DIRICHLET DENSITY EXTENDS ASYMPTOTIC DENSITY IN MULTIPLICATIVE SYSTEMS

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Abstract. Dirichlet density extends global asymptotic density in an additive number system $\mathcal{A}$ whose generating series $\mathbf{A}(x)$ diverges at its radius of convergence (see [1], p. 50). This note shows that the parallel result holds for a multiplicative number system $\mathcal{A}$, namely if the abcissa of convergence α of the generating series $\mathbf{A}(x)$ is finite and $\mathbf{A}(\alpha) = \infty$ then Dirichlet density extends global asymptotic density. Using this result it is proved that global asymptotic density Δ satisifies

$$\Delta\big(\mathsf{P}_1^{\geq m_1} \cdots \mathsf{P}_k^{\geq m_k}\mathsf{A}\big) \;=\; \Delta\big(\mathsf{P}_1^{\geq m_1}\mathsf{A}\big) \cdots \Delta\big(\mathsf{P}_k^{\geq m_k}\mathsf{A}\big),$$

where the P_i are pairwise disjoint sets of indecomposables, provided partition sets of $\mathcal{A}$ have global asymptotic density. This answers a question in [2].

1. Introduction

The relevant notation from [1] is presented in this section. Given a Dirichlet series

$$\mathbf{R}(x) \;=\; \sum_{n \geq 1} r(n) n^{-x},$$

the *global count function* for $\mathbf{R}(x)$ is

$$R(x) \;=\; \sum_{n \leq x} r(n). \tag{1.1}$$

A *multiplicative number system*

$$\mathcal{A} \;=\; (\mathsf{A}, \mathsf{P}, \cdot, 1, \|\ \|)$$

consists of a countable free monoid $(\mathsf{A}, \cdot, 1)$, freely generated by its set P of indecomposable elements, with a multiplicative norm $\|\ \|$ from A to the positive integers, that is,

$$\begin{aligned} \|\mathsf{a} \cdot \mathsf{b}\| &= \|\mathsf{a}\| \cdot \|\mathsf{b}\| \\ \|\mathsf{a}\| &= 1 \quad \text{if and only if} \quad \mathsf{a} = 1, \end{aligned}$$

with the property that only finitely many elements of A can have the same norm. α is the abscissa of convergence of $\mathbf{A}(x)$.

For $\mathsf{B} \subseteq \mathsf{A}$ the *generating series* $\mathbf{B}(x)$ for B is the Dirichlet series

$$\mathbf{B}(x) \;=\; \sum_{n \geq 1} b(n) n^{-x}$$

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where $b(n)$ is the number of elements in B of norm n. The (global) *asymptotic density* $\Delta(\mathsf{B})$ is defined by

$$\Delta(\mathsf{B}) \;=\; \lim_{n\to\infty} \frac{B(n)}{A(n)},$$

providing this limit exists. The *Dirichlet density* $\partial(\mathsf{B})$ of B is given by

$$\partial(\mathsf{B}) \;=\; \lim_{x\to\alpha+} \frac{\mathbf{B}(x)}{\mathbf{A}(x)},$$

provided the limit exists.*

Let

$$\mathbf{B}^+(x) \;=\; \sum_{n\geq 1} B(n)n^{-x},$$

and let $\mathbf{A}^+(x)$ have abcissa of convergence α^+. A modified Dirichlet density of B is given by the following limit when it exists:

$$\partial^+(\mathsf{B}) \;=\; \lim_{x\to\alpha^+ +} \frac{\mathbf{B}^+(x)}{\mathbf{A}^+(x)}.$$

Lemma 1 (Proposition 9.11 in [1])**.** *If α^+ is finite and $\mathbf{A}^+(\alpha^+) = \infty$ then*

$$\Delta \subseteq \partial^+,$$

that is, for any $\mathsf{B} \subseteq \mathsf{A}$, if $\Delta(\mathsf{B})$ exists then so does $\partial^+(\mathsf{B})$, and they are equal.

