Outline

A Primer on Boole's Algebra of Logic for Classes

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"The quasimathematical methods of Dr. Boole especially are so magical and abstruse, that they appear to pass beyond the comprehension and criticism of most other writers, and are calmly ignored." — William Stanley Jevons, 1869 in *Substitution of Similars*.

The Paradigm for an Algebra of Logic

	LOGIC		ALGEBRA
PREMISSES	Propositions	$\xrightarrow{\text{Translate}}$	Equations
			$\downarrow$
CONCLUSIONS	Propositions	$\xleftarrow{\text{Translate}}$	Equations

An Example

	LOGIC		ALGEBRA
PREMISSES	All x is y All y is z	$\xrightarrow{\text{Translate}}$	$x = x \cdot y$ $y = y \cdot z$
			$\downarrow$
CONCLUSIONS	All x is z	$\xleftarrow{\text{Translate}}$	$x = x \cdot z$

Early Insights into Boole's Algebra of Classes

> *That the symbolic processes of algebra, invented as tools of numerical calculation, should be competent to express every act of thought, and to furnish the grammar and dictionary of an all-containing system of logic, would not have been believed until it was proved.*

—Augustus De Morgan, in *A Budget of Paradoxes*
(See the MacTutor article on Boole)

> *The algebra which Boole himself used was simply ordinary numerical algebra as applied to a collection of quantities each of which was assumed to be subject to the quadratic equation* $x(l - x) = 0$*, and Boole showed how this hypothesis could be applied to the solution of many logical problems.*

—C.S. Peirce, from a Nachlass article of 1904 on Huntington's postulates, in Vol. IV of Peirce's collected papers.

Boole's Algebra of Classes (Quick Overview)

Boole used the high-school algebra of numbers, to create an algebra of classes.

His system was based on the following symbols:

Operations	$+, -, \times$	Variables	$x, y, z, \ldots$
Constants	$0, 1$	Equality	$=$

The variables were to be interpreted as classes;
0 would denote the empty class, 1 the universe.

Boole's starting point was to define the operations $+, -, \times$ on classes,
to state the laws and rules of inference for this algebra,
and then to study valid equational arguments

$$\underbrace{p_1(\mathbf{x}) = q_1(\mathbf{x}), \ldots, p_k(\mathbf{x}) = q_k(\mathbf{x})}_{\textit{Premisses}} \;\therefore\; \underbrace{p(\mathbf{x}) = q(\mathbf{x})}_{\textit{Conclusion}}.$$

First Step: Defining Multiplication $A \cdot B$ of Classes

Let us start by assuming, **as Boole likely did**, that valid equations and equational arguments for numbers, say for the integers $\mathbb{Z}$, are valid for this algebra of classes.

For the time being let us consider just one of Boole's operations on classes, namely the product of two classes:

DEFINITION: $AB \ := \ \text{“}A \cap B\text{”}.$

This satisfies the familiar commutative law $AB = BA$ and the associative law $A(BC) = (AB)C$.

It also introduces the non-numerical **idempotent law**

$$\boxed{A^2 = A}$$

which plays a key role in Boole's algebra.

Which Numbers can be Names of Classes?

Having decided to incorporate the basic operations of numerical algebra into an algebra of logic, can we incorporate some of the symbols n for numbers as names of classes?

We would like that such an n, when interpreted as a class, obey the laws of numbers, and vice-versa.

This just means we want $n^2 = n$, so **the only candidates are** $n = 0$ **and** $n = 1$.

Which Classes Could be Named by 0?

To preserve the number law $x \cdot 0 = 0$,

we want the corresponding class law to hold, that is,

for every class A we want $A \cap 0 = 0$.

The only class that 0 could be is the empty class.

Definition. $\boxed{0 := \text{“}\emptyset\text{”}}$ (Nothing; the “empty class”).

Which Classes Could be Named by 1?

From the number law $x \cdot 1 = x$ we find that

1 could only be the universe U.

Definition. $\boxed{1 := \text{“}U\text{''}}$ (the Universe).

So far we have defined the product of two classes

and the interpretation of 0 and 1 as classes.

Before defining $+$ and $-$,

let's consider the expressive power of what we have.

Schröder's Translations in the 1890s

Soon we will see that $1 - A$ denotes the complement of A in Boole's system.

This allows us to express Aristotle's four kinds of categorical propositions by equations and negated equations:

(A)	All x is y.	$x = xy$
(E)	No x is y.	$xy = 0$
(I)	Some x is y.	$xy \neq 0$
(O)	Some x is not y.	$x(1 - y) \neq 0$

However, Boole only used equations in his translations of categorical propositions.

