ADMISSIBLE DIRICHLET SERIES

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Abstract. We propose a definition of admissible Dirichlet series as the analog of Hayman's 1956 definition of admissible power series.

1. Introduction

In 1956 Hayman [6] defined *admissible functions*—they are analytic in a neighborhood of 0 and one can use the saddle point method to estimate the coefficients of the power series expansion of such functions. They include the functions e^z and $\exp \dfrac{1}{1-z}$, are closed under product (of series with the same radius of convergence) and under exponentiation.

In this paper a notion of admissibility for functions that have Dirichlet series expansions is proposed. We believe that this is a viable analog of Hayman's definition because (1) this notion of admissible generalizes the conditions of Tenenbaum in [9], (2) there is a fundamental theorem (Theorem 7) that is the analog of Hayman's fundamental theorem, and (3) a product of admissible Dirichlet series (with the same abscissa of convergence) is again admissible.

2. Definition of Admissible

Theorem 1. *Suppose the function* $\mathbf{F}(s)$

(A1) *has a Dirchlet series expansion* $\mathbf{F}(s) = \sum_{n\geq 1} f(n)n^{-s}$, *where the coefficients* $f(n)$ *are nonnegative real, with* $f(1) > 0$,

(A2) *has abscissa of (absolute) convergence* $\alpha \in [0, \infty)$, *and*

(A3) $\mathbf{F}(s)$ *has no zeros in its halfplane of convergence.*

Then there exists a Dirichlet series $\mathbf{H}(s)$ *with real coefficients such that* $\mathbf{F}(s) = e^{\mathbf{H}(s)}$ *for* $\sigma > \alpha$ *where* $s = \sigma + it$.

Proof. This is a slight specialization of Theorem 11.14 in Apostol [1]. □

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This paper benefited from discussions with R. Warlimont after he had received our preliminary version [3]. He suggested the use of an integral condition in the definition of admissible—the definition continued to evolve after this suggestion. Also he found that $\exp\left(\zeta(s)^k\right)$ are examples of admissible functions.

Let $\mathbb{R}$ be the set of real numbers. Assuming that $\mathbf{F}(s)$ satisfies (A1)–(A3) and $\mathbf{F}(s) = e^{\mathbf{H}(s)}$, we will need the basic facts about a Taylor series expansion with remainder of $\mathbf{H}(\sigma + it)$ about $t = 0$. With

$$\mathbf{a}(s) := \mathbf{H}'(s) \quad \mathbf{b}(s) := \mathbf{H}''(s) \quad \mathbf{c}(s) := \mathbf{H}'''(s) \tag{1}$$

we have, for $\sigma > \alpha$ and $t \in \mathbb{R}$,

$$\mathbf{H}(\sigma + it) \;=\; \mathbf{H}(\sigma) + i\mathbf{a}(\sigma)t - \frac{\mathbf{b}(\sigma)}{2}t^2 + \mathsf{R}(\sigma + it) \tag{2}$$

with the remainder term given by

$$\mathsf{R}(\sigma + it) \;=\; -\frac{i}{2}\int_0^t \mathbf{c}(\sigma + iv)(t - v)^2 dv. \tag{3}$$

Thus we can write

$$\frac{\mathbf{F}\big(\sigma + it\big)}{\mathbf{F}(\sigma)} \;=\; \exp\Big(i\mathbf{a}(\sigma)t - \frac{\mathbf{b}(\sigma)}{2}t^2 + \mathsf{R}\big(\sigma + it\big)\Big) \tag{4}$$

$$\frac{\Big|\mathbf{F}\big(\sigma + it\big)\Big|}{\mathbf{F}(\sigma)} \;=\; \exp\Big(-\frac{\mathbf{b}(\sigma)}{2}t^2\Big) \cdot \Big|\exp\Big(\mathsf{R}\big(\sigma + it\big)\Big)\Big|. \tag{5}$$

Definition 2. Suppose $\mathbf{F}(s)$ satisfies (A1)–(A3), with $\mathbf{F}(s) = e^{\mathbf{H}(s)}$. Let $\mathbf{a}(s)$ and $\mathbf{b}(s)$ be the first two derivatives of $\mathbf{H}(s)$ as in (1), let $\mathsf{R}(\sigma + it)$ be the remainder term of the Taylor expansion for $\mathbf{H}(\sigma + it)$ as in (2), and suppose there is a function $\delta : (\alpha, \beta) \to (0, 1)$, for some $\beta > \alpha$, such that as $\sigma \to \alpha+$

(A4) $\delta(\sigma) \;\to\; 0$

(A5) $\sigma^2\mathbf{b}(\sigma) \;\to\; \infty$

(A6) $\mathbf{b}(\sigma) \cdot \exp\Big(-\mathbf{b}(\sigma)\delta(\sigma)^2\Big) \;\to\; 0$

(A7) $\mathsf{R}(\sigma + it) \;\to\; 0$ uniformly for $|t| \le \delta(\sigma)$

(A8) $\dfrac{\sigma\sqrt{\mathbf{b}(\sigma)}}{\mathbf{F}(\sigma)}\displaystyle\int_{|t|\ge\delta(\sigma)} \big|\mathbf{F}(\sigma + it)\big| \frac{dt}{\sigma^2 + t^2} \;\to\; 0\,.$

Then we say $\mathbf{F}(s)$ is **admissible**, as **witnessed** by $\delta(\sigma)$.

Remark 3. Except for §5 we can replace (A6) and (A8) by the following, giving a more general notion of admissible:

(A6-) $\mathbf{b}(\sigma)\delta(\sigma)^2 \;\to\; \infty$

(A8-) $\dfrac{\sigma\sqrt{\mathbf{b}(\sigma)}}{\mathbf{F}(\sigma)}\displaystyle\int_{|t|\ge\delta(\sigma)} \mathbf{F}(\sigma + it)x^{it}\frac{dt}{(\sigma + it)(\sigma + 1 + it)} \;\to\; 0$
uniformly for $x > 0$

The full strength of (A6) and (A8) are used to prove the product theorem in §5.

3. Asymptotic Estimates and Regular Variation

In this section we assume that $\mathbf{F}(s)$ is admissible, witnessed by $\delta(\sigma)$. (A5) implies

$$\mathbf{b}(\sigma) \;\to\; \infty \qquad \text{as } \sigma \to \alpha+, \tag{6}$$

so there is a $\beta > \alpha$ such that

$$\mathbf{b}(\sigma) > 0 \qquad \text{for } \sigma \in (\alpha, \beta). \tag{7}$$

We will consistently use β as a number in (α, ∞) such that $\mathbf{b}(\sigma) > 0$ on (α, β), keeping the original requirement that $\delta(\sigma)$ be defined on (α, β). From (A6) and (6) we have

$$\mathbf{b}(\sigma)\delta(\sigma)^2 \ \to\ \infty \qquad \text{as } \sigma \to \alpha+. \tag{8}$$

Definition 4. The partial sums of the coefficients of $\mathbf{F}(s)$ and its integral are denoted as follows:

$$F(x) \ := \ \sum_{n \le x} f(n)$$
$$\widehat{F}(x) \ := \ \int_1^x F(u)du.$$

Hayman [6] makes a direct application of Cauchy's integral formula to express the coefficients of a power series. Tenenbaum [9] makes a direct application of Perron's integral formula to express $F(x)$. The next lemma, where the Perron formula is used to express $\widehat{F}(x)$, is used to derive a formula that leads to the verification of regular variation at infinity for $F(x)$. [1]

Lemma 5. *For $x > 0$ and $c > \alpha$,*

$$\widehat{F}(x) \ = \ \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathbf{F}(s)\ \frac{x^{s+1}}{s(s+1)}\ ds\,.$$

Proof. See [7] or Lemma 11.22 in [2]. □

An elementary estimate will also be needed.

Lemma 6. *For $h, \lambda > 0$ and $\kappa \in \mathbb{R}$,*

$$\int_{-h}^{h} e^{i\kappa - \lambda u^2} du \ = \ \sqrt{\frac{\pi}{\lambda}} e^{-\kappa^2/4\lambda}\Big(1 + \varepsilon(h, \kappa, \lambda)\Big),$$

where

$$\big|\varepsilon(h, \kappa, \lambda)\big| \ < \ \frac{2}{h\sqrt{\lambda}}\,.$$

The following gives the fundamental formula for $\widehat{F}(x)$. It is this form, rather than the asymptotics that can be obtained by specializing σ to be the saddle point σ_x, that leads to a verification of regular variation at infinity.

