BOOLE'S METHOD I. A MODERN VERSION

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Abstract. A rigorous, modern version of Boole's algebra of logic is presented, based partly on the 1890s treatment of Ernst Schröder.

1. Preamble to Papers I and II

The sophistication and mathematical depth of Boole's approach to the logic of classes is not commonly known, not even among logicians. It includes much, much more than just the basic operations and equational laws for an algebra of classes. Indeed, aside from possibly a few tricks to speed up computations, Boole considered his algebra of logic to be the perfect completion of the fragmentary Aristotelian logic. Whereas the latter consisted of a small catalog of valid arguments, Boole's system offered a method (consisting of algebraic algorithms) to determine

(B1) the strongest possible conclusion $\varphi(\vec{A}\,)$ from *any* given finite collection of premisses $\varphi_i(\vec{A},\vec{B}\,)$ concerning classes $A_1,\ldots,A_m$, $B_1,\ldots,B_n$, and

(B2) the expression of any class A_i in terms of the other classes in *any* given finite collection of premisses $\varphi_i(\vec{A}\,)$ concerning classes $A_1,\ldots,A_m$.

Boole's algebra of logic was developed well before concerns were raised about possible paradoxes in the study of classes. To maintain contact with Boole's writings, as well as with modern set-theoretic foundations and notations, we will simply treat the words 'class' and 'set' as equivalent. To put everything into a more modern form, simply change the word 'class' everywhere into the word 'set'.

With Boole's algebraic approach, the mastery of the logic of classes changed dramatically from the requirement of memorizing a finite and very incomplete catalog of valid arguments in Aristotelian logic to the requirement of learning:

(a) how to translate class-propositions into class-equations, and vice-versa,

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(b) the axioms and rules of inference for Boole's algebra of logic, and
(c) the fundamental theorems of Boole's algebra of logic.

Boole never precisely stated which ordinary language statements $\varphi_i(\vec{A}\,)$ qualified as class-propositions, that is, propositions about classes, although he gave many examples. By 1890 Schröder concluded that any class-proposition was equivalent to a *basic formula* in the modern Boolean algebra of sets, that is, either to an equational assertion $p(\vec{A}\,) = q(\vec{A}\,)$, or the negation $p(\vec{A}\,) \neq q(\vec{A}\,)$ of an equational assertion. The four forms of categorical propositions from Aristotelian logic are readily seen to satisfy this condition:

Form	Ordinary language	Equational form
A	All A is B	$A = A \cap B$
E	No A is B	$A \cap B = \emptyset$
I	Some A is B	$A \cap B \neq \emptyset$
O	Some A is not B	$A \cap B' \neq \emptyset$

Boole allowed more complex assertions, such as 'All A is B or C', which can be expressed by $A = A \cap (B \cup C)$. The converse, that every basic formula $\beta(A_1, \ldots, A_m)$ in the modern Boolean algebra of sets can be expressed by a proposition in ordinary language, is not so clear—ordinary language suffers from not using parentheses to group terms. For example, it is cumbersome to express $A \cap (B \cup (C \cap D))) = \emptyset$ in ordinary language; but with parentheses it is easy, namely 'The class A intersected with the class (B unioned with the class (C intersected with the class D)) is empty'. Without parentheses one needs the cumbersome method of introducing new symbols, for example, 'There are classes E and F such that F is the intersection of C and D, and E is the union of B and F, and A and E are disjoint'.

We will simply assume that class-propositions correspond precisely to basic formulas in the modern Boolean algebra of classes. Furthermore we assume that the reader knows how to translate between class-propositions and basic formulas.

The word 'algebra' has two major meanings in mathematics—we first learn to think of algebra as *procedures*, such as finding the roots of a quadratic equation; later we learn that it can also refer to a *structure* such as the ring of integers $\mathbb{Z} =$

$(\mathbb{Z}, +, \cdot, -, 0, 1)$, or the power set algebra $\mathbf{PS}(U) = (\mathcal{PS}(U), \cup, \cap, ', \emptyset, U)$ of subsets of U.