Before defining addition and subtraction of classes

two important CAUTIONS are noted.

Two Cautions

<u>CAUTION 1</u>: **Equational arguments have a single equation as the conclusion.**

Consider $\boxed{xy = 0 \;\therefore\; x = 0 \;\text{ or }\; y = 0}$, a valid argument in $\mathbb{Z}$,

This is NOT an equational argument!!

(And it is certainly not valid in Boole's algebra of classes.)

<u>CAUTION 2</u>: **The idempotent law is for variables, and not, in general, for compound terms.**

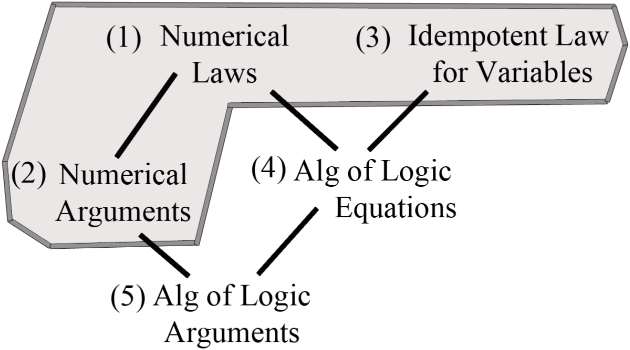
The following reasoning is valid in Boole's algebra of classes:

$$\boxed{(x+x)^2 = x+x} \implies 4x^2 = 2x \implies 4x = 2x \implies \boxed{x = 0}$$

Assuming $(x+x)^2 = x+x$ is a consequence of the idempotent law leads to disaster!

Boole's Equational System (Overview)

The following diagram shows the ingredients that go into Boole's equational logic.



NOTE: (1) and (2) are closed under (uniform) substitutions (of terms for variables), but not (3).

Boole's Equational System (Examples)

(1) Commutative, Associative, Distributive Laws, Etc.

(2) $\boxed{nx = 0 \ \therefore x = 0}$, for $n = 1, 2, \ldots$

$$\boxed{x^2 + y^2 = 0 \ \therefore \ x = 0}$$

(3) $\boxed{x^2 = x}$ for x a variable

(4) $\boxed{x^m = x^n}$ for $1 \leqslant m \leqslant n$

$$\boxed{(1 - x)^2 = 1 - x}$$

$$\boxed{(x + y)^2 = x + y + 2xy}$$

(5) $\boxed{(x + y)^2 = x + y \ \therefore \ xy = 0}$

$$\boxed{p(x, \mathbf{y}) = 0 \ \therefore \ p(1, \mathbf{y}) \cdot p(0, \mathbf{y}) \ = \ 0}$$

Defining $A + B$

$\boxed{\text{Suppose } A + B \text{ is defined}}$ for given classes A and B.

Then

(1) $(A + B)^2 = \boxed{A + B}$

Also,

(2) $(A + B)^2 = A^2 + 2AB + B^2 = \boxed{A + B + 2AB}$

Combining (1) and (2) we obtain

$$A + B = A + B + 2AB,$$

so $\boxed{AB = 0}$.

Thus if $A + B$ is defined we must have $A \cap B = \emptyset$.

This means **addition** must be a **partial** operation on classes.

Defining $A + B$ (Cont'd)

Since $A + B$ is assumed to be defined,

let $C = A + B$ and $D = A \cup B$. We evaluate CD two ways.

$$\begin{aligned}
\boxed{CD} &= (A + B) \cdot D \\
&= A \cdot D \; + \; B \cdot D \\
&= A \cap (A \cup B) \; + \; B \cap (A \cup B) \; = \; \boxed{A + B}.
\end{aligned}$$

$$\begin{aligned}
\boxed{CD} &= C \cap (A \cup B) \\
&= [C \cap A] \; \cup \; [C \cap B] \\
&= [C \cdot A] \; \cup \; [C \cdot B] \\
&= (A^2 + BA) \cup (AB + B^2) \\
&= \boxed{A \cup B} \quad \text{since } AB = 0.
\end{aligned}$$

Thus $\boxed{A + B = A \cup B}$.

In summary, if $A + B$ is defined, then

$A \cap B = \emptyset$ and $A + B = A \cup B$.

Defining $A - B$

Likewise we can show that $\boxed{\text{if } A - B \text{ is defined}}$ then $\boxed{B \subseteq A}$ and $\boxed{A - B = A \setminus B}$.