Theorem 7. *For $x > 0$ and $\sigma > \alpha$*

$$\widehat{F}(x) \ = \ \frac{x^{\sigma+1}\,\mathbf{F}(\sigma)}{\sigma(\sigma+1)\sqrt{2\pi \mathbf{b}(\sigma)}}\left(\exp\left(\frac{-\big(\mathbf{a}(\sigma) + \log x\big)^2}{2\mathbf{b}(\sigma)}\right) \ + \ R(x, \sigma)\right)$$

where

$$R(x, \sigma) \ \to \ 0 \qquad \text{as } \sigma \to \alpha+, \ \text{uniformly for } x > 0.$$

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[1] Oppenheim ([7], [8]) appears to state this lemma for a particular choice of $\mathbf{F}(s)$, the zeta function connected with 'Factorisatio Numerorum', but it is a general result. On pages 210–211 of [2], one of the sources cited for Lemma 5, the several occurrences of $\exp(\mathbf{S}(z))$ need to be replaced by $\mathbf{S}(z)$.

Proof. For $x > 0$ and $\sigma > \alpha$

$$\begin{aligned}
\widehat{F}(x) &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathbf{F}(s)\ \frac{x^{s+1}}{s(s+1)}\ ds \qquad \text{by Lemma 5}\\
&= \frac{x^{\sigma+1}}{2\pi}\int_{-\infty}^{\infty} \frac{\mathbf{F}(\sigma+it)x^{it}}{(\sigma+it)(\sigma+1+it)}\ dt\\
&= \frac{x^{\sigma+1}}{2\pi}\big(J_1(\sigma,x)+J_2(\sigma,x)\big)
\end{aligned}$$

where

$$\begin{aligned}
J_1(\sigma,x) &= \int_{-\delta(\sigma)}^{\delta(\sigma)} \frac{\mathbf{F}(\sigma+it)x^{it}}{(\sigma+it)(\sigma+1+it)}\ dt\\
J_2(\sigma,x) &= \int_{|t|\geq\delta(\sigma)} \frac{\mathbf{F}(\sigma+it)x^{it}}{(\sigma+it)(\sigma+1+it)}\ dt.
\end{aligned}$$

Since

$$\big|J_2(\sigma,x)\big| \ \leq\ \int_{|t|\geq\delta(\sigma)} \big|\mathbf{F}(\sigma+it)\big|\frac{dt}{\sigma^2+t^2}$$

by (A8) we immediately have

$$J_2(\sigma,x) \ =\ \frac{\sqrt{2\pi}\ \mathbf{F}(\sigma)}{\sigma(\sigma+1)\sqrt{\mathbf{b}(\sigma)}}\,\mathrm{o}(1)$$

as $\sigma \to \alpha+$, uniformly for $x > 0$.

Let us collect some simple facts before estimating $J_1(\sigma, x)$. We easily have

$$\frac{\sigma+1}{\sigma+1+it} \ =\ 1+\mathrm{o}(1) \tag{9}$$

as $\sigma \to \alpha+$, uniformly for $|t| \leq \delta(\sigma)$, since

$$\frac{\sigma+1}{\sigma+1+it} \ =\ 1-\frac{it}{\sigma+1+it}$$

and

$$\left|\frac{it}{\sigma+1+it}\right| \ \leq\ \frac{\delta(\sigma)}{\sigma+1} \ =\ \mathrm{o}(1) \qquad \text{by (A4).}$$

Also

$$\left|\frac{\sigma}{\sigma+it}-1\right| \ =\ \left|\frac{it}{\sigma+it}\right| \ \leq\ \frac{|t|}{\sigma}. \tag{10}$$

Let

$$a(\sigma,x) \ =\ \mathbf{a}(\sigma)+\log x.$$

Then by (4), (A7), (9) and (10), for $\sigma \in (\alpha, \beta)$ and $x > 0$

$$
\begin{aligned}
J_1(\sigma, x) &= \int_{|t|\le\delta(\sigma)} \frac{\mathbf{F}(\sigma+it)x^{it}}{(\sigma+it)(\sigma+it+1)}dt \\
&= \frac{\mathbf{F}(\sigma)}{\sigma(\sigma+1)} \int_{|t|\le\delta(\sigma)} \exp\Big(ia(\sigma,x)t - \frac{\mathbf{b}(\sigma)}{2}t^2 + \mathsf{R}(\sigma+it)\Big) \\
&\qquad\qquad\qquad\qquad \cdot \frac{\sigma(\sigma+1)}{(\sigma+it)(\sigma+1+it)}dt \\
&= \frac{\mathbf{F}(\sigma)}{\sigma(\sigma+1)} \int_{|t|\le\delta(\sigma)} \exp\Big(ia(\sigma,x)t - \frac{\mathbf{b}(\sigma)}{2}t^2\Big)\Big(1 + \mathrm{o}(1) + \mathrm{O}\Big(\frac{|t|}{\sigma}\Big)\Big)dt \\
&= \underbrace{\frac{\mathbf{F}(\sigma)}{\sigma(\sigma+1)} \int_{|t|\le\delta(\sigma)} \exp\Big(ia(\sigma,x)t - \frac{\mathbf{b}(\sigma)}{2}t^2\Big)\big(1+\mathrm{o}(1)\big)dt}_{J_{11}(\sigma,x)} \\
&\quad + \underbrace{\frac{\mathbf{F}(\sigma)}{\sigma(\sigma+1)} \int_{|t|\le\delta(\sigma)} \exp\Big(ia(\sigma,x)t - \frac{\mathbf{b}(\sigma)}{2}t^2\Big)\mathrm{O}\Big(\frac{|t|}{\sigma}\Big)dt}_{J_{12}(\sigma,x)}.
\end{aligned}
$$

For $J_{12}(\sigma, x)$ we have, for $\sigma \in (\alpha, \beta)$ and $x > 0$,

$$
\begin{aligned}
\big|J_{12}(\sigma,x)\big| &= \mathrm{O}\Big(\frac{\mathbf{F}(\sigma)}{\sigma(\sigma+1)} \int_{|t|\le\delta(\sigma)} \exp\Big(-\frac{\mathbf{b}(\sigma)}{2}t^2\Big)\frac{|t|}{\sigma}dt\Big) \\
&= \mathrm{O}\Big(\frac{\mathbf{F}(\sigma)}{\sigma^2(\sigma+1)} \int_0^\infty \exp\Big(-\frac{\mathbf{b}(\sigma)}{2}t^2\Big)tdt\Big) \\
&= \mathrm{O}\Big(\frac{\mathbf{F}(\sigma)}{\sigma(\sigma+1)\sqrt{\mathbf{b}(\sigma)}}\Big(\frac{1}{\sigma\sqrt{\mathbf{b}(\sigma)}}\Big)\Big).
\end{aligned}
$$

Thus by (A5)

$$
J_{12}(\sigma,x) = \frac{\sqrt{2\pi}\,\mathbf{F}(\sigma)}{\sigma(\sigma+1)\sqrt{\mathbf{b}(\sigma)}}\,\mathrm{o}(1)
$$

as $\sigma \to \alpha+$, uniformly for $x > 0$.