2. Dirichlet Density Extends Global Asymptotic Density

Let $\mathcal{A}$ be a multiplicative number system. This section shows that $\alpha < \infty$ and $\mathbf{A}(\alpha) = \infty$ imply $\partial^+ = \partial$, and thus $\Delta \subseteq \partial$.

Lemma 2. $\qquad \alpha^+ = \alpha + 1.$

Proof. Let $A^+(n) = A(1) + \cdots + A(n)$. Then, by Proposition 7.3 (b) of [1],

$$\alpha^+ \;=\; \limsup_{n\to\infty} \frac{\log A^+(n)}{\log n}.$$

Since

$$\frac{n-1}{2}A(n/2) \;\leq\; \sum_{n/2\leq m\leq n} A(m) \;\leq\; A^+(n) \;\leq\; nA(n)$$

it follows that

$$\frac{\log A(n/2) \,+\, \log(n-1) \,-\, 1}{\log n} \;\leq\; \frac{\log A^+(n)}{\log n} \;\leq\; \frac{\log A(n) \,+\, \log n}{\log n}.$$

Taking the limsup as $n \to \infty$ gives

$$\alpha + 1 \;\leq\; \alpha^+ \;\leq\; \alpha + 1.$$

□

Thus α is finite iff α^+ is finite. *From now on it is assumed that α and α^+ are finite.*

Lemma 3. *For $x > \alpha$ and $\mathsf{B} \subseteq \mathsf{A}$,* $\displaystyle\lim_{n\to\infty} \frac{B(n)}{n^x} = 0.$

*The notation $x \to \alpha+$ means that x approaches α from the right.

Proof. Let α_{B} be the abcissa of convergence of $\mathbf{B}(x)$. Then

$$\limsup_{n\to\infty} \frac{\log B(n)}{\log n} \;=\; \begin{cases} 0 & \text{if } \mathsf{B} \text{ is finite} \\ \alpha_{\mathsf{B}} & \text{else.} \end{cases}$$

Since both 0 and α_{B} are at most α and $\alpha < x$ it follows, for n sufficiently large, that

$$0 \;\le\; B(n) \;<\; n^{(\alpha+x)/2},$$

and thus eventually

$$0 \;\le\; \frac{B(n)}{n^x} \;<\; n^{(\alpha-x)/2}.$$

Now note that the right side goes to 0 as $n \to \infty$ since $\alpha - x < 0$. □

Lemma 4. *For $x > \alpha$ and $\mathsf{B} \subseteq \mathsf{A}$,*

$$\mathbf{B}(x) \;=\; x \int_1^\infty \frac{B(t)}{t^{x+1}} dt.$$

Proof. Use summation by parts on $\mathbf{B}(x) = \sum_{n\ge 1} b(n) n^{-x}$ (see [3], Theorem 421 on p. 346) and Lemma 3 above. □

Lemma 5. *For $x > \alpha$ and $\mathsf{B} \subseteq \mathsf{A}$,*

$$\left|\mathbf{B}^+(x+1) - \frac{1}{x}\mathbf{B}(x)\right| \;\le\; \mathbf{B}^+(x+2).$$

Proof.

$$\begin{aligned} \left|\mathbf{B}^+(x+1) - \frac{1}{x}\mathbf{B}(x)\right| &= \left|\sum_{n=1}^\infty \frac{B(n)}{n^{x+1}} - \int_1^\infty \frac{B(t)}{t^{x+1}} dt\right| \\ &\le \sum_{n=1}^\infty B(n) \left|\frac{1}{n^{x+1}} - \int_n^{n+1} \frac{dt}{t^{x+1}}\right| \\ &\le \sum_{n=1}^\infty B(n) \left|\frac{1}{n^{x+1}} - \frac{1}{(n+1)^{x+1}}\right| \\ &\le (x+1) \sum_{n=1}^\infty B(n) \frac{1}{n^{x+2}} \\ &= (x+1)\mathbf{B}^+(x+2). \end{aligned}$$