This leads us to Boole's definitions:

$$
\begin{array}{rcl}
A \cdot B & := & A \cap B \\
A + B & := & \begin{cases} A \cup B & \text{if } A \cap B = \emptyset \\ \text{undefined otherwise.} \end{cases} \\
A - B & := & \begin{cases} A \setminus B & \text{if } B \subseteq A \\ \text{undefined otherwise.} \end{cases} \\
0 & := & \emptyset \\
1 & := & U.
\end{array}
$$

REMARK: Dropping the 'undefined' restrictions gives models for a version of the Modern Algebra of Classes.

But then one cannot freely use the equational algebra of numbers.

Expressing Modern Operations in Boole's System

Boole explicitly expressed our modern operations of *union* ($\cup$), *intersection* ($\cap$), *complement* ($'$) and *symmetric difference* ($\triangle$) by **totally defined idempotent** terms in his system:

- $A \cap B \;=\; \boxed{AB}$
- $A' \;=\; \boxed{1 - A}$.
- $A \cup B \;=\; \boxed{A + (1 - A)B} \;=\; (A + B) - AB$
- $A \triangle B \;=\; \boxed{A(1 - B) + (1 - A)B} \;=\; (A + B) - 2AB$

For two of the operations, $\cup$ and $\triangle$, alternate expressions have been derived using the algebra of numbers.

These alternate terms are **uninterpretable** (undefined) for some values of A and B. But they are still idempotent in Boole's algebra of classes.

Boole said that using uninterpretable terms is OK—the important thing is to be **faithful to the laws and rules of inference**!

Interpretable Results Using Uninterpretables!

Let's use (partially) uninterpretable terms in Boole's algebra to find properties of the symmetric difference $x \triangle y$, a total operation.

We start by using $x \triangle y = (x + y) - 2xy$.

1. $x \triangle x \;=\; (x + x) - 2x \;=\; 0$
2. $x \triangle y \;=\; (x + y) - 2xy \;=\; (y + x) - 2yx \;=\; y \triangle x$
3. $x \triangle (y \triangle z) \;=\; x \triangle \big((y + z) - 2yz\big)$

$$
\begin{aligned}
&= \Big(x + \big((y + z) - 2yz\big)\Big) - 2x\big((y + z) - 2yz\big) \\
&= (x + y + z) - 2(xy + xz + yz) + 4xyz && \text{(number algebra)} \\
&= z \triangle (x \triangle y) && \text{(by symmetry)} \\
&= (x \triangle y) \triangle z && \text{(by item 2).}
\end{aligned}
$$

Boole's Rule of 0 and 1

After presenting his laws and rules of inference, Boole suddenly offered a radically new foundational principle for his work, which we call the **Rule of 0 and 1**.

Boole said that **the laws, axioms and processes** of his algebra of logic were exactly the same as those for numbers when the variables can only assume the values 0 and 1. (*LT*, p. 37).

> *Let us conceive, then, of an Algebra in which* **the symbols x, y, z, etc. admit indifferently of the values 0 and 1**, *and of these values alone. The laws, the axioms, and the processes, of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of interpretation will alone divide them.* **Upon this principle the method of the following work is established.**

The Rule of 0 and 1 (Cont'd)

To make Boole's rule precise, let $\boxed{\mathbb{Z} \models_{01} \varphi(\mathbf{x})}$ mean $\varphi(\mathbf{x})$ holds in $\mathbb{Z}$ whenever the variables are restricted to the two values 0 and 1.

The RULE of 0 and 1
An equational argument $\boxed{\varepsilon_1(\mathbf{x}), \ldots, \varepsilon_k(\mathbf{x}) \ \therefore \varepsilon(\mathbf{x})}$ is correct in

Boole's algebra of logic iff $\boxed{\mathbb{Z} \ \models_{01} \ \varepsilon_1(\mathbf{x}) \ \wedge \cdots \wedge \ \varepsilon_k(\mathbf{x}) \ \rightarrow \ \varepsilon(\mathbf{x})}$

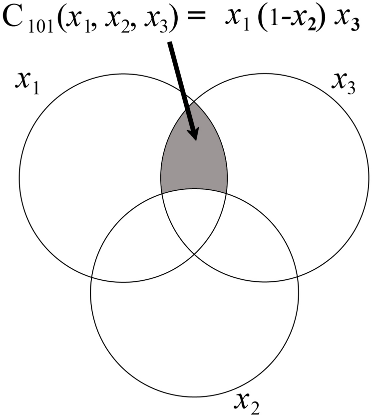
The latter is equivalent to

$$\boxed{\mathbb{Z} \ \models \ \bigwedge_\sigma \Big[\varepsilon_1(\sigma) \ \wedge \cdots \wedge \ \varepsilon_k(\sigma) \ \rightarrow \ \varepsilon(\sigma)\Big]}$$

with σ ranging over strings of 0s and 1s.

