI has a smallest submodel, namely the subalgebra generated by the constant 1.

The elements of this submodel are just 1,2,3, etc., where

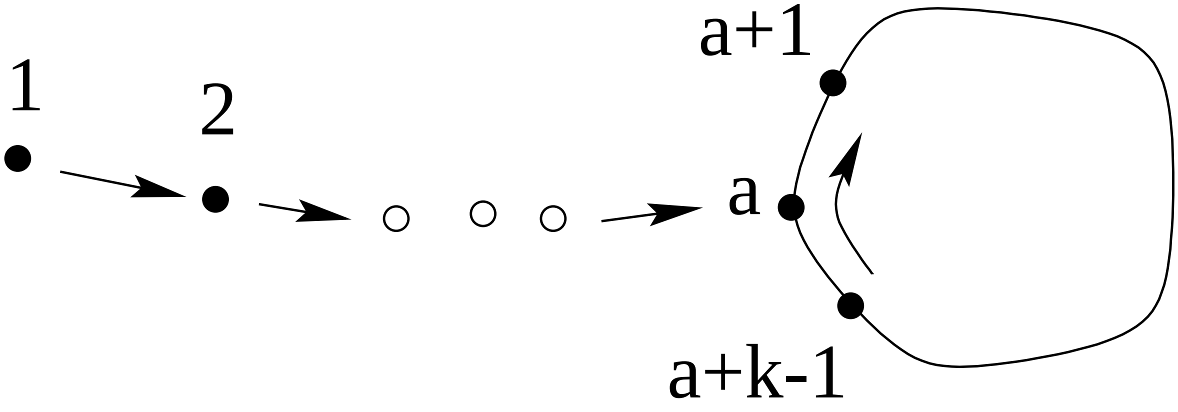
$$\begin{array}{rcl} 2 & := & 1+1 \\ 3 & := & 2+1 \\ & \text{etc.} & \end{array}$$

since these elements include 1 and are closed under addition, multiplication and exponentiation.

We call these elements the **integers** of the model.

If the set of integers of a model of HSI is infinite then it must look like $\mathbf{N}$, the **free** HSI algebra freely generated by $\emptyset$.

On the other hand if the set of integers is finite then it must give a submodel that looks like:



The Integers $\mathrm{N}_{a,k}$

satisfy the identity

$$a \approx a + k.$$

Not every positive a and k gives a quotient of $\mathbf{N}$ as pictured in the $\mathbf{N}_{a,k}$ above—each picture supports addition and multiplication, but not necessarily exponentiation.

Theorem The finite quotients of $\mathbf{N}$ are the $\mathbf{N}_{a,k}$ where $a, k \in N$ satisfy (for all primes p):

$$\begin{aligned} p^e | k &\Rightarrow e \leq a \\ p | k &\Rightarrow (p-1) | k. \end{aligned}$$

The next corollary gives a complete list of the five "circle" integer HSI-algebras, i.e., those with $a = 1$, and hence no "tail".

Corollary $\mathbf{N}_{1,k}$ is a quotient of $\mathbf{N}$ iff $k \in \{1, 2, 6, 42, 1806\}$.

These give the examples of **rings** $\mathbf{Z}/(k)$ that support exponentiation (with $0^0 = 0$).

Already in the study of the $\mathbf{N}_{a,k}$ we run into difficult questions.

Given $a \in N$ define the sequence of primes $\Sigma_a = (p_1, p_2, \cdots)$ by

- $p_1 = 2$;

- given $p_1, \ldots, p_i$, let p_{i+1} be the smallest prime p which is greater than p_i and such that $(p-1)|(p_1 \cdots p_i)^a$, assuming such a p exists. If no such p exists then Σ_a terminates with p_i.

Proposition Given a positive integer a, there are infinitely many $\mathbf{N}_{a,k}$ iff the sequence of primes Σ_a is infinite.

By the above result $\Sigma_1 = (2, 3, 7, 43)$, a finite sequence.

Problem Is Σ_a finite for all (any) $a > 1$?

About 20% of the primes below 1,000,000 are in $\Sigma_2 = (2, 3, 5, 7, 11, 13, 19, 23, \ldots, 999667, 999727, \ldots)$ — so even if Σ_2 is finite, a computer enumeration does not look feasible.

The Two-Element Models of HSI

1.

$+$	1	a
1	1	1
a	1	a

$\times$	1	a
1	1	a
a	a	a

$\uparrow$	1	a
1	1	1
a	a	1

2.