□

Lemma 6. *If $\mathbf{A}(\alpha) = \infty$ then $\partial = \partial^+$.*

Proof. Given $\mathsf{B} \subseteq \mathsf{A}$ let

$$\begin{aligned} f(x) &= \mathbf{A}^+(x+1) - \frac{1}{x}\mathbf{A}(x) \\ g(x) &= \mathbf{B}^+(x+1) - \frac{1}{x}\mathbf{B}(x). \end{aligned}$$

Then, by Lemma 5, both $f(x)$ and $g(x)$ are bounded for $x \in (\alpha, \alpha+1)$, and one has

$$\frac{\mathbf{B}^+(x+1)}{\mathbf{A}^+(x+1)} \;=\; \frac{\mathbf{B}(x)+xg(x)}{\mathbf{A}(x)+xf(x)}.$$

Thus, in view of Lemma 2, $\partial^+(\mathsf{B})$ exists iff $\partial(\mathsf{B})$ exists, and if so they are equal. □

Lemma 6 answers the question posed at the bottom of p. 162 of [1].

Lemma 7. $\mathbf{A}(\alpha) = \infty$ *iff* $\mathbf{A}^+(\alpha^+) = \infty$.

Proof. If $\alpha = 0$ then

$$\begin{aligned} \mathbf{A}(\alpha) &= \sum_n a(n) \;=\; \infty \\ \mathbf{A}^+(\alpha^+) &= \sum_n A(n)/n \;=\; \infty. \end{aligned}$$

Now suppose $\alpha > 0$. For $x > \alpha$ Lemma 5 gives

$$\mathbf{A}^+(x+1) \;\le\; \frac{1}{x}\mathbf{A}(x) \;+\; (x+1)\mathbf{A}^+(x+2),$$

so, taking the limsup as $x \to \alpha+$ gives, in view of Lemma 2,

$$\mathbf{A}^+(\alpha^+) \;\le\; \frac{1}{\alpha}\mathbf{A}(\alpha) \;+\; (\alpha+1)\mathbf{A}^+(\alpha^+ + 1).$$

As $\mathbf{A}^+(\alpha^+ + 1) < \infty$ it follows that

$$\mathbf{A}^+(\alpha^+) = \infty \quad \text{implies} \quad \mathbf{A}(\alpha) = \infty.$$

Likewise

$$\frac{1}{x}\mathbf{A}(x) \;\le\; \mathbf{A}^+(x+1) \;+\; (x+1)\mathbf{A}^+(x+2)$$

gives

$$\mathbf{A}(\alpha) = \infty \quad \text{implies} \quad \mathbf{A}^+(\alpha^+) = \infty.$$

□

Theorem 1. *If $\mathbf{A}(\alpha) = \infty$ then $\Delta \subseteq \partial$, that is, Dirichlet density extends global asymptotic density.*

Proof. This follows from Lemma 1, Lemma 6 and Lemma 7. □

The parallel result for additive number systems was proved in Proposition 3.13 of [1].

3. An Application to Density of Partition Sets

Let $\mathcal{A}$ be a multiplicative number system. For B_1 and B_2 subsets of A the product $\mathsf{B}_1\mathsf{B}_2$ is defined in the obvious way:

$$\mathsf{B}_1\mathsf{B}_2 \;=\; \{\mathsf{b}_1\mathsf{b}_2 : \mathsf{b}_i \in \mathsf{B}_i\}.$$