From Lemma 6 we have, for $\sigma \in (\alpha, \beta)$ and $x > 0$,

$$
\begin{aligned}
J_{11}(\sigma,x) &= \frac{\mathbf{F}(\sigma)}{\sigma(\sigma+1)} \int_{|t|\le\delta(\sigma)} \exp\Big(ia(\sigma,x)t - \frac{\mathbf{b}(\sigma)}{2}t^2\Big)\big(1+\mathrm{o}(1)\big)dt \\
&= \frac{\mathbf{F}(\sigma)}{\sigma(\sigma+1)}\sqrt{\frac{\pi}{\mathbf{b}(\sigma)/2}}\Big(\exp\Big(\frac{-a(\sigma,x)^2}{2\mathbf{b}(\sigma)}\Big) \\
&\qquad\qquad + \varepsilon\big(\delta(\sigma), a(\sigma,x), \mathbf{b}(\sigma)/2\big) + \mathrm{o}(1)\Big) \\
&= \frac{\sqrt{2\pi}\,\mathbf{F}(\sigma)}{\sigma(\sigma+1)\sqrt{\mathbf{b}(\sigma)}}\Big(\exp\Big(\frac{-\big(\mathbf{a}(\sigma)+\log(x)\big)^2}{2\mathbf{b}(\sigma)}\Big) + \mathrm{o}(1)\Big)
\end{aligned}
$$

as $\sigma \to \alpha+$, uniformly for $x > 0$ since by Lemma 6 and (8)

$$
\Big|\varepsilon\big(\delta(\sigma), a(\sigma,x), \mathbf{b}(\sigma)/2\big)\Big| < \frac{2}{\delta(\sigma)\sqrt{\mathbf{b}(\sigma)/2}} = \mathrm{o}(1).
$$

Combining these results we have

$$\begin{aligned} J(\sigma,x) \;&:=\; J_1(\sigma,x) \;+\; J_2(\sigma,x) \\ &=\; \frac{\sqrt{2\pi}\,\mathbf{F}(\sigma)}{\sigma(\sigma+1)\sqrt{\mathbf{b}(\sigma)}}\left(\exp\left(\frac{-\big(\mathbf{a}(\sigma)+\log(x)\big)^2}{2\mathbf{b}(\sigma)}\right) \;+\; \mathrm{o}(1)\right) \end{aligned}$$

as $\sigma \to \alpha+$, uniformly for $x > 0$, and the proof of the theorem is completed by observing that

$$\widehat{F}(x) \;=\; \frac{x^{\sigma+1}}{2\pi}\,J(\sigma,x).$$

□

Corollary 8. $\mathbf{a}(\sigma)$ *is strictly increasing on* (α,β) *and as* $\sigma \to \alpha+$

(a) $\mathbf{a}(\sigma) \;\to\; -\infty$

(b) $\dfrac{\mathbf{a}(\sigma)^2}{\mathbf{b}(\sigma)} \;\to\; \infty$

(c) $\big(\sigma-\alpha\big)\cdot\mathbf{a}(\sigma) \to -\infty.$

Proof. We know that $\mathbf{a}'(\sigma) = \mathbf{b}(\sigma) > 0$ on (α,β), so $\mathbf{a}(\sigma)$ is strictly increasing on (α,β). Now $\widehat{F}(1) = 0$, so by Theorem 7 with $x = 1$ we have for $\sigma \in (\alpha,\beta)$

$$0 \;=\; \frac{\mathbf{F}(\sigma)}{\sigma(\sigma+1)\sqrt{2\pi\mathbf{b}(\sigma)}}\left(\exp\left(\frac{-\mathbf{a}(\sigma)^2}{2\mathbf{b}(\sigma)}\right) \;+\; R(1,\sigma)\right)$$

so

$$0 \;=\; \exp\left(\frac{-\mathbf{a}(\sigma)^2}{2\mathbf{b}(\sigma)}\right) \;+\; R(1,\sigma).$$

Therefore

$$\exp\left(\frac{-\mathbf{a}(\sigma)^2}{2\mathbf{b}(\sigma)}\right) \;\to\; 0 \qquad \text{as } \sigma \to \alpha+,$$

and thus (b) holds. Since $\mathbf{a}(\sigma)$ decreases on (α,β) as $\sigma \to \alpha+$ and $\mathbf{b}(\sigma) \to \infty$ by (6) we see that (a) follows from (b).

To prove (c) we first use (a) to choose a $\gamma \in (\alpha,\beta)$ so that $\mathbf{a}(\sigma) < 0$ for $\sigma \in (\alpha,\gamma)$. Then for $\alpha < \sigma_1 < \sigma < \gamma$ we have, by the mean value theorem,

$$\begin{aligned} \frac{1}{\mathbf{a}(\sigma)} - \frac{1}{\mathbf{a}(\sigma_1)} \;&=\; -\frac{\mathbf{a}'(\xi)}{\mathbf{a}(\xi)^2}(\sigma-\sigma_1) \quad \text{for some } \xi \in (\sigma_1,\sigma) \\ &=\; -\frac{\mathbf{b}(\xi)}{\mathbf{a}(\xi)^2}(\sigma-\sigma_1) \\ &=\; \mathrm{o}(\sigma-\sigma_1) \quad \text{as } \sigma \to \alpha+ \\ &=\; \mathrm{o}(\sigma-\alpha) \quad \text{as } \sigma \to \alpha+. \end{aligned}$$

Letting $\sigma_1 \to \alpha+$

$$\frac{1}{\mathbf{a}(\sigma)} \;=\; \mathrm{o}(\sigma-\alpha) \qquad \text{as } \sigma \to \alpha+$$

so $(\sigma-\alpha)\cdot\mathbf{a}(\sigma) \to -\infty$ as $\sigma \to \alpha+$.

□

In view of Corollary 8(a), from now on we will assume that β was chosen small enough that $\mathbf{a}(\sigma) < 0$ for $\sigma \in (\alpha,\beta)$.

Notice that $\mathbf{a}(\sigma) \to -\infty$ as $\sigma \to \alpha+$ implies that for x sufficiently large the equation $\mathbf{a}(\sigma) + \log x = 0$ has, by the continuity of $\mathbf{a}(\sigma)$, a solution. In particular since $\mathbf{a}'(\sigma)$ is positive on (α, β), for

$$x \ \geq \ x_0 \ := \ \exp\big(- \mathbf{a}(\beta)\big) \ + \ 1$$

one has a unique solution in (α, β).

Definition 9. For $x \geq x_0$ (as just described) let σ_x be the unique solution for $\sigma \in (\alpha, \beta)$ to the equation

$$\mathbf{a}(\sigma) + \log x \ = \ 0. \tag{11}$$

The function σ_x is strictly decreasing on $[x_0, \infty)$ and

$$\sigma_x \to \alpha+ \quad \text{as} \quad x \to \infty \tag{12}$$

since $\lim\limits_{x\to\infty} \mathbf{a}(\sigma_x) = -\lim\limits_{x\to\infty} \log x = -\infty$.

Also note that if one puts $\sigma = \sigma_x$ (where $x \geq x_0$) in the expression for $\widehat{F}(x)$ in Theorem 7 then it simplifies to

$$\widehat{F}(x) \ = \ \frac{x^{\sigma_x+1}\,\mathbf{F}(\sigma_x)}{\sigma_x(\sigma_x+1)\sqrt{2\pi\mathbf{b}(\sigma_x)}}\big(1 + R(x, \sigma_x)\big),$$

where $R(x, \sigma_x) \to 0$ as $x \to \infty$. So we have the following.

Corollary 10.

$$\widehat{F}(x) \ \sim \ \frac{x^{\sigma_x+1}\,\mathbf{F}(\sigma_x)}{\sigma_x(\sigma_x+1)\sqrt{2\pi\mathbf{b}(\sigma_x)}} \quad \textit{as } x \to \infty.$$

The choice of $\sigma = \sigma_x$ is what is commonly meant by 'finding the saddlepoint', and the resulting formula for $\widehat{F}(x)$ is the result of 'applying the saddlepoint method'. In reality the value $s = \sigma_x$ is usually only near a saddle point of the integrand of the integral in Lemma 5, that is, a point s where the derivative of the integrand vanishes. By choosing the line of integration of this integral to pass through (a point near) the saddle point one hopes to concentrate the value of the integral in a small neighborhood of the real axis. Indeed, that is what happens for admissible functions. In the proof of Theorem 7, the value of $\widehat{F}(x)$ is concentrated in the integral $J_1(x)$ when $\sigma = \sigma_x$ as $x \to \infty$, leading to Corollary 10 above.

Corollary 11. *As* $\sigma \to \alpha+$

(a) $\dfrac{\mathbf{F}(\sigma)}{\sigma\sqrt{\mathbf{b}(\sigma)}} \to \infty$ *and*

(b) $\mathbf{F}(\sigma) \to \infty$.