$+$	1	a
1	1	1
a	1	a

$\times$	1	a
1	1	a
a	a	a

$\uparrow$	1	a
1	1	1
a	a	a

3.

$+$	1	a
1	1	a
a	a	a

$\times$	1	a
1	1	a
a	a	a

$\uparrow$	1	a
1	1	1
a	a	a

4.

$+$	1	2
1	2	2
2	2	2

$\times$	1	2
1	1	2
2	2	2

$\uparrow$	1	2
1	1	1
2	2	2

$\mathbf{N_{2,1}}$

5.

$+$	1	2
1	2	1
2	1	2

$\times$	1	2
1	1	2
2	2	2

$\uparrow$	1	2
1	1	1
2	2	2

$\mathbf{N_{1,2}}$

Clearly algebras (4) and (5) satisfy all the identities of $\mathbf{N}$ as they are quotients of $\mathbf{N}$.

Problem Are any of the algebras (1)–(3) in the variety generated by $\mathbf{N}$?

1.
$$\begin{array}{c|cc} + & 1 & a \\ \hline 1 & 1 & 1 \\ a & 1 & a \end{array} \qquad \begin{array}{c|cc} \times & 1 & a \\ \hline 1 & 1 & a \\ a & a & a \end{array} \qquad \begin{array}{c|cc} \uparrow & 1 & a \\ \hline 1 & 1 & 1 \\ a & a & 1 \end{array}$$

2.
$$\begin{array}{c|cc} + & 1 & a \\ \hline 1 & 1 & 1 \\ a & 1 & a \end{array} \qquad \begin{array}{c|cc} \times & 1 & a \\ \hline 1 & 1 & a \\ a & a & a \end{array} \qquad \begin{array}{c|cc} \uparrow & 1 & a \\ \hline 1 & 1 & 1 \\ a & a & a \end{array}$$

3.
$$\begin{array}{c|cc} + & 1 & a \\ \hline 1 & 1 & a \\ a & a & a \end{array} \qquad \begin{array}{c|cc} \times & 1 & a \\ \hline 1 & 1 & a \\ a & a & a \end{array} \qquad \begin{array}{c|cc} \uparrow & 1 & a \\ \hline 1 & 1 & 1 \\ a & a & a \end{array}$$

By taking the variety generated by each of these two-element models of HSI we easily have many other recognizable models of HSI:

In four of the cases below exponentiation is the *first projection* function π (defined by $\pi(a, b) = a$).

- Let $\mathbf{H} = \langle H, \vee, \wedge, \rightarrow, 0, 1\rangle$ be a Heyting algebra.

  Then $\mathbf{H}^{\star} = \langle H, \vee, \wedge, \leftarrow, 1\rangle$ is an HSI-algebra, where $a \leftarrow b$ is defined to be $b \rightarrow a$.

- Let $\mathbf{D} = \langle D, \vee, \wedge, 1\rangle$ be a distributive lattice with 1.

  Then $\langle D, \vee, \wedge, \pi, 1\rangle$ is an HSI-algebra.

- Let $\mathbf{S} = \langle S, \wedge, 1 \rangle$ be a semilattice with 1.

  Then $\langle S, \wedge, \wedge, \pi, 1 \rangle$ is an HSI-algebra.

- Let $\mathbf{S} = \langle S, \wedge, 0, 1 \rangle$ be a semilattice with 0,1.

  Then $\langle S, f, \wedge, \pi, 1 \rangle$ is an HSI-algebra, where f is the binary constant map whose value is always 0.

- Let $\mathbf{R} = \langle R, +, \times, 0, 1 \rangle$ be a Boolean ring.

  Then $\langle R, +, \times, \pi, 1 \rangle$ is an HSI-algebra.

Are there any exotic identities of $\mathbb{N}$?

By the 1960s there was an interest in determining if HSI actually axiomatizes the equational theory of $\mathbb{N}$.

Although the identities were known to be decidable, this did not give a way to determine if there are any exotic identities of $\mathbb{N}$.

In 1981 **Wilkie**, in a never published manuscript, claimed that the identity

$$\boxed{\begin{aligned}&\big((1+x)^y+(1+x+x^2)^y\big)^x\cdot\big((1+x^3)^x+(1+x^2+x^4)^x\big)^y\\ &\approx\big((1+x)^x+(1+x+x^2)^x\big)^y\cdot\big((1+x^3)^y+(1+x^2+x^4)^y\big)^x\end{aligned}}$$

which we call $W(x,y)$, is indeed an exotic identity of $\mathbb{N}$; that is, it holds in $\mathbb{N}$ but cannot be derived from HSI.