We write $\boxed{s(\mathbf{x}) \equiv t(\mathbf{x})}$ if $\mathbb{Z} \models_{01} s(\mathbf{x}) = t(\mathbf{x})$.

The Constituents $C_\sigma(\mathbf{x})$



For x a variable let $\quad C_0(x) = 1 - x, \quad C_1(x) = x.$

Given $\mathbf{x} := x_1, \ldots, x_m$ let σ be a string of 0's and 1's of length m.

$\boxed{C_\sigma(\mathbf{x}) := C_{\sigma_1}(x_1) \cdots C_{\sigma_m}(x_m)}$ is a **constituent** of $\mathbf{x}$.

Properties of Constituents $C_\sigma(\mathbf{x})$

MAIN PROPERTIES OF CONSTITUENTS:

$$C_\sigma(\mathbf{x}) \cdot C_\tau(\mathbf{x}) \equiv \begin{cases} C_\sigma(\mathbf{x}) & \text{if } \sigma = \tau \\ 0 & \text{if } \sigma \neq \tau \end{cases}$$

$$C_\sigma(\tau) \equiv \begin{cases} 1 & \text{if } \sigma = \tau \\ 0 & \text{if } \sigma \neq \tau \end{cases}$$

$$1 \equiv \sum_\sigma C_\sigma(\mathbf{x})$$

$$mC_\sigma(\mathbf{x}) \equiv nC_\sigma(\mathbf{x}) \quad \Leftrightarrow \quad m = n$$

Expansion Theorem

THEOREM: $\boxed{t(\mathbf{x}) \;\equiv\; \sum_\sigma t(\sigma)\, C_\sigma(\mathbf{x})}$

COR: $t(\mathbf{x})C_\sigma(\mathbf{x}) \equiv t(\sigma)C_\sigma(\mathbf{x})$

COR: $s(\mathbf{x}) \equiv t(\mathbf{x})$ iff $s(\sigma) = t(\sigma)$, for all σ

COR: $t(\mathbf{x})^2 \equiv t(\mathbf{x})$ iff $t(\sigma) \in \{0, 1\}$ for all σ.

LEMMA: Every equation $r(\mathbf{x}) = s(\mathbf{x})$ is equivalent to an equation in the form $t(\mathbf{x}) = 0$. (Put $t(\mathbf{x}) = r(\mathbf{x}) - s(\mathbf{x})$.)

COR: *Every* equation is equivalent to

(1) setting certain constituents equal to 0, as well as to

(2) a single totally interpretable equation.

$$
\begin{aligned}
t(\mathbf{x}) = 0 \quad &\text{iff} \quad \bigwedge \big\{ C_\sigma(\mathbf{x}) = 0 \;:\; t(\sigma) \neq 0 \big\} \\
&\text{iff} \quad 0 \;=\; \sum \big\{ C_\sigma(\mathbf{x}) \;:\; t(\sigma) \neq 0 \big\}.
\end{aligned}
$$

Reduction Theorem

A system of equations can always be reduced to a single equation.

THEOREM: A system of equations

$$t_1(\mathbf{x}) = 0,\ \ldots,\ t_k(\mathbf{x}) = 0$$

is equivalent to the single equation

$$t_1(\mathbf{x})^2 + \cdots + t_k(\mathbf{x})^2 \ = \ 0.$$

Note that if the $t_i(\mathbf{x})$ are idempotent terms, that is, $t_i(\mathbf{x})^2 \equiv t_i(\mathbf{x})$, then one can choose the single equation to be

$$t_1(\mathbf{x}) + \cdots + t_k(\mathbf{x}) \ = \ 0.$$

Elimination Theorem

THEOREM: The complete result of eliminating **x** from the single equation $t(\mathbf{x}, \mathbf{y}) = 0$ is

$$\prod_{\sigma} t(\sigma, \mathbf{y}) \; = \; 0.$$

Syllogisms are special cases of Elimination.

For example, the conclusion of $\dfrac{\begin{array}{c}\text{All A is B} \\ \text{All B is C}\end{array}}{\text{All A is C}}$ is the result of eliminating B from the premisses.