Proof. Note that $\widehat{F}(2) > 0$ (since $f(1) > 0$). Then for $\sigma \in (\alpha, \beta)$, by Theorem 7

$$\widehat{F}(2) \ = \ \frac{2^{\sigma+1}\,\mathbf{F}(\sigma)}{\sigma(\sigma+1)\sqrt{2\pi\mathbf{b}(\sigma)}}\left(\exp\left(\frac{-\big(\mathbf{a}(\sigma)+\log 2\big)^2}{2\mathbf{b}(\sigma)}\right) \ + \ R(2, \sigma)\right). \tag{13}$$

By Corollary 8(a) there is a $\gamma \in (\alpha, \beta)$ such that $\mathbf{a}(\sigma)$ is negative on (α, γ), and thus nonzero. For $\sigma \in (\alpha, \gamma)$ we then have

$$\frac{\big(\mathbf{a}(\sigma)+\log 2\big)^2}{\mathbf{b}(\sigma)} \ = \ \frac{\mathbf{a}(\sigma)^2}{\mathbf{b}(\sigma)}\left(1 + \frac{2\log 2}{\mathbf{a}(\sigma)} + \frac{(\log 2)^2}{\mathbf{a}(\sigma)^2}\right).$$

By Corollary 8 the right hand side of this equation goes to ∞ as $\sigma \to \alpha+$, so

$$\exp\left(\frac{-\big(\mathbf{a}(\sigma)+\log 2\big)^2}{2\mathbf{b}(\sigma)}\right) \ \to\ 0 \qquad \text{as } \sigma \to \alpha+.$$

From Theorem 7 we know $R(2,\sigma) \to 0$ as $\sigma \to \alpha+$; and clearly

$$\frac{2^{\sigma+1}}{(\sigma+1)\sqrt{2\pi}} \ \to\ \frac{2^{\alpha+1}}{(\alpha+1)\sqrt{2\pi}} \ <\ \infty \qquad \text{as } \sigma \to \alpha+.$$

The left side of (13) is a positive constant, so it follows that part (a) of this Corollary must hold: $\dfrac{\mathbf{F}(\sigma)}{\sigma\sqrt{\mathbf{b}(\sigma)}} \to \infty$ as $\sigma \to \alpha+$. Then part (a) and (A5) give $\mathbf{F}(\sigma) \to \infty$ as $\sigma \to \alpha+$, which is part (b). □

The next corollary shows that as $\sigma \to \alpha+$ we have $\mathbf{F}(\sigma)$ growing much faster than any power of $\mathbf{a}(\sigma)$ or $\mathbf{b}(\sigma)$. This leads in turn to the fact that $\mathbf{F}(\sigma)$ grows much faster than any power of $\sigma - \alpha$. Consequently $\mathbf{F}(s)$ cannot have a pole at α.

Corollary 12.

(a) *For all $\varepsilon > 0$,*

$$\mathbf{a}(\sigma) \ =\ \mathrm{o}\big(\mathbf{F}(\sigma)^\varepsilon\big) \qquad \textit{and} \qquad \mathbf{b}(\sigma) \ =\ \mathrm{o}\big(\mathbf{F}(\sigma)^\varepsilon\big) \qquad \textit{as } \sigma \to \alpha+.$$

(b) *For all $r \in \mathbb{R}$,*

$$\big(\sigma-\alpha\big)^r\mathbf{F}(\sigma) \ \to\ \infty \qquad \textit{as } \sigma \to \alpha+.$$

Proof. We break the proof of (a) into two claims.

Claim 1: For all $\varepsilon > 0$ and all $\gamma \in (\alpha, \beta)$ there is a $\sigma \in (\alpha, \gamma)$ such that

$$\frac{|\mathbf{a}(\sigma)|}{\mathbf{F}(\sigma)^\varepsilon} \ <\ 1.$$

Assume not. Then we can choose $\varepsilon > 0$ and $\gamma \in (\alpha, \beta)$ such that for all $\sigma \in (\alpha, \gamma)$

$$\frac{|\mathbf{a}(\sigma)|}{\mathbf{F}(\sigma)^\varepsilon} \ \geq\ 1. \tag{14}$$

Then for $\alpha < \sigma_1 < \sigma < \gamma$ by the mean value theorem

$$\begin{aligned}
\frac{1}{\mathbf{F}(\sigma)^\varepsilon} - \frac{1}{\mathbf{F}(\sigma_1)^\varepsilon} \quad &= \quad \frac{-\varepsilon\mathbf{F}'(\xi)}{\mathbf{F}(\xi)^{1+\varepsilon}}(\sigma-\sigma_1) \quad \text{for some } \xi \in (\sigma_1, \sigma)\\
&= \quad -\varepsilon\frac{\mathbf{a}(\xi)}{\mathbf{F}(\xi)^\varepsilon}(\sigma-\sigma_1)\\
&= \quad \varepsilon\frac{|\mathbf{a}(\xi)|}{\mathbf{F}(\xi)^\varepsilon}(\sigma-\sigma_1)\\
&\geq \quad \varepsilon(\sigma-\sigma_1) \quad \text{by (14).}
\end{aligned}$$

Letting $\sigma_1 \to \alpha+$ gives $1/\mathbf{F}(\sigma)^\varepsilon \geq \varepsilon(\sigma-\alpha)$; so

$$\mathbf{F}(\sigma) \ \leq\ \left(\frac{1}{\varepsilon(\sigma-\alpha)}\right)^{1/\varepsilon} \qquad \text{for } \sigma \in (\alpha,\gamma). \tag{15}$$

We can also assume that $\gamma \in (\alpha, \beta)$ is such that $(\sigma-\alpha)\big|\mathbf{a}(\sigma)\big| > 2/\varepsilon$ for $\sigma \in (\alpha, \gamma)$ by Corollary 8(c). So for $\alpha < \sigma < \sigma_2 < \gamma$

$$-\frac{\mathbf{F}'(u)}{\mathbf{F}(u)} \ =\ -\mathbf{a}(u) \ >\ \frac{2}{\varepsilon(u-\alpha)} \qquad \text{for } u \in [\sigma, \sigma_2]$$

which implies that

$$-\int_\sigma^{\sigma_2} \frac{\mathbf{F}'(u)}{\mathbf{F}(u)} du \;>\; \int_\sigma^{\sigma_2} \frac{2}{\varepsilon(u-\alpha)} du,$$

that is,

$$-\Big(\log \mathbf{F}(\sigma_2) - \log \mathbf{F}(\sigma)\Big) \;>\; \frac{2}{\varepsilon} \log\Big(\frac{\sigma_2-\alpha}{\sigma-\alpha}\Big).$$

From this inequality and (15)

$$\log \mathbf{F}(\sigma_2) + \frac{2}{\varepsilon}\log\Big(\frac{\sigma_2-\alpha}{\sigma-\alpha}\Big) \;<\; \log \mathbf{F}(\sigma) \;\le\; \frac{1}{\varepsilon}\Big(\log\frac{1}{\varepsilon} + \log\frac{1}{\sigma-\alpha}\Big).$$

Thus

$$\begin{aligned}\frac{1}{\varepsilon}\log\frac{1}{\sigma-\alpha} \quad &> \quad \log \mathbf{F}(\sigma_2) + \frac{2}{\varepsilon}\log(\sigma_2-\alpha) - \frac{1}{\varepsilon}\log\frac{1}{\varepsilon} + \frac{2}{\varepsilon}\log\frac{1}{\sigma-\alpha} \\ &= \quad C + \frac{2}{\varepsilon}\log\frac{1}{\sigma-\alpha}\end{aligned}$$

where C is independent of σ. Hence

$$1 \;>\; \frac{C\varepsilon}{\log\big(1/(\sigma-\alpha)\big)} \;+\; 2 \;\to\; 2 \quad \text{as } \sigma \to \alpha+,$$

which is a contradiction, proving Claim 1.

Claim 2: For all $\varepsilon > 0$ there is a $\gamma \in (\alpha, \beta)$ such that

$$\frac{|\mathbf{a}(\sigma)|}{\mathbf{F}(\sigma)^\varepsilon} < 1 \quad \text{for } \sigma \in (\alpha, \gamma]. \tag{16}$$

Let $\varepsilon > 0$ be given. From Claim 1 and Corollary 8(b) we know that there exists a $\gamma \in (\alpha, \beta)$ such that

$$\frac{|\mathbf{a}(\gamma)|}{\mathbf{F}(\gamma)^\varepsilon} < 1 \quad \text{and} \quad \frac{\mathbf{b}(\sigma)}{\mathbf{a}(\sigma)^2} < \varepsilon \quad \text{for } \sigma \in (\alpha, \gamma]. \tag{17}$$

We will show this γ is such that (16) holds. Otherwise there is a $\sigma \in (\alpha, \gamma)$ such that $|\mathbf{a}(\sigma)|/\mathbf{F}(\sigma)^\varepsilon \ge 1$. By the intermediate value theorem there must be a $\sigma \in (\alpha, \gamma)$ such that $|\mathbf{a}(\sigma)|/\mathbf{F}(\sigma)^\varepsilon = 1$. Letting σ_1 be the largest such σ in (α, γ) we have

$$\frac{|\mathbf{a}(\sigma_1)|}{\mathbf{F}(\sigma_1)^\varepsilon} = 1 \quad \text{and} \quad \frac{|\mathbf{a}(\sigma)|}{\mathbf{F}(\sigma)^\varepsilon} < 1 \quad \text{for } \sigma \in (\sigma_1, \gamma].$$

Equivalently

$$|\mathbf{a}(\sigma_1)| - \mathbf{F}(\sigma_1)^\varepsilon = 0 \quad \text{and} \quad |\mathbf{a}(\sigma)| - \mathbf{F}(\sigma)^\varepsilon < 0 \quad \text{for } \sigma \in (\sigma_1, \gamma]. \tag{18}$$

As $|\mathbf{a}(\sigma)| = -\mathbf{a}(\sigma)$ on (α, β), from (18) we have

$$\frac{d}{d\sigma}\Big(-\mathbf{a}(\sigma) - \mathbf{F}(\sigma)^\varepsilon\Big)\Big|_{\sigma=\sigma_1} \;\le\; 0.$$

Hence

$$\begin{aligned} 0 \quad &\le \quad \mathbf{a}'(\sigma_1) + \varepsilon \mathbf{F}'(\sigma_1)\mathbf{F}(\sigma_1)^{\varepsilon-1} \\ &= \quad \mathbf{b}(\sigma_1) + \varepsilon \mathbf{a}(\sigma_1)\mathbf{F}(\sigma_1)^\varepsilon \\ &= \quad \mathbf{b}(\sigma_1) - \varepsilon \mathbf{a}(\sigma_1)^2.\end{aligned}$$

By (17) $\mathbf{b}(\sigma_1) < \varepsilon \mathbf{a}(\sigma_1)^2$. This is a contradiction, proving Claim 2.

From Claim 2 we immediately have $\mathbf{a}(\sigma) = \mathrm{O}\big(\mathbf{F}(\sigma)^{\varepsilon/2}\big)$, and thus by Corollary 11(b) $\mathbf{a}(\sigma) = \mathrm{o}\big(\mathbf{F}(\sigma)^{\varepsilon}\big)$. Then from Corollary 8(b)

$$\mathbf{b}(\sigma) \;=\; \mathrm{o}\big(\mathbf{a}(\sigma)^2\big) \;=\; \mathrm{o}\big(\mathbf{F}(\sigma)^{\varepsilon}\big) \quad \text{as } \sigma \to \alpha+.$$

This finishes the proof of (a).

Part (b) is now a trivial consequence of part (a) and Corollary 8(c).

□

Remark 13. Corollary 12(b) readily shows many Dirichlet series satisfying (A1)–(A3) are not admissible.

(a) $\zeta(s)^k$, $k = 1, 2, \ldots$, is not admissible as it has a pole at its abscissa $\alpha = 1$.

(b) The zeta function

$$\prod_{j=1}^{k} \Big(1 - n_j^{-s}\Big)^{-m_j}$$

of a finitely generated multiplicative number system is not admissible as it has a pole at its abscissa $\alpha = 0$.

Corollary 14. *The function $\widehat{F}(x)$ grows much faster than $x^{\alpha+1}$, namely*

$$\lim_{x\to\infty} \frac{\widehat{F}(x)}{x^{\alpha+1}} \;=\; \infty.$$

Proof. This is clear from Corollary 10, Corollary 11(a) and (12). □

Definition 15. For $\alpha \in \mathbb{R}$, a real-valued function $g(x)$ that is eventually defined on the reals and eventually positive is said to have **regular variation at infinity with index** α, written simply as $g(x) \in \mathsf{RV}_\alpha$, if for any $y > 0$

$$\lim_{x\to\infty} \frac{g(xy)}{g(x)} \;=\; y^{\alpha}. \tag{19}$$

Corollary 16. $\widehat{F}(x) \in \mathsf{RV}_{\alpha+1}$.

Proof. We assume $x, y > 0$. In the expressions for $\widehat{F}(xy)$ and $\widehat{F}(x)$ given by Theorem 7 let $\sigma = \sigma_x$ (for x sufficiently large) and divide to obtain

$$\begin{aligned} \frac{\widehat{F}(xy)}{\widehat{F}(x)} \quad &= \quad (y^{\sigma_x+1}) \frac{\exp\Big(-(\log y)^2/\big(2\mathbf{b}(\sigma_x)\big)\Big) \;+\; R(xy,\sigma_x)}{1 + R(x,\sigma_x)} \\ &\to \quad y^{\alpha+1} \quad \text{as } x \to \infty \end{aligned} \tag{20}$$

since both $R(x, \sigma_x) \to 0$ and $R(xy, \sigma_x) \to 0$ as $x \to \infty$ by Theorem 7 and (12); and since $\mathbf{b}(\sigma_x) \to \infty$ as $x \to \infty$ by (6) and (12). □

Lemma 17. *Let $\mathbf{G}(s) \;=\; \sum_{n\ge1} g(n)/n^s$ be a Dirichlet series with nonnegative real coefficients and abscissa $\alpha \ge 0$, and let $G(x) = \displaystyle\sum_{1\le n\le x} g(n)$, $\widehat{G}(x) = \displaystyle\int_1^x G(u)du$.*

If $\widehat{G}(x) \in \mathsf{RV}_{\alpha+1}$ then

(a) $G(x) \in \mathsf{RV}_\alpha$, *and*

(b) $G(x) \;\sim\; \dfrac{\alpha+1}{x}\, \widehat{G}(x)$.

Proof. This is an immediate consequence of Lemma 11.21 from [2]. □

Corollary 18. *$F(x) \in \mathsf{RV}_\alpha$ and*

$$F(x) \ \sim\ \frac{\alpha+1}{x}\,\widehat{F}(x) \ \sim\ \frac{x^{\sigma_x}\ \mathbf{F}(\sigma_x)}{\sigma_x\sqrt{2\pi\mathbf{b}(\sigma_x)}}$$

as $x \to \infty$.

Proof. By Corollary 10, Corollary 16, Lemma 17 and (12). □

Corollary 19. *The function $F(x)$ grows much faster than x^α, namely*

$$\lim_{x\to\infty}\frac{F(x)}{x^\alpha} \ =\ \infty.$$

Proof. By Corollary 11(a), Corollary 18 and (12). □

From this Corollary it is immediate that $\zeta(s)$ is not admissible (a fact already noted in Remark 13).

4. Tenenbaum's Condtions

A version of admissibility conditions for Dirichlet series due to Tenenbaum [9], 1988, is given in the following.[2]

Definition 20. Suppose $\mathbf{F}(s)$ satisfies conditions (A1)–(A3) and there is a function $T : (\alpha,\beta) \to (0,\infty)$, for some $\beta > \alpha$, such that as $\sigma \to \alpha+$

- **(T1)** $\sigma^2\mathbf{b}(\sigma) \ \to\ \infty$
- **(T2)** $\dfrac{\mathbf{b}(\sigma)^3}{\mathbf{c}(\sigma)^2} \ \to\ \infty$
- **(T3)** $T(\sigma)\,\dfrac{\big|\mathbf{F}(\sigma+it)\big|}{\mathbf{F}(\sigma)} \ \le\ 1,\quad$ for $\sigma \in (\alpha,\beta)$ and for $\delta(\sigma) \le |t| \le T(\sigma)$, where $\delta(\sigma) \ =\ \big|\mathbf{b}(\sigma)\mathbf{c}(\sigma)\big|^{-1/5}$
- **(T4)** $\dfrac{\sqrt{\mathbf{b}(\sigma)}}{T(\sigma)} \ \to\ 0$
- **(T5)** $\big|\mathbf{c}(\sigma+it)\big| \le \big|\mathbf{c}(\sigma)\big| \qquad$ for $\sigma \in (\alpha,\beta)$ and $t \in \mathbb{R}$
- **(T6)** $\liminf\limits_{\sigma\to\alpha+}\big|\mathbf{c}(\sigma)\big| \ >\ 0.$

Then we say that $\mathbf{F}(s)$ is **T-admissible**, as **witnessed** by $T(\sigma)$.

Tenenbaum uses $T(\sigma) = \sigma\mathbf{b}(\sigma)/\varepsilon(\sigma)$ where $\varepsilon(\sigma) \to 0$ as $\sigma \to \alpha+$. This choice of $T(\sigma)$ makes condition (T4) unnecessary. Furthermore he gives an error term that is important to his applications in number theory, especially to the function $\psi(x, y)$.