Wilkie's proof was purely syntactic, using an induction on the length of a supposed derivation of $W(x, y)$ from HSI.

Wilkie's proof was soon replaced by a very simple proof of Gurevič (published in 1985). Using the subterms of $W(x, y)$, and a little tweaking, Gurevič constructed a 59-element algebra satisfying HSI but not $W(x, y)$.

In a 1990 paper by Gurevič we find the following remark:

> C.W. Henson once asked if there are countermodels to Tarski's question (whether all valid identities in signature $(+, \cdot, \uparrow)$ were derivable) of a very small size, say, 5. Currently I don't know; my own record was 33 elements and I heard a rumour that someone had pushed the record further to 28 elements.

Since then there has been considerable progress in the search for the smallest counterexample to $W(x,y)$:

R. Gurevič	59 elements		1985
	⋮		
R. Gurevič	33 elements		($\leq$ 1990)
S. Burris	28 elements		(1988)
	⋮		
S. Burris	16 elements		(1990)
S. Lee	15 elements		(1991)
S. Burris S. Lee	15 elements	$\geq$ 7 elements	1992
M. Jackson	14 elements	$\geq$ 8 elements	1996
S. Burris K. Yeats	13 elements		(2001)
S. Burris K. Yeats	12 elements		(2001)

Conjecture: The 12 element algebra is the smallest counterexample to Wilkie's identity.

$+$	1	2	3	4	a	b	c	d	e	f	g	h
1	2	3	4	4	2	3	d	3	3	3	3	4
2	3	4	4	4	3	4	3	4	4	4	4	4
3	4	4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4	4	4
a	2	3	4	4	b	4	b	3	h	3	3	4
b	3	4	4	4	4	4	4	4	4	4	4	4
c	d	3	4	4	b	4	b	3	3	3	3	4
d	3	4	4	4	3	4	3	4	4	4	4	4
e	3	4	4	4	h	4	3	4	4	3	h	4
f	3	4	4	4	3	4	3	4	3	4	3	4
g	3	4	4	4	3	4	3	4	h	3	4	4
h	4	4	4	4	4	4	4	4	4	4	4	4

$\times$	1	2	3	4	a	b	c	d	e	f	g	h
1	1	2	3	4	a	b	c	d	e	f	g	h
2	2	4	4	4	b	4	b	4	4	4	4	4
3	3	4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4	4	4
a	a	b	4	4	c	b	c	b	h	4	4	4
b	b	4	4	4	b	4	b	4	4	4	4	4
c	c	b	4	4	c	b	c	b	4	4	4	4
d	d	4	4	4	b	4	b	4	4	4	4	4
e	e	4	4	4	h	4	4	4	4	4	h	4
f	f	4	4	4	4	4	4	4	4	4	4	4
g	g	4	4	4	4	4	4	4	h	4	4	4
h	h	4	4	4	4	4	4	4	4	4	4	4

$\uparrow$	1	2	3	4	a	b	c	d	e	f	g	h
1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	4	4	4	4	4	4	4	f	4	4	4
3	3	4	4	4	e	4	4	4	g	4	e	h
4	4	4	4	4	4	4	4	4	4	4	4	4
a	a	c	c	c	c	c	c	c	c	c	c	c
b	b	4	4	4	4	4	4	4	4	4	4	4
c	c	c	c	c	c	c	c	c	c	c	c	c
d	d	4	4	4	f	4	4	4	4	4	4	4
e	e	4	4	4	4	4	4	4	h	4	4	4
f	f	4	4	4	4	4	4	4	4	4	4	4
g	g	4	4	4	h	4	4	4	4	4	h	4
h	h	4	4	4	4	4	4	4	4	4	4	4

How We Found the 12-Element Example

The method used first a lengthy search for possible **cores** of such an example.

If $W(x, y)$ fails in a model of HSI at (a, b) then we call the $\widehat{\text{HSI}}$-subalgebra generated by $\{a\}$ a core of the model.

Burris and Lee found a number of conditions that cores must satisfy, for example, there must be at least three integers.

So first we searched for possible cores. Then we tried to expand the cores to models of HSI that failed $W(x, y)$. Again we had some conditions that b must satisfy if $W(a, b)$ is to fail to hold, for example b cannot be in the core generated by a (Jackson).

C-programs were written and a few months of computing needed...

The Search for Natural Counterexamples

The counterexamples to Wilkie that we know are quite intricate and certainly not easy to remember.