Given a nonempty subset Q of the indecomposables P and a positive integer m let

$$\mathsf{Q}^{\ge m}$$

be the set of all elements of A that can be written as a product involving at least m factors from Q, and only factors from Q. Then let

$$\mathsf{Q}^{\geq 0} \;=\; \{1\} \cup \mathsf{Q}^{\geq 1}.$$

If $\mathsf{P}_1, \ldots, \mathsf{P}_k$ are nonempty and pairwise disjoint subsets of P, and if $m_1, \ldots, m_k$ are nonnegative integers, then

$$\mathsf{P}_1^{\geq m_1} \cdots \mathsf{P}_k^{\geq m_k} \mathsf{A} \tag{3.1}$$

is the set of all elements of A that can be written as a product involving at least m_i factors from P_i, for $1 \leq i \leq k$. If the P_i also cover P then one can omit the factor A, that is,

$$\mathsf{P}_1^{\geq m_1} \cdots \mathsf{P}_k^{\geq m_k} \mathsf{A} \;=\; \mathsf{P}_1^{\geq m_1} \cdots \mathsf{P}_k^{\geq m_k}.$$

Sets of this form are special cases of *partition sets* defined in [1]. If the P_i do not cover P then one has

$$\mathsf{P}_1^{\geq m_1} \cdots \mathsf{P}_k^{\geq m_k} \mathsf{A} \;=\; \mathsf{P}_1^{\geq m_1} \cdots \mathsf{P}_k^{\geq m_k} \Big(\mathsf{P} \smallsetminus \bigcup_{i=1}^{k} \mathsf{P}_i\Big)^{\geq 0},$$

and thus $\mathsf{P}_1^{\geq m_1} \cdots \mathsf{P}_k^{\geq m_k} \mathsf{A}$ is again a partition set.*

For $\mathsf{Q} \subseteq \mathsf{P}$ define

$$\partial_{\mathsf{Q}}\big(\mathsf{Q}^{\geq m}\big) \;=\; \lim_{x \to \alpha+} \frac{\mathbf{Q}^{\geq m}(x)}{\mathbf{Q}^{\geq 0}(x)}.$$

Theorem 2. *Suppose $\mathcal{A}$ is such that all sets of the form (3.1) have global asymptotic density, and $\mathbf{A}(\alpha) = \infty$. Then*

$$\Delta\big(\mathsf{P}_1^{\geq m_1} \cdots \mathsf{P}_k^{\geq m_k} \mathsf{A}\big) \;=\; \Delta\big(\mathsf{P}_1^{\geq m_1} \mathsf{A}\big) \cdots \Delta\big(\mathsf{P}_k^{\geq m_k} \mathsf{A}\big),$$

where the P_i are nonempty pairwise disjoint sets of indecomposables.

Proof. There are two cases to consider, depending on whether or not the P_i cover P. We will give the details for the case that they do not, the other case being a minor variation. Let $\mathsf{P}_{k+1} = \mathsf{P} \smallsetminus \big(\mathsf{P}_1 \cup \cdots \cup \mathsf{P}_k\big)$, a nonempty subset of P, and let $m_{k+1} = 0$. Then

$$\begin{aligned}
\Delta\Big(\prod_{i=1}^{k} \mathsf{P}_i^{\geq m_i} \mathsf{A}\Big) &= \Delta\Big(\prod_{i=1}^{k+1} \mathsf{P}_i^{\geq m_i}\Big) \\
&= \partial\Big(\prod_{i=1}^{k+1} \mathsf{P}_i^{\geq m_i}\Big) \qquad \text{by Theorem 1} \\
&= \prod_{i=1}^{k+1} \partial_{\mathsf{P}_i}\Big(\mathsf{P}_i^{\geq m_i}\Big) \qquad \text{by Theorem 9.37, p. 172 of [1]} \\
&= \prod_{i=1}^{k+1} \lim_{x \to \alpha+} \frac{\mathbf{P}_i^{\geq m_i}(x)}{\mathbf{P}_i^{\geq 0}(x)} \\
&= \prod_{i=1}^{k+1} \lim_{x \to \alpha+} \frac{\mathbf{P}_i^{\geq m_i}(x) \cdot (\mathbf{P} \setminus \mathbf{P}_i)^{\geq 0}(x)}{\mathbf{P}_i^{\geq 0}(x) \cdot (\mathbf{P} \setminus \mathbf{P}_i)^{\geq 0}(x)}
\end{aligned}$$