Theorem 21. *If $\mathbf{F}(s)$ is T-admissible then it is admissible.*

Proof. Let $\mathbf{F}(s) = \exp\big(\mathbf{H}(s)\big)$ be a T-admissible Dirichlet series as witnessed by $T(\sigma) : (\alpha,\beta) \to (0,\infty)$. (T1) shows that

$$\mathbf{b}(\sigma) \to \infty \qquad \text{as } \sigma \to \alpha+, \tag{21}$$

so we can assume that $\mathbf{b}(\sigma)$ is positive on (α,β). From (21) and (T4) it is clear that

$$T(\sigma) \to \infty \qquad \text{as } \sigma \to \alpha+.$$

[2] Tenenbaum actually uses $T(\sigma) = \sigma\mathbf{b}(\sigma)/\varepsilon$, which makes our (T4) unnecessary, and he uses the saddlepoint σ_x instead of σ in (T1)–(T3), which he labels as (H2)–(H4).

By (21) and (T6) one has

$$\delta(\sigma) \to 0 \quad \text{as } \sigma \to \alpha+,$$

so (A4) holds. As (T1) is (A5) we only need to verify that (A6)–(A8) hold.

For (A7) we have for $\sigma \in (\alpha, \beta)$ and $|t| \le \delta(\sigma)$

$$\begin{aligned}
\big|\mathsf{R}(\sigma+it)\big| &\le \big|\mathbf{c}(\sigma)t^3\big| \qquad \text{by (3)}\\
&\le \big|\mathbf{c}(\sigma)\cdot\delta(\sigma)^3\big|\\
&= |\mathbf{c}(\sigma)|\cdot\big|\mathbf{b}(\sigma)\mathbf{c}(\sigma)\big|^{-3/5}\\
&= \left(\frac{\mathbf{c}(\sigma)^2}{\mathbf{b}(\sigma)^3}\right)^{1/5}\\
&\to 0 \qquad \text{as } \sigma \to \alpha+ \quad \text{by (T2).}
\end{aligned}$$

For (A6) we have from (T3) for $\sigma \in (\alpha, \beta)$

$$T(\sigma)\,\frac{\big|\mathbf{F}\big(\sigma+i\delta(\sigma)\big)\big|}{\mathbf{F}(\sigma)} \;\le\; 1.$$

Multiplying this by (T4) gives

$$\sqrt{\mathbf{b}(\sigma)}\,\frac{\big|\mathbf{F}\big(\sigma+i\delta(\sigma)\big)\big|}{\mathbf{F}(\sigma)} \;\to\; 0,$$

which, in view of (5) and the fact that (A7) holds, gives (A6).

Finally (A8) is verified as follows, where $\sigma \in (\alpha, \beta)$:

$$\begin{aligned}
&\sigma\sqrt{\mathbf{b}(\sigma)}\int_{|t|\ge\delta(\sigma)}\frac{\big|\mathbf{F}(\sigma+it)\big|}{\mathbf{F}(\sigma)}\,\frac{dt}{\sigma^2+t^2}\\
\le\;& \sigma\sqrt{\mathbf{b}(\sigma)}\int_{\delta(\sigma)\le|t|\le T(\sigma)}\frac{\big|\mathbf{F}(\sigma+it)\big|}{\mathbf{F}(\sigma)}\,\frac{dt}{\sigma^2+t^2} \;+\; 2\sigma\sqrt{\mathbf{b}(\sigma)}\int_{T(\sigma)}^{\infty}\frac{dt}{t^2}\\
\le\;& \frac{\sqrt{\mathbf{b}(\sigma)}}{T(\sigma)}\left(\sigma\int_{\delta(\sigma)\le|t|\le T(\sigma)}\frac{dt}{\sigma^2+t^2}\right) \;+\; \frac{2\sigma\sqrt{\mathbf{b}(\sigma)}}{T(\sigma)} \qquad \text{by (T3)}\\
=\;& \mathrm{o}(1) \qquad \text{by (T4).}
\end{aligned}$$

□

The conditions of Tenenbaum have proved to be very practical, giving the asymptotics for many naturally occurring examples of Dirichlet series to which the saddlepoint method applies.

Example 22. The function

$$\mathbf{F}(s) \;:=\; e^{\zeta(s)}$$

is readily proved to be T-admissible, witnessed by $T(\sigma) = \mathbf{b}(\sigma)$, after noting

- $\zeta(s) \;=\; \dfrac{1}{s-1} + g(s)$, where $g(s)$ is holomorphic
- there is a constant $C > 0$ such that for $\sigma \in [1, 2]$ and $|t| \ge 1$ we have $\big|\zeta(\sigma+it)\big| \;\le\; C\log|t|$.

From the T-admissibility of $\exp\big(\zeta(s)\big)$ one easily has the T-admissibility of

$$\mathbf{F}_\lambda(s) \;:=\; \exp\big(\zeta(s-\lambda)\big) \;=\; \exp\Big(\sum_{n=1}^{\infty} n^\lambda\cdot n^{-s}\Big)$$

for $\lambda \geq 0$.[3]

Example 23. Tenenbaum studies the counting functions $\psi(x,y)$ for the zeta functions

$$\zeta(s,y) \ := \ \prod_{p\le y} \big(1-p^{-s}\big)^{-1}.$$

As noted in Remark 13, the functions $\zeta(s,y)$ are not admissible. These functions satisfy all the conditions for being T-admissible except (T3), and for y in a suitable range (depending on x) they satisfy (T3) provided $\sigma = \sigma_x$. This leads to asymptotics for $\psi(x,y)$ as x and y tend to infinity with y suitably constrained.

Example 24. The function

$$\mathbf{F}_k(s) \ := \ \exp\Big(\frac{1}{1-k^{-s}}\Big)$$

is admissible for $k = 2,\dots$, but not T-admissible. It is clear that each $\mathbf{F}_k(s)$ satisfies (A1)–(A3), and has a Dirichlet series expansion with abscissa of convergence $\alpha = 0$. To see that $\mathbf{F}_k(s)$ is not T-admissible note that

$$\frac{\big|\mathbf{F}_k(\sigma+it)\big|}{\mathbf{F}_k(\sigma)}$$

is, for each $\sigma > 0$, positive and periodic as a function of t, and thus does not uniformly go to 0 on $[\delta(\sigma), T(\sigma)]$ as $\sigma \to \alpha+$. Consequently $\mathbf{F}_k(s)$ does not satisfy condition (T3).

To show that $\mathbf{F}_k(s)$ is admissible let

$$\begin{aligned} \delta_k(\sigma) &= (k^\sigma - 1)^{7/5} \\ T_k(\sigma) &= (k^\sigma - 1)^{-3}. \end{aligned}$$

One has

$$\begin{aligned} \mathbf{a}_k(\sigma) &= -\frac{k^\sigma}{\big(k^\sigma-1\big)^2}\log k \\ \mathbf{b}_k(\sigma) &= \frac{k^{2\sigma}+k^\sigma}{\big(k^\sigma-1\big)^3}\big(\log k\big)^2 \\ \mathbf{c}_k(\sigma) &= -\frac{k^{3\sigma}+4k^{2\sigma}+k^\sigma}{\big(k^\sigma-1\big)^4}\big(\log k\big)^3. \end{aligned}$$

Verifying (A4)–(A6) is routine. For (A7) we proceed as in the proof of Theorem 21, namely for $|t| \le \delta(\sigma)$ one has

$$\big|\mathsf{R}(\sigma+it)\big| \ \le \ \big|\mathbf{c}(\sigma)\delta(\sigma)^3\big| \ \to \ 0 \qquad \sigma \to \alpha+.$$

[3]The asymptotics for $F_\lambda(x)$ are also analyzed in §11.5 of [2] by the saddlepoint method, after changing the path of the Perron integral. (See Footnote 1 for errata to §11.5.)

R. Warlimont [12] first pointed out the example of $\exp(\zeta(s))$ to us. Later he found related examples of admissible functions, such as $\exp\big(\zeta(s)^k\big)$, that subsequently turned out to be T-admissible as well.