One can hope for a natural counterexample along the lines of Birkhoff's algebra of posets.

Unfortunately the examples that we have constructed share features of Birkhoff's algebra that lead to $W(x, y)$ being satisfied:

1. Elements of the algebra are structures that decompose into **sums of components**.

2. Exponentiation is given by certain maps, and such decompose into maps on the components.

3. Product is given by Cartesian product.

Three Problems

Problem Is there is natural counterexample to $W(x, y)$?

Problem Is there a natural HSI-algebra that is not in $V(\mathbf{N})$?

Problem Is there an algebra with fewer than 12 elements that satisfies HSI but is not in the variety generated by $\mathbf{N}$?

The Equational Theory of $\mathbf{N}$
is not Finitely Axiomatizable

Gurevič showed in 1990 there is no finite set of identities that axiomatize the identities of $\mathbf{N}$.

He showed that the following collection of Wilkie style identities in one variable x are true of $\mathbf{N}$ but cannot be derived from any finite subset of the identities true of $\mathbf{N}$ (n is odd in the following):

$$(P^x+Q_n^x)^{2^x}\cdot(R_n^{2^x}+S_n^{2^x})^x = (P^{2^x}+Q_n^{2^x})^x\cdot(R_n^x+S_n^x)^{2^x}$$

where

$$\begin{aligned} P &= 1+x \\ Q &= 1+x+\cdots+x^{n-1} \\ R &= 1+x^n \\ S &= 1+x^2+\cdots+x^{2n-2}. \end{aligned}$$

Although much of his proof of the non-finitely axiomatizable result is an elementary study of the forms in which terms and equational proofs can be expressed, at one point he needs to go into the complex plane and examine **analytic continuations around singularities**.

Some Positive Results

Considerable effort has been expended to find nice sets S of terms such that any equation true of $\mathbf{N}$ that has both sides from S will be a consequence of HSI.

C.W. Henson and **L.A. Rubel** used Nevanlinna theory in 1984 to show that S can be the set of terms that only use exponentiation of variables or constants.

Gurevič extended this work in 1993 by showing that one can allow exponentiation of polynomials.

The Smallest Exotic Identity

The one variable terms are well-ordered by comparing their behaviour near infinity, so one can ask:

What is the "smallest" one variable identity true of $\mathbf{N}$ but not derivable from HSI?

Gurevic **conjectures** that the answer is:

$$(P^x+Q^x)^{2^x}\cdot(R^{2^x}+S^{2^x})^x = (P^{2^x}+Q^{2^x})^x\cdot(R^x+S^x)^{2^x}$$

where

$$\begin{aligned} P &= 1+x \\ Q &= 2+x \\ R &= 2+x+x^3 \\ S &= 4+x^2+x^3. \end{aligned}$$

Decidability

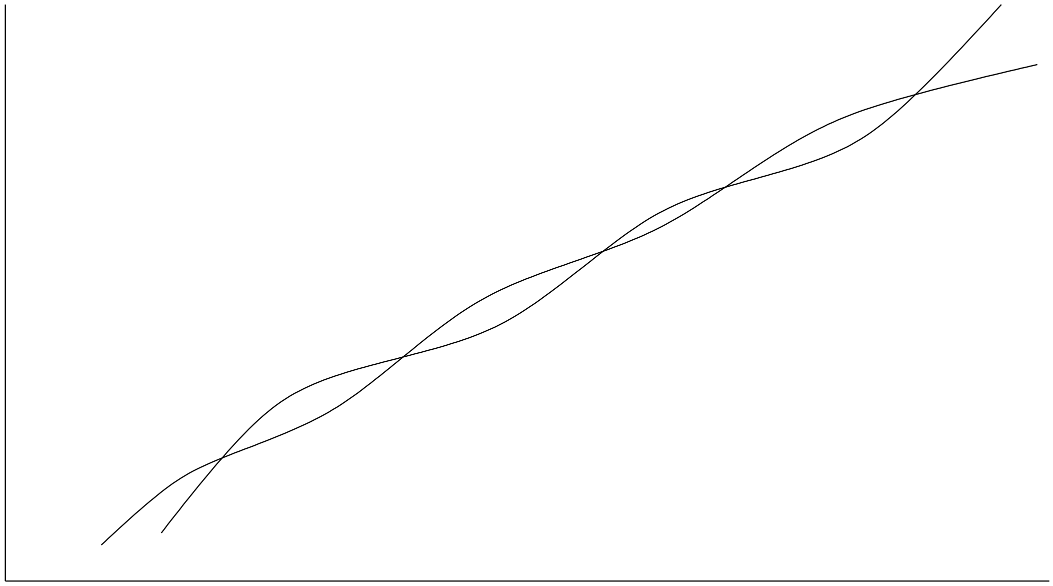
G.H. Hardy's 1911 investigations of the asymptotic behaviour of "logarithmico-exponential" functions (**L-functions**), as written up in his treatise "Orders of Infinity", provided the starting point for investigations into the decidability of the equational theory of $\mathbf{N}$.