Solution Theorem

THEOREM: To solve $\boxed{q(\mathbf{x}) \cdot y = p(\mathbf{x})}$, write $\boxed{y = \dfrac{p(\mathbf{x})}{q(\mathbf{x})}}$.

The fraction has no meaning, so apply the Expansion Theorem:

$$y \;=\; \sum_{\sigma} \frac{p(\sigma)}{q(\sigma)} C_{\sigma}(\mathbf{x}).$$

Replace coefficients $\dfrac{n}{n}$ with $n \neq 0$ by 1.

Coefficients $\dfrac{0}{0}$ become arbitrary parameters v.

Coefficients $\dfrac{0}{n}$, with $n \neq 0$, are replaced by 0.

For all other coefficients put $C_{\sigma}(\mathbf{x}) = 0$. This gives the necessary and sufficient side-conditions on $\mathbf{x}$ for the solution to exist.

Solution Theorem (Cont'd)

In summary, let

$$\begin{aligned} J_1 &:= \{\sigma \;:\; p(\sigma) = q(\sigma) \neq 0\} \\ J_2 &:= \{\sigma \;:\; p(\sigma) = q(\sigma) = 0\} \\ J_3 &:= \{\sigma \;:\; 0 \neq p(\sigma) \neq q(\sigma)\} \end{aligned}$$

Then the general solution to the equation is

$$y \;=\; \sum\{C_\sigma(\mathbf{x}) : \sigma \in J_1\} \;+\; v \cdot \sum\{C_\sigma(\mathbf{x}) : \sigma \in J_2\}$$

provided one has

$$C_\sigma(\mathbf{x}) = 0 \quad \text{for} \quad \sigma \in J_3.$$

If $p(\mathbf{x})$ and $q(\mathbf{x})$ are idempotent then one has a simpler expression:

$$y = q(\mathbf{x}) + v\big(1 - p(\mathbf{x})\big), \text{ provided } q(\mathbf{x}) \cdot \big(1 - p(\mathbf{x})\big) = 0.$$

Issues with Boole's Algebra of Logic

(1) **Can one really use the algebra of numbers for an algebra of logic?**

Boole thought it was amazing that the laws of numbers and of logic differed in only one item, the idempotent law.

He said it was possibly beyond the capability of man to understand why. (LT, p. 11)

Indeed for more than a century no one could justify Boole's algebra of logic — but this story belongs to my second talk (tomorrow)!

Issues (Cont'd)

(2) **Can one really use uninterpretables to derive true results about interpretables?**

Boole said absolutely, by his Principles of Symbolical Reasoning.

As a well-known example, he cited the derivation of trigonometric identities by using the 'uninterpretable' $\sqrt{-1}$.

In general Boole's Principles are not correct—ordinary equational reasoning with partial algebras can give false conclusions!

But in Boole's algebra, ordinary equational reasoning gives correct results!

Issues (Cont'd)

(3) **How does one justify Boole's formal use of division, with coefficients like $\frac{0}{0}$ and $\frac{1}{0}$?**

Solving $q(\mathbf{x})y \;=\; p(\mathbf{x})$ is equivalent to solving the system

$$\boxed{q(\sigma)yC_\sigma(\mathbf{x}) \;=\; p(\sigma)C_\sigma(\mathbf{x})} \qquad (\text{all } \sigma).$$

If $q(\sigma) = 0$ but $p(\sigma) \neq 0$ then clearly $C_\sigma(\mathbf{x}) = 0$.

This is just Boole's requirement that if $\dfrac{p(\sigma)}{q(\sigma)} = \dfrac{n}{0}$, with $n \neq 0$, then one sets $C_\sigma(\mathbf{x}) = 0$.

Etc.

Boole's expansion with fractional coefficients is useful as *a mnemonic device*.

How was Boole's Algebra of Logic Received?

They loved the beauty and perfection of his main theorems.

They did not like his "algebra of numbers" approach.

Within a decade a movement was underway to replace Boole's approach with one based solely on union, complement and intersection.

All four of Boole's main theorems could be comfortably formulated to fit the new 'Boolean Algebra".

This was first done by E. Schröder in his 1877 monograph "Operationskreis des Logikkalkuls".

During the years 1910–1940 Harvard scholars succeeded in applying the name "Boolean Algebra" to this alternate approach, as C.S. Peirce had always thought the right thing to do.

THANK YOU!

$\boxed{\text{THE END}}$