Boole's Equational Treatment of Particular Statements

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Obstacles to Understanding Boole's Algebra of Logic

• Boole was using **ordinary algebra** with **idempotent symbols** ($X^2 = X$)

He was not using modern Boolean algebra nor Boolean rings.

His work was a blend of partial semantics with formal algebraic manipulation.

0	Empty class
1	Universe
XY	Intersection of X and Y
$X + Y$	Union of X and Y if disjoint
$X - Y$	Difference of X and Y if $Y \subseteq X$

Thus $1 - X$ is the **complement** of X.

- **Modern semantics** (for categorical propositions) was **not available** at the time of Boole's work.

Consider the following passage from De Morgan's *Formal Logic*, 1847, pp. 110:

> On looking into any writer on logic, we shall see that *existence* is claimed for the significations of all the names. Never, in the statement of a proposition, do we find any reason left for the alternative, *suppose there should be no such thing*.

Regarding the meaning of ‘All X is Y’
De Morgan goes on to say,

> If neither X nor Y exist, I will not . . . attempt to settle what nonexistent things agree or disagree.

Modern semantics was explicitly introduced by Peirce in 1880 when he stated that ‘All X is Y’ holds when X does not exist.

Peirce notes that the rule of limitation fails in his system.

Peirce’s semantics was adopted by Schröder in his famous 3 volumes: Algebra der Logik.

Boole's Translation of Categorical Statements into Equations

Affirmative Propositions

		1847	1848/1854
A	All X is Y	$X = XY$	$X = VY$
I	Some X is Y	$V = XY$	$VX = VY$

For the Negative Propositions just replace Y by $1 - Y$ in the above.

		1847	1848/1854
E	No X is Y	$X = X(1 - Y)$	$X = V(1 - Y)$
O	Some X is not Y	$V = X(1 - Y)$	$VX = V(1 - Y)$

EXAMPLE: (Conversion by Limitation)

$$\frac{\text{All } X \text{ is } Y}{\text{Some } Y \text{ is } X.}$$

Boole's 1847 Derivation

Given: $X = XY$ (equivalently, $X(1 - Y) = 0$).

Solve this for X to obtain $X = VY$.

[Verify this by substituting into the previous equation $X(1 - Y) = 0$.]

Boole's 1854 Derivation

Given: $X = VY$.

Multiply both sides by V to obtain $VX = VVY$.

Use the idempotence of V to obtain $VX = VY$.

The Theme of This Talk

Boole's use of **equations** for the **particular** propositions

	1847	1848/1854
I	$V = XY$	$VX = VY$
O	$V = X(1 - Y)$	$VX = V(1 - Y)$

has been unjustly dismissed by logicians, starting with the rejection by C.S. Peirce.

Boole's system can be easily patched to give a robust system (that will be called BF_3).

Boole Freed From every Flaw

The system BF_3 :

A	All X is Y	$X = XY$
I	Some X is Y	$V = VXY$

For the negative propositions replace Y by $1 - Y$.

The choice for **A** is the same as Boole 1847.

The choice for **I** lies between Boole 1847 and Boole 1848/1854 since:

$V = XY$ implies $V = VXY$

implies $VX = VY$.

Kneale and Kneale, *The Development of Logic*, 1962, p. 411

Hailperin, *Boole's Logic and Probability*, 1986, p. 112

OBJECTION: Boole has special rules for interpreting V as 'some', for example:

Some X is not Y	$VX = V(1 - Y)$
	algebraically leads to
Some Y is not X	$VY = V(1 - X)$

VY is not to be interpreted as 'some Y'.

ANSWER: In BF_3 there are no special rules limiting when one can interpret the symbols as 'some'.

In BF_3 one has

Some X is not Y $\quad V = VX(1-Y)$

does not lead to

Some Y is not X $\quad V = VY(1-X)$

Furthermore we note that in BF_3 any letter can play the role of V.

EXAMPLE: (Limitation)

All X is Y $\quad X = XY$

leads to

Some X is Y $\quad X = X(XY)$

C.S. Peirce, *On the agebra of logic*, 1880, [p. 22]

OBJECTION: 'Some X is not Y' yields, by Boole's algebraic system, 'Some Y is not X'.

ANSWER: (See previous two slides):

(1) Peirce does not realize that Boole has restrictions on when one can interpret V as 'some'.

In Peirce's defence: In *Laws* Boole does not formulate his restrictions on using V as 'some' until Chapter XV, the last chapter on logic.

(2) Cannot make this inference in BF_3 .

E. Schröder, *Algebra der LogiK Vol. II*, 1891, [p. 92]

OBJECTION: He gives a rigorous proof that one cannot use a system of equations to express 'Some X is Y'.

ANSWER:

(1): He is using modern semantics. His proof uses 0,1 as possible values for X,Y,...

In a footnote he says that further details are needed if the value 0 is excluded.

His proof uses substitution—he actually proves no single equation $E(X,Y) = 0$ can uniformly express 'Some X is Y' using modern semantics.

(2) A minor modification of BF_3 works for modern semantics.

T. Hailperin, *Boole's Logic and Probability*, 1986

He notes (p. 110) that Boole always converts $X = VY$ into $X(1 - Y) = 0$, but he goes on to say that

OBJECTION: "in the case of the particular categorical there is no easy way out."

ANSWER:

(1) He is using the modern semantics.

(2) There is an easy way out: $V = VXY$

(3) Actually Boole does not always eliminate the V in the translations of universal statements — see Chap. XV of *Laws of Thought* where Boole gives a uniform treatment of syllogisms.