This leaves (A8), which is usually the challenging part of the verification of admissibility. First note that

$$\begin{aligned}\frac{\sigma\sqrt{\mathbf{b}_k(\sigma)}}{\mathbf{F}_k(\sigma)}\int_{|t|\geq T_k(\sigma)}\left|\mathbf{F}_k(\sigma+it)\right|\frac{dt}{\sigma^2+t^2} &\leq \sigma\sqrt{\mathbf{b}_k(\sigma)}\int_{|t|\geq T_k(\sigma)}\frac{dt}{t^2}\\ &= \frac{\sigma\sqrt{\mathbf{b}_k(\sigma)}}{T_k(\sigma)} \to 0 \quad \text{as } \sigma\to\alpha+.\end{aligned}$$

Thus we only need to show that

$$\frac{\sigma\sqrt{\mathbf{b}_k(\sigma)}}{\mathbf{F}_k(\sigma)}\int_{\delta(\sigma)\leq|t|\leq T_k(\sigma)}\left|\mathbf{F}_k(\sigma+it)\right|\frac{dt}{\sigma^2+t^2} \to 0 \quad \text{as } \sigma\to\alpha+.$$

Substituting $\tau = t\log k$, we need to show

$$\frac{\sigma\sqrt{\mathbf{b}_k(\sigma)}}{\mathbf{F}_k(\sigma)}\int_{\delta(\sigma)\log k\leq|\tau|\leq T_k(\sigma)\log k}\left|\mathbf{F}_k\Big(\sigma+i\frac{\tau}{\log k}\Big)\right|\frac{(\log k)d\tau}{\big(\sigma\log k\big)^2+\tau^2} \to 0$$

as $u\to 1+$. Letting $u = k^\sigma$ it suffices to show

$$\int_{(u-1)^{7/5}\log k}^{(u-1)^{-3}\log k}\frac{\log u}{(u-1)^{3/2}}\exp\left(\frac{2\big(\cos(\tau)-1\big)}{(u-1)\big(u^2-2u\cos(\tau)+1\big)}\right)\frac{d\tau}{(\log u)^2+\tau^2} \to 0$$

as $u\to 1+$.

One can do this by noting that as $u\to 1+$ the integrand rapidly and uniformly approaches 0 outside neighborhoods of radius $(u-1)^{7/5}$ about the points $\tau = 2m\pi$, indeed much faster than $(u-1)^3$. Thus it suffices to show that

$$\int_U\frac{\log u}{(u-1)^{3/2}}\exp\left(\frac{2\big(\cos(\tau)-1\big)}{(u-1)\big(u^2-2u\cos(\tau)+1\big)}\right)\frac{d\tau}{(\log u)^2+\tau^2} \to 0 \tag{22}$$

as $u\to 1+$, where U is the union of the intervals

$$\big[2m\pi-(u-1)^{7/5},\ (2m\pi+(u-1)^{7/5}\big]$$

about the points $2m\pi$, $m\geq 1$, such that $2m\pi-(u-1)^{7/5} < (u-1)^{-3}$. The integral in (22) is bounded by

$$2\zeta(2)\int_0^{(u-1)^{7/5}}\frac{\log u}{(u-1)^{3/2}}\exp\left(\frac{2\big(\cos(\tau)-1\big)}{(u-1)\big(u^2-2u\cos(\tau)+1\big)}\right)d\tau. \tag{23}$$

Let $J(u,\tau)$ be the integrand in (23). Then

$$\begin{aligned}\int_0^{(u-1)^{7/5}}J(u,\tau)d\tau &= \int_0^{(u-1)^{3/2}}J(u,\tau)d\tau + \int_{(u-1)^{3/2}}^{(u-1)^{7/5}}J(u,\tau)d\tau\\ &\leq J(u,0)\cdot(u-1)^{3/2} + J\big(u,(u-1)^{3/2}\big)\cdot(u-1)^{7/5}\\ &\to 0 \qquad \text{as } u\to 1+.\end{aligned}$$

This proves $\mathbf{F}_k(s)$ is admissible, and thus the class of admissible functions is wider than the class of T-admissible functions.

5. Closure under Product

The goal of this section is to prove that the product of two admissible functions $\mathbf{F}_1(s)$ and $\mathbf{F}_2(s)$ with the same abscissa of convergence is again admissible.

Theorem 25. *Suppose $\mathbf{F}_1(s)$ and $\mathbf{F}_2(s)$ are admissible with the same abscissa of convergence α. Then $\mathbf{F}_1(s) \cdot \mathbf{F}_2(s)$ is admissible.*

Proof. We assume $\mathbf{F}_j(s), \delta_j(\sigma), \mathbf{b}_j(s)$ satisfy (A1)–(A8) for $j = 1, 2$, and we assume $\beta_j > \alpha$ chosen such that $\mathbf{b}_j(\sigma) > 0$ for $\sigma \in (\alpha, \beta_j)$.

Let

$$\begin{aligned} \mathbf{F}(s) &:= \mathbf{F}_1(s) \cdot \mathbf{F}_2(s) \\ \beta &:= \min(\beta_1, \beta_2) \\ \delta(\sigma) &:= \min\big(\delta_1(\sigma), \delta_2(\sigma)\big) \quad \text{for } \sigma \in (\alpha, \beta). \end{aligned}$$

We have

$$\begin{aligned} \mathbf{F}_j(s) &= e^{\mathbf{H}_j(s)} \qquad (j = 1, 2) \\ \mathbf{F}(s) &= e^{\mathbf{H}(s)} \\ \mathbf{H}(s) &= \mathbf{H}_1(s) + \mathbf{H}_2(s) \\ \mathbf{a}(s) &= \mathbf{a}_1(s) + \mathbf{a}_2(s) \\ \mathbf{b}(s) &= \mathbf{b}_1(s) + \mathbf{b}_2(s) \\ \mathsf{R}(s) &= \mathsf{R}_1(s) + \mathsf{R}_2(s). \end{aligned}$$

It is easy to check that (A1)–(A5) hold for $\mathbf{F}(s)$.

Next,

$$\begin{aligned} \mathsf{R}(\sigma + it) &= \mathsf{R}_1(\sigma + it) \; + \; \mathsf{R}_2(\sigma + it) \\ &\to 0 \qquad \text{uniformly for } |t| \le \delta(\sigma) \end{aligned}$$

since each of the R_j satisfy (A7) and since $\delta(\sigma) \le \delta_j(\sigma)$ for $j = 1, 2$. So (A7) also holds for $\mathbf{F}$.

To prove (A6) and (A8) for $\mathbf{F}$ we first observe that for $\sigma > \alpha$, for $t \in \mathbb{R}$ and for $j = 1, 2$

$$\big|\mathbf{F}_j(\sigma + it)\big| \;\le\; \mathbf{F}_j(\sigma), \tag{24}$$

and for $\sigma \in (\alpha, \beta)$ and $j = 1, 2$

$$\mathbf{b}_j(\sigma) \;>\; 0 \tag{25}$$

$$\mathbf{b}(\sigma) \;=\; \mathbf{b}_1(\sigma) + \mathbf{b}_2(\sigma) \;\le\; 2\max\big(\mathbf{b}_1(\sigma), \mathbf{b}_2(\sigma)\big). \tag{26}$$

From (24) we have for $\sigma > \alpha$, for $t \in \mathbb{R}$ and for $j = 1, 2$

$$\frac{\big|\mathbf{F}(\sigma + it)\big|}{\mathbf{F}(\sigma)} \;=\; \frac{\big|\mathbf{F}_1(\sigma + it)\big|}{\mathbf{F}_1(\sigma)} \cdot \frac{\big|\mathbf{F}_2(\sigma + it)\big|}{\mathbf{F}_2(\sigma)} \;\le\; \frac{\big|\mathbf{F}_j(\sigma + it)\big|}{\mathbf{F}_j(\sigma)}. \tag{27}$$

Choose $\gamma_1 \in (\alpha, \beta)$ such that for $\sigma \in (\alpha, \gamma_1)$ and $j = 1, 2$

$$\mathbf{b}_j(\sigma)\delta_j(\sigma)^2 \;>\; 1. \tag{28}$$

This is possible by (8).

Now suppose that we are given $\varepsilon \in (0, 1)$.

Choose $\gamma_2 \in (\alpha, \gamma_1)$ such that for $\sigma \in (\alpha, \gamma_2)$ and $j = 1, 2$

$$\mathbf{b}_j(\sigma) \cdot \exp\Big(- \mathbf{b}_j(\sigma)\delta_j(\sigma)^2 \Big) \ < \ \varepsilon^2 \tag{29}$$

$$\sigma\sqrt{\mathbf{b}_j(\sigma)} \int_{|t| \geq \delta_j(\sigma)} \frac{\big|\mathbf{F}_j(\sigma + it)\big|}{\mathbf{F}_j(\sigma)} \frac{dt}{\sigma^2 + t^2} \ < \ \varepsilon. \tag{30}$$

We can do this because the $\mathbf{F}_j$ satisfy (A6) and (A8).