L-functions are described by expressions constructed using a single variable x, the reals as **constants**, the **rational** functions $+, -, \times, \div$, **radicals** $\sqrt[m]{\ }$, **exp** and **log**.

<u>Hardy's Requirement</u>: An L-function must be eventually defined on the reals.

A key result of Hardy was that the relation **is eventually less than or equal** ($\preceq$) gives a linear ordering on these functions.

Thus two distinct L-functions cannot equal one another for arbitrarily large values, as suggested by the following diagram:



A. Ehrenfeucht showed in 1973 that the class Sk of term functions for the $T(x)$ in the language of HSI is **well-ordered** by $\preceq$.

Query Is ε_0 the order type of $(Sk, \preceq)$?

To prove the ordering result Hardy introduced the following concepts:

(1) The **order** of an L-term is the maximal number of nested exponentials and logs in it.

(2) An **integral** L-term of order at most n is one of the form

$$\sum_i \alpha_i(x) \cdot e^{\beta_i(x)} \cdot \prod_j \log \gamma_{ij}(x)$$

where the $\alpha_i(x)$, $\beta_i(x)$ and $\gamma_i(x)$ are of order at most $n-1$.

Proof by the usual induction over formulas does not seem to work.

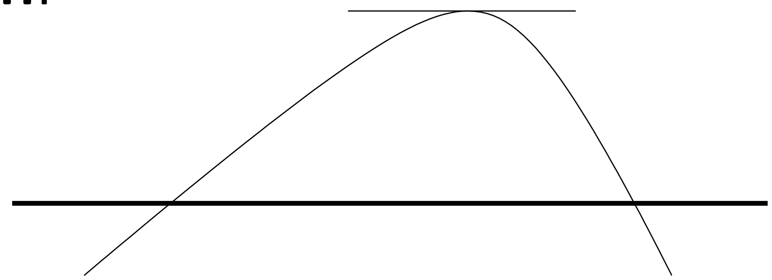
Hardy proves his result by a multi-stage induction, using the fact that:

Every L-function of order at most n is an algebraic expression in integral functions of order at most n.

One of his most important ways of reducing a case to the induction hypothesis is to differentiate:

$T(x)$ **has arbitrarily large roots implies** $T'(x)$ **has arbitrarily large roots.**

This follows of course from **Rolle's Theorem**:



In 1969 **D. Richardson** built on Hardy's work to show that the **one variable** equational theory of $\mathbf{N}$ is decidable, that is, one can decide if $S(x) \approx T(x)$ holds in the natural numbers.

He worked with **multivariable** terms using

$+$, $-$, $\times$, $\div$, exp, log$|\ |$, and 0,1.

[He dropped the radicals used by Hardy.]

Let us call this class **LR** (the logarithmico-exponential terms of Richardson).

Problem Can one decide if $T(x)$ is eventually defined, for $T(x)$ a one variable LR term?

Dominance Problem Given an oracle for the previous problem, can one decide if $T(x) \succeq 0$, for $T(x)$ an LR term?

Problem Is there a decision procedure for $S(x) \preceq T(x)$, where $S(x), T(x)$ are terms in the language of $\mathbf{N}$?

Richardson showed that if a decision procedure for the above problem exists then one can decide equality for the **exponential constants**, that is the variable free terms in the language $+, \times, \div, \uparrow, 1$.

Gurevic showed that if one considers only $S(x), T(x) \prec 2^{x^2}$ then the above problem is decidable.

Note that if $T(x, \vec{y}) \in$ LR and $\vec{b} \in \mathbb{R}$ are such that $T(x, \vec{b})$ is defined on an interval I, then

$T(x, \vec{b})$ defines a function in $C^{\infty}(I)$.

Richardson's main idea was to associate with each LR term T a finite sequence of LR terms

$$T_0, \ldots, T_k$$

which give important information about the number of distinct zeros of $T(x, \vec{b})$.