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*It is easy to show that all partition sets of $\mathcal{A}$ have global asymptotic density iff all sets of the form (3.1) have global asymptotic density. By a result of Warlimont, these conditions guarantee that $\alpha < \infty$.

$$\begin{aligned}
&= \prod_{i=1}^{k+1} \lim_{x\to\alpha+} \frac{\big(\mathbf{P}_i^{\geq m_i}(\mathbf{P}\setminus\mathbf{P}_i)^{\geq 0}\big)(x)}{\mathbf{A}(x)} \qquad \text{by Cor. 9.31, p. 170 of [1]}\\
&= \prod_{i=1}^{k+1} \lim_{x\to\alpha+} \frac{\big(\mathbf{P}_i^{\geq m_i}\mathbf{A}\big)(x)}{\mathbf{A}(x)}\\
&= \prod_{i=1}^{k} \lim_{x\to\alpha+} \frac{\big(\mathbf{P}_i^{\geq m_i}\mathbf{A}\big)(x)}{\mathbf{A}(x)}\\
&= \prod_{i=1}^{k} \partial\big(\mathsf{P}_i^{\geq m_i}\mathsf{A}\big)\\
&= \prod_{i=1}^{k} \Delta\big(\mathsf{P}_i^{\geq m_i}\mathsf{A}\big) \qquad \text{by Theorem 1.}
\end{aligned}$$

□

This answers Question 4, p. 493, from [1] in the affirmative.* One can also write the conclusion of this theorem as

$$\Delta\Big(\bigcap_{i=1}^{k}\mathsf{P}_i^{\geq m_i}\mathsf{A}\Big) \;=\; \Delta\Big(\prod_{i=1}^{k}\mathsf{P}_i^{\geq m_i}\mathsf{A}\Big) \;=\; \prod_{i=1}^{k}\Delta\big(\mathsf{P}_i^{\geq m_i}\mathsf{A}\big),$$

showing that sets of the form $\mathsf{P}_i^{\geq m_i}\mathsf{A}$ behave like independent events, provided the P_i are disjoint.

From this result it easily follows, for a multiplicative number system satisfying the hypotheses of Theorem 2, that the values of $\Delta(\mathsf{B})$ for partition sets B are completely determined by the values of $\Delta(\mathsf{Q}^{\geq m}\mathsf{A})$, where $\mathsf{Q}\subseteq\mathsf{P}$ and m is a nonnegative integer.

Problem 1. *If $\mathcal{A}$ is a multiplicative number system such that each set of the form $\mathsf{Q}^{\geq m}\mathsf{A}$ has global asymptotic density, where $\mathsf{Q}\subseteq\mathsf{P}$, does it follow that that all sets of the form (3.1) have global asymptotic density?*

An affirmative answer would imply that all partition sets of $\mathcal{A}$ have global asymptotic density.

References

[1] Stanley N. Burris, *Number Theoretic Density and Logical Limit Laws.* Math. Surveys and Monographs **86**, Amer. Math. Soc., 2001.

[2] Stanley Burris and András Sárközy, *Fine spectra and limit laws I: First-order laws.* Can. J. Math. **49** (1997), 468 - 498.

[3] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford Press, 5th ed., 1979.

*A similar theorem holds for additive number systems for which partition sets have asymptotic density, namely

$$\delta\Big(\sum_{i=1}^{k}(\geq m_i)\mathsf{P}_i+\mathsf{A}\Big) \;=\; \prod_{i=1}^{k}\delta\big((\geq m_i)\mathsf{P}_i+\mathsf{A}\big).$$

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