Choose $\gamma \in (\alpha, \gamma_2)$ such that for $\sigma \in (\alpha, \gamma)$ and $j = 1, 2$

$$\frac{\big|\mathbf{F}_j(\sigma + it)\big|}{\mathbf{F}_j(\sigma)} \ < \ 2\exp\big(- \mathbf{b}_j(\sigma)t^2/2\big) \qquad \text{for } |t| \leq \delta_j(\sigma). \tag{31}$$

In view of (5) we can do this because the $\mathbf{F}_j$ satisfy (A7).

Claim: For $\sigma \in (\alpha, \gamma)$

$$\mathbf{b}(\sigma) \cdot \exp\Big(- \mathbf{b}(\sigma)\delta(\sigma)^2 \Big) \ < \ 2\varepsilon^2 \tag{32}$$

$$\sigma\sqrt{\mathbf{b}(\sigma)} \int_{|t| \geq \delta(\sigma)} \frac{\big|\mathbf{F}(\sigma + it)\big|}{\mathbf{F}(\sigma)} \frac{dt}{\sigma^2 + t^2} \ < \ 12\varepsilon. \tag{33}$$

This will prove that (A6) and (A8) hold for $\mathbf{F}$.

We start by fixing $\sigma \in (\alpha, \gamma)$.

Case (i): $\quad \delta_2(\sigma) \ \leq \ \delta_1(\sigma)$.

Then $\delta(\sigma) = \delta_2(\sigma)$.

Subcase (ia): $\quad \mathbf{b}_1(\sigma) \ \leq \ \mathbf{b}_2(\sigma)$.

Then by (26) and (29)

$$\begin{aligned} \mathbf{b}(\sigma) \cdot \exp\Big(- \mathbf{b}(\sigma)\delta(\sigma)^2 \Big) \quad &< \quad 2\mathbf{b}_2(\sigma) \cdot \exp\Big(- \mathbf{b}_2(\sigma)\delta_2(\sigma)^2 \Big) \\ &< \quad 2\varepsilon^2. \end{aligned}$$

Also by (26), (27) for $j = 2$ and (30) for $j = 2$ we have

$$\begin{aligned} &\sigma\sqrt{\mathbf{b}(\sigma)} \int_{|t| \geq \delta(\sigma)} \frac{\big|\mathbf{F}(\sigma + it)\big|}{\mathbf{F}(\sigma)} \frac{dt}{\sigma^2 + t^2} \\ \leq \quad &\sigma\sqrt{2\mathbf{b}_2(\sigma)} \int_{|t| \geq \delta_2(\sigma_2)} \frac{\big|\mathbf{F}_2(\sigma + it)\big|}{\mathbf{F}_2(\sigma)} \frac{dt}{\sigma^2 + t^2} \\ < \quad &\sqrt{2}\varepsilon \ < \ 12\varepsilon. \end{aligned}$$

Subcase (ib): $\quad \mathbf{b}_2(\sigma) \ \leq \ \mathbf{b}_1(\sigma)$.

By (29) for $j = 2$

$$\sqrt{\mathbf{b}_2(\sigma)} \ \exp\big(- \mathbf{b}_2(\sigma)\delta_2(\sigma)^2/2\big) \ < \ \varepsilon,$$

so

$$\sqrt{\mathbf{b}_1(\sigma)} \ \exp\big(- \mathbf{b}_1(\sigma)\delta_2(\sigma)^2/2\big) \ < \ \varepsilon \tag{34}$$

since

$$\sqrt{x} \ \exp\big(- x\delta_2(\sigma)^2/2\big)$$

is decreasing for $x > 1/\delta_2(\sigma)^2$, and since by Subcase (ib) and (28) for $j = 2$

$$\mathbf{b}_1(\sigma) \ \geq \ \mathbf{b}_2(\sigma) \ > \ \frac{1}{\delta_2(\sigma)^2}.$$

Then by (26) and (34)

$$\begin{aligned}\mathbf{b}(\sigma) \cdot \exp\Big(- \mathbf{b}(\sigma)\delta(\sigma)^2\Big) \quad &< \quad 2\mathbf{b}_1(\sigma) \cdot \exp\Big(- \mathbf{b}_1(\sigma)\delta_2(\sigma)^2\Big) \\ &< \quad 2\varepsilon^2.\end{aligned}$$

Also from (34) we have

$$\sqrt{\mathbf{b}_1(\sigma)} \ \exp\big(- \mathbf{b}_1(\sigma)t^2/2\big) \ < \ \varepsilon \qquad \text{for } \delta_2(\sigma) \leq |t|.$$

Combined with (31) for $j = 1$ this gives

$$\sqrt{\mathbf{b}_1(\sigma)} \, \frac{\big|\mathbf{F}_1(\sigma + it)\big|}{\mathbf{F}_1(\sigma)} \ < \ 2\varepsilon \qquad \text{for } \delta_2(\sigma) \leq |t| \leq \delta_1(\sigma). \tag{35}$$

By (26) and (27) for $j = 1$

$$\begin{aligned} &\sigma\sqrt{\mathbf{b}(\sigma)} \, \int_{|t|\geq\delta(\sigma)} \frac{\big|\mathbf{F}(\sigma + it)\big|}{\mathbf{F}(\sigma)} \, \frac{dt}{\sigma^2 + t^2} \\ \leq \quad &\sigma\sqrt{2\mathbf{b}_1(\sigma)} \, \int_{|t|\geq\delta_2(\sigma)} \frac{\big|\mathbf{F}_1(\sigma + it)\big|}{\mathbf{F}_1(\sigma)} \, \frac{dt}{\sigma^2 + t^2} \\ = \quad &\sqrt{2}\Big(J_1(\sigma) \ + \ J_2(\sigma)\Big),\end{aligned}$$

where

$$\begin{aligned} J_1(\sigma) \quad &= \quad \sigma\sqrt{\mathbf{b}_1(\sigma)} \, \int_{\delta_2(\sigma)\leq|t|\leq\delta_1(\sigma)} \frac{\big|\mathbf{F}_1(\sigma + it)\big|}{\mathbf{F}_1(\sigma)} \, \frac{dt}{\sigma^2 + t^2} \\ J_2(\sigma) \quad &= \quad \sigma\sqrt{\mathbf{b}_1(\sigma)} \, \int_{|t|\geq\delta_1(\sigma)} \frac{\big|\mathbf{F}_1(\sigma + it)\big|}{\mathbf{F}_1(\sigma)} \, \frac{dt}{\sigma^2 + t^2}.\end{aligned}$$

By (35)

$$J_1(\sigma) \ \leq \ 2\varepsilon\sigma \int_{\delta_2(\sigma)\leq|t|\leq\delta_1(\sigma)} \frac{dt}{\sigma^2 + t^2} \ < \ 2\pi\varepsilon,$$

and by (30)

$$J_2(\sigma) \ < \ \varepsilon.$$

Thus

$$\sigma\sqrt{\mathbf{b}(\sigma)} \, \int_{|t|\geq\delta(\sigma)} \frac{\big|\mathbf{F}(\sigma + it)\big|}{\mathbf{F}(\sigma)} \, \frac{dt}{\sigma^2 + t^2} \ < \ \sqrt{2}\big(2\pi + 1\big)\varepsilon \ < \ 12\varepsilon,$$

and the claim is proved in Case (i).

Case (ii), where $\delta_1(\sigma) \leq \delta_2(\sigma)$, is handled likewise. So (A6) and (A8) hold for $\mathbf{F}$, and the theorem is proved.

$\square$

6. Open questions

Problem 1. Is the sum of two admissible functions also admissible?

Problem 2. Is the product of any two admissible functions also admissible?

Problem 3. Given two admissible functions $\mathbf{F}_j(x) = \exp\big(\mathbf{H}_j(s)\big)$ is the function $\exp\big(\mathbf{H}_1(s) \cdot \mathbf{H}_2(s)\big)$ admissible?

Problem 4. If $\mathbf{F}(s)$ is admissible, does it follow that $e^{\mathbf{F}(s)}$ is also admissible?

We suspect, by analogy with Hayman's work, that this is true.

Problem 5. Can the notion of admissible be extended to include $\zeta(s)$?

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