Let us say that a term $T(x, \vec{y})$

has the property $\mathcal{R}(x, k)$ if

there is a sequence of LR terms

$$T_0(x, \vec{y}), \ \ldots, \ T_k(x, \vec{y})$$

such that

(1) $T_0(x, \vec{y}) = T(x, \vec{y})$ and $\frac{\partial}{\partial x} T_k(x, \vec{y}) = 0$

(2) For every interval $I \subseteq \mathbb{R}$ and every tuple $\vec{b} \in \mathbb{R}^n$, if $T(x, \vec{b})$ is defined on I then, for each $a \in I$ and each $i < k$, $T_i(a, \vec{b})$ is defined and

$$T_{i+1}(a, \vec{b}) = 0 \qquad \Leftrightarrow \qquad \frac{\partial}{\partial x} T_i(a, \vec{b}) = 0.$$

Examples

(1) Let $T(x)$ be a $\boxed{\text{polynomial of degree } k}$. Then

$$T(x),\ T'(x),\ \ldots,\ T^{(k)}$$

is a sequence that shows $T(x)$ has $\mathcal{R}(x,k)$.

(2) Let $\boxed{T(x) = e^{P(x)} - 1}$, where $P(x)$ is a polynomial of degree k. This has property $\mathcal{R}(x,k)$ since the following sequence satisfies the conditions:

$$e^{P(x)} - 1, P'(x), \ldots, P^{(k)}(x).$$

(3) If $Q(x)$ is also a polynomial, say of degree ℓ, then for $\boxed{T(x) = \exp(P(x)/Q(x)) - 1}$ let $R(x) = P(x)Q'(x) - Q(x)P'(x)$. The sequence

$$e^{P(x)/Q(x)} - 1, R(x), R'(x), \ldots, R^{(k+\ell-1)}(x)$$

shows that $T(x)$ has the property $\mathcal{R}(x, k+\ell-1)$.

Suppose that $T(x,\vec{y})$ is an LR-term that has property $\mathcal{R}(x,k)$.

It is easy to show that for any interval I and any tuple $\vec{b}$, if $T(x,\vec{b})$ is defined in I then

either $T(x,\vec{b}) = 0$ **on** I**, or** $T(x,\vec{b})$ **has at most** k **distinct roots in** I**.**

Richardson's main result is that for each LR-term $T(\vec{x})$ and each variable x_i from $\vec{x}$

there is an effective procedure to find a nonnegative integer k **such that** $T(\vec{x})$ **has property** $\mathcal{R}(x_i,k)$**.**

From this the decidability of the **one variable identities** of N follows immediately.

Initially unaware of Richardson's work, **A. Macintyre** proved the decidability of the **full** equational theory of $\mathbb{N}$. This was published in 1981.

His method is: at each step of Hardy's induction proof in which $T(x, \vec{b})$ is not identically 0 on the interval I, to effectively compute an upper bound on the the number of roots of $T(x, \vec{b})$ in I.

Then he uses a simple inductive argument to go from one variable to several variables.

We note that Macintyre's proof requires an excursion into the complex plane, something that the other two published proofs do not.

A third proof of the decidability of the equational theory of $\mathbf{N}$ was given by Gurevič in 1985 using ideas of A. G. Khovanski. This proof has a strong resemblance to that of Richardson, with the emphasis on building a **chain of terms** $T_0, \ldots, T_k$. In Gurevič's paper they must satisfy

$$T_0 = T, \quad T_k \in Q, \quad \frac{\partial T_j}{\partial x} = \frac{\beta_j T_j + T_{j+1}}{\alpha_j}$$

By solving these linear differential equations one sees that in any given interval of definition I of T_0 in which $T_j(x, \vec{b})$ is not identically 0, the number of roots in I can drop by at most one when passing from T_j to T_{j+1}.

In the same paper Gurevic examines the equational consequences of any finite set Σ of identities true of $\mathbf{N}$ that contains the identities

$$1^x \approx 1, \qquad x^1 \approx 1 \cdot x \approx x \cdot 1 \approx x.$$

He gives a decision procedure that takes as input any such Σ and identity $s \approx t$ and determines if $\Sigma \vdash s \approx t$.

The method is to describe a congruence of the term algebra generated by the variables of Σ, s and t, that yields a finite quotient whose size is effectively bounded and in which $s \approx t$ will fail if it is not a consequence of Σ.

Thus, for every such Σ, **the equational consequences of Σ is a decidable set of identities.**

In particular, this shows that **the equational consequences of the axioms HSI are decidable.**

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