When Boole introduced and refined his algebra of logic for classes, from 1847 to 1854, he was primarily interested in procedures to determine the items in (B1) and (B2) above. Given a finite list of class-propositions φ_i for the premisses, the first step was to convert them into *equations* $p_i = q_i$. (Note: Schröder thought it was necessary to use basic formulas, not just equations. Boole believed he only needed equations.) Then he gave algebraic algorithms for (B1) and (B2) in the setting of equations. The result was then translated back into ordinary language to give the desired class-proposition conclusion.

Remark 1.1. *The reader can find a detailed presentation of Boole's algorithms with examples, but without proofs, in the article* George Boole*, in the online Stanford Encyclopedia of Philosophy* [5].

Boole's version of the algebra of logic for classes was significantly different from what we now call Boolean algebra—but it led directly to modern Boolean algebra, thanks to Jevons [9] replacing Boole's partial operations by total operations. Scholars had from the very beginning at least three major concerns about Boole's system:

(1) It seemed unduly and mysteriously tied to the algebra of numbers, the so-called *common algebra*—for the fundamental operations on classes, and the fundamental constants, Boole chose the symbols $+$, $\cdot$, $-$, 0 and 1, symbols traditionally reserved for the algebra of numbers. (Boole also used division, but only in a very special setting.) His manipulation of equations was dictated by the procedures used in common algebra, **with one addition:** multiplication was idempotent for *class-symbols*, that is, $A^2 = A$ for any class-symbol A.

(2) Boole interpreted 0 as the empty class, 1 as the universe, and the multiplication of classes as their intersection. But his operations of addition $(+)$ and subtraction $(-)$ on classes were *partial operations*, not total operations; that is, they were only partially defined. If two classes A and B had elements in common, then $A + B$ was simply not defined (or, as Boole said, $A + B$ was not interpretable). Likewise, if B was not a subclass of A, then $A - B$ was not defined; otherwise $A - B$ was $A \cap B'$, the class of elements in A but not in B.

The difficulty readers had with Boole's partial operations was that Boole applied the processes of common algebra to equations without being concerned about whether the terms were defined or not. In modern universal algebra we know that the usual rules of equational inference (Birkhoff's five rules) are *correct* and *complete* for the equational logic of *total* algebras, that is, algebras with fundamental operations that are totally defined on the domain of the algebra. Unfortunately these properties may not hold when working with partial algebras.

With Boole's system, the question was whether or not the application of the usual rules of equational inference always leads to correct results when one starts with meaningful premisses and ends with a meaningful conclusion, but not all the equations appearing in the intermediate steps are meaningful. (Boole claimed that the answer was 'yes'.)

(3) Boole claimed that he could translate particular propositions into equations by introducing a new symbol V. For example, 'Some A is B' was translated initially by $V = AB$, and later by $VA = VB$.

Items (1) and (2) remained troublesome issues for more than a century, until the appearance of Hailperin's book [8] in 1976. He set these concerns aside by noting that each partial algebra $\mathbf{B}(U) = (\mathcal{PS}(U), +, \cdot, -, 0, 1)$ in Boole's setting could be embedded in a total algebra of signed multi-sets; this is equivalent to saying that $\mathbf{B}(U)$ can be embedded in the ring $\mathbb{Z}^U$ (see, for example, [7]). Regarding item (3), Schröder 'proved' that one had to use negated equations for propositions with existential import. (This approach made the introduction of a new symbol V quite unnecessary). Item 3 has remained a concern...we will show that Boole's view, that only equations are needed, is actually correct as well (that is, after we make a very small adjustment to his translations between class-properties and class-equations).

Boole's algorithms are powerful tools in the study of classes, and they carry over almost verbatim to the setting of modern Boolean algebra. We will adapt Boole's algebra of logic for classes (his theorems and algorithms) to the modern setting in this paper, essentially along the lines laid out in the 1890s by Schröder. This will allow the reader to understand and judge the importance of Boole's work, without the hinderance of possibly many nagging concerns regarding whether or not one has properly understood all the nuances of meaning in Boole's writings. This modern version of Boole's work will include a discussion of item (3) above, showing that indeed one only needs equations (refuting Schröder's claim to have proved the contrary).

In the second paper we turn to Boole's original system (compactly presented in the aforementioned SEP article) and provide full details of the proofs (using the results of this first paper), including addressing item (3) above. The controversy-free presentation in this first paper will hopefully make it easier for the reader to focus in the second paper on how the concerns regarding (1) and (2) in Boole's system are overcome. Furthermore this first paper sets the stage for how we will resolve the concerns about item (3) in Boole's system.

In closing this Preamble, we would like to mention that, in [7], Boole's claim that his "Rule of 0 and 1" is sufficient to prove his theorems is vindicated.

1.1. **Introduction.** More specifically, Boole's algebra of logic (1847/1854) offered

- a translation of propositions into equations,
- an algorithm for eliminating symbols in the equations,
- an algorithm for solving for a variable, and
- a reverse translation, from conclusion equations to conclusion propositions.

This algebra of logic has long puzzled readers for many reasons, including:

(a) its foundation, which appears to be the 'common' algebra, namely the algebra of numbers, augmented by idempotent variables,
(b) the appearance of uninterpretable terms in various procedures,
(c) a strange division procedure,
(d) a dubious encoding of propositions as equations, especially the particular propositions (using his famous V), and
(e) dubious proofs of the main theorems.

Yet the system seemed, by and large, to work just as Boole said it would.[1] The mechanical details of Boole's method of using algebra to analyze arguments are given in considerable detail in the article "George Boole" in the online Stanford Encyclopedia of Philosophy (see [5]). Now we turn to the justification of his method.

[1] In 1864 Jevons [9] modified Boole's system, giving the basic structure that would develop into modern Boolean algebra.

This first paper gives a compact yet rigorous modern version of Boole's algebra of logic. It is based in good part on Volumes I and II of Schröder's *Algebra der Logik* [10], published in the 1890s. These results, along with the remarkable insights of Hailperin ([8] 1976/1986), are used in the second paper ([6]) to likewise give a compact, rigorous presentation of Boole's original algebra of logic.[2]

2. Background

For purposes of indexing we prefer to use sets of the form $\widetilde{n} := \{1, \ldots, n\}$ instead of the usual finite ordinals $n := \{0, \ldots, n-1\}$.

Given a universe U, the **power set** $PS(U)$ of U is the set of subsets of U. The **power-set algebra** $\mathbf{PS}(U)$ is the algebra $(PS(U), \cup, \cap, ', \emptyset, U)$ of subsets of U. $\mathcal{PSA}$ is the collection of power-set algebras $\mathbf{PS}(U)$ with $U \neq \emptyset$.

For the syntactic side of power-set algebra we use the **operation symbols** $\cup$ (**union**), $\cap$ (**intersection**) and $'$ (**complement**); and the **constants** 0 (the **empty set**) and 1 (the **universe**). There is a countably infinite set X of **variables**, and the **$\mathcal{PSA}$-terms** $p(\mathbf{x}) := p(x_1, \ldots, x_k)$ are constructed from the above operation symbols, constant symbols and variables, in the usual way by induction:

- variables and constants are $\mathcal{PSA}$-terms;
- if p and q are $\mathcal{PSA}$-terms then so are (p'), $(p \cup q)$, and $(p \cap q)$.

We adopt the usual convention of not writing outer parentheses. We often write $p \cdot q$, or simply pq, instead of $p \cap q$. It will be assumed that intersection takes precedence over union, for example, $p \cup qr$ means $p \cup (q \cap r)$. It will be convenient to adopt the **abbreviations** $p \subseteq q$ and $q \supseteq p$ for $p = pq$, or equivalently, $pq' = 0$.

(First-order) $\mathcal{PSA}$-formulas are defined inductively:

[2] Brown [4] has given a fairly compact treatment of the development of the algebra used by Boole, showing that a certain ring (a ring of polynomials modulo idempotent generators) satisfies Boole's theorems. However this does not show that Boole's algebra of logic gives a correct calculus of classes, as Boole claimed, and the author seems to suggest. Boole's algebra of logic has partial operations, and one cannot simply apply Birkhoff's rules of equational logic to partial algebras. Hailperin [8] extended Boole's partial algebra to a total algebra of signed multisets, and for such an algebra Birkhoff's rules apply—this, or some step connecting Boole's partial algebra to Birkhoff's rules, is missing in Brown's treatment. Hailperin's work falls short of being complete by the absence of his justification to Boole's use of equations to express particular propositions–we address this in our second paper.

- $\mathcal{PSA}$**-equations**, that is, expressions of the form $(p = q)$, where p and q are $\mathcal{PSA}$-terms, are $\mathcal{PSA}$-formulas
- if φ is a $\mathcal{PSA}$-formula then so is $(\neg\,\varphi)$
- if φ_1 and φ_2 are $\mathcal{PSA}$-formulas, then so are $(\varphi_1 \wedge \varphi_2)$, $(\varphi_1 \vee \varphi_2)$, $(\varphi_1 \to \varphi_2)$, and $(\varphi_1 \leftrightarrow \varphi_2)$
- if φ is a $\mathcal{PSA}$-formula and $x \in X$, then $\big((\forall x)\varphi\big)$ and $\big((\exists x)\varphi\big)$ are $\mathcal{PSA}$-formulas.

Again we adopt the usual convention of not writing outer parentheses. The notation $(\exists \mathbf{x})$ stands for $(\exists x_1)\cdots(\exists x_k)$, where $\mathbf{x}$ is the list $x_1, \ldots, x_k$.

An **interpretation** I **into** $\mathbf{PS}(U)$ is a mapping $I : X \to PS(U)$ that is extended by induction to all terms as follows:

- $I(0) := \emptyset$, $I(1) := U$
- $I(p') := I(p)'$
- $I(p \cup q) := I(p) \cup I(q)$
- $I(pq) := I(p) \cap I(q)$.

I is a $\mathcal{PSA}$**-interpretation** if it is an interpretation into some $\mathbf{PS}(U)$.

The notion of a (first-order) $\mathcal{PSA}$-formula φ being true under an interpretation I into $\mathbf{PS}(U)$, written $I(\varphi) = \mathsf{TRUE}$, is recursively defined as follows, where $I(\varphi) = \mathsf{FALSE}$ means $I(\varphi) \neq \mathsf{TRUE}$:

- $I(p = q) = \mathsf{TRUE}$ iff $I(p) = I(q)$;
- $I(\neg\varphi) = \mathsf{TRUE}$ iff $I(\varphi) = \mathsf{FALSE}$;
- $I(\varphi \vee \psi) = \mathsf{TRUE}$ iff either $I(\varphi) = \mathsf{TRUE}$ or $I(\psi) = \mathsf{TRUE}$;
- $I(\varphi \wedge \psi) = \mathsf{TRUE}$ iff both $I(\varphi) = \mathsf{TRUE}$ and $I(\psi) = \mathsf{TRUE}$;
- $I(\varphi \to \psi) = \mathsf{TRUE}$ iff $I(\varphi) = \mathsf{FALSE}$ or $I(\psi) = \mathsf{TRUE}$;
- $I(\varphi \leftrightarrow \psi) = \mathsf{TRUE}$ iff both $I(\varphi \to \psi) = \mathsf{TRUE}$ and $I(\psi \to \varphi) = \mathsf{TRUE}$;
- $I\big((\forall x)\varphi\big) = \mathsf{TRUE}$ iff for each interpretation $\widehat{I}$ into $\mathbf{PS}(U)$ that agrees with I on $X \smallsetminus \{x\}$, one has $\widehat{I}(\varphi) = \mathsf{TRUE}$;
- $I\big((\exists x)\varphi\big) = \mathsf{TRUE}$ iff for some interpretation $\widehat{I}$ into $\mathbf{PS}(U)$ that agrees with I on $X \smallsetminus \{x\}$, one has $\widehat{I}(\varphi) = \mathsf{TRUE}$.

Note that $I(p \subseteq q) = \mathsf{TRUE}$ iff $I(q \supseteq p) = \mathsf{TRUE}$ iff $I(p) \subseteq I(q)$.

Some additional notation that we will use is:

- $\mathcal{PSA} \models \varphi$, read $\mathcal{PSA}$ **satisfies** φ, means $I(\varphi) = \mathsf{TRUE}$ for every interpretation I into a member of $\mathcal{PSA}$.
- $\varphi_1, \ldots, \varