Boolean Products of Indecomposables

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Recently Vaggione [2] proved that discriminator varieties V are characterized by the two properties: FHP and $V = \mathrm{I}\Gamma^a(V_{\mathrm{DI}})$. He asked if one could replace FHP by the weaker property BFC. In this note we point out that BFC + $V = \mathrm{I}\Gamma^a(V_{\mathrm{DI}})$ leads to every directly indecomposable being simple, and furthermore the class V_{S} of simples in V is defined by a set of universal sentences.

The proof that V_{DI} is axiomatized by universal sentences is similar to the proof in [1]. First suppose that $\mathbf{B} \le \mathbf{D} \in V_{\mathrm{DI}}$, and we want to show that $\mathbf{B} \in V_{\mathrm{DI}}$. We assume that $\mathbf{B}$ is not trivial. Let $\mathbf{A}$ be the subalagebra of $\mathbf{D}[\mathcal{C}]^\star$ consisting of those f which satisfy $f(x_0) \in B$. Then $\mathbf{A} \underset{a}{=} \mathcal{PS}(\mathbf{A})$ as $\mathbf{A}$ is a Boolean product with clopen equalizers, and the stalks on a dense set are directly indecomposable. From our assumptions we know $\mathbf{A} \cong \mathbf{A}' \in \Gamma^a(V_{\mathrm{DI}})$, so we can assume $\mathrm{Triv}(\mathbf{A}') = \emptyset$. Then $\mathbf{A}' \underset{a}{=} \mathcal{PS}(\mathbf{A}')$, so $\mathcal{PS}(\mathbf{A}) \underset{a}{=} \mathcal{PS}(\mathbf{A}')$. This guarantees that $\mathbf{B} \in V_{\mathrm{DI}}$, so V_{DI} is closed under subalgebras.

To show V_{DI} is closed under ultraproducts let $\mathbf{D}_i \in V_{\mathrm{DI}}$, for $i \in I$. Then we argue as in [1] that the stalks of $\mathcal{PS}(\prod_{i\in I}\mathbf{D}_i)$ are the ultraproducts $\prod_{i\in I}\mathbf{D}_i/\mathcal{U}$. By hypothesis there is a $\mathbf{B}$ such that

$$\mathcal{PS}(\prod_{i\in I}\mathbf{D}_i) \cong \mathbf{B} \in \Gamma^a(V_{\mathrm{DI}}).$$

We can assume $\mathrm{Triv}(\mathbf{B}) = \emptyset$. Thus $\mathcal{PS}(\mathbf{B}) \underset{a}{=} \mathbf{B}$, so

$$\mathcal{PS}(\prod_{i\in I}\mathbf{D}_i) \underset{a}{=} \mathcal{PS}(\mathbf{B}),$$

and thus each of the ultraproducts $\prod_{i\in I}\mathbf{D}_i/\mathcal{U}$ is in V_{DI}.

Combining the above two results we see that V_{DI} is a class of algebras defined by universal sentences.

Now suppose $\alpha : \mathbf{D} \to \mathbf{B} \in V_{\mathrm{S}}$ where α is not 1-1 and $\mathbf{B}$ is not trivial. Let $\mathbf{A}$ be the modification of $\mathbf{B}[\mathcal{C}]^\star$ obtained by replacing the stalk $\mathbf{B}$ at x_0 by $\mathbf{D}$, and letting the elements of $\mathbf{A}$ be obtained by taking the elements f of $\mathbf{B}[\mathcal{C}]^\star$ by any g that agrees with f off x_0, and such that $\alpha(g(x_0)) = f(x_0)$. Then $\mathbf{A}$ is in $\Gamma(V_{\mathrm{DI}}) \setminus \Gamma^a(V_{\mathrm{DI}})$. Choose $\mathbf{A}'$ such that $\mathbf{A} \cong \mathbf{A}' \in \Gamma^a(V_{\mathrm{DI}})$. We can assume that $\mathrm{Triv}(\mathbf{A}') = \emptyset$.

Then $\mathbf{A}' \underset{a}{=} \mathcal{PS}(\mathbf{A}') \underset{a}{=} \mathcal{PS}(\mathbf{A})$. For ϕ a factor congruence of $\mathbf{A}$ we have $\phi_x \in \{\Delta, \nabla\}$. For x such that $\phi_x = \nabla$ we can choose $f, g \in A$ such that $f \neq g$ on a neighborhood M of x. Then $\phi_x = \nabla$ on M, so $\phi_x = \nabla$ on an open subset of X. Likewise $\overline{\phi}_x = \nabla$ on an open subset of X, where $\phi, \overline{\phi}$ is a pair of factor congruences of $\mathbf{A}$. Consequently $\phi_x = \Delta$ on a clopen set N, $\overline{\phi}_x = \Delta$ on $X \setminus N$, and this leads to ϕ being a transparent factor congruence. Thus $\mathbf{A} \underset{a}{=} \mathcal{PS}(\mathbf{A})$, so $\mathbf{A} \underset{a}{=} \mathbf{A}'$. But this is a contradiction as one belongs to $\Gamma^a(V_{\mathrm{DI}})$, but not the other.

From the last paragraph we see that every finitely generated $\mathbf{A} \in V_{\mathrm{DI}}$ must be simple (otherwise it has a proper nontrivial simple quotient). And then every $\mathbf{A} \in V_{\mathrm{DI}}$ is simple (as finitely generated subalgebras are in V_{DI}, and hence simple). Thus we have proved the following:

Theorem 1. *Let V be a variety with BFC such that $V = \mathrm{I}\Gamma^a(V_{\mathrm{DI}})$. Then V_{DI} consists only of simple algebras, and it is a class defined by universal sentences.*

References

[1] D. Bigelow and S. Burris, *Boolean algebras of factor congruences.* Acta Sci. Math. **54** (1990), 11–20.

[2] D. Vagggione, *A characterization of discriminator varieties.* Proc. Amer. Math. Soc. **129** (2001), 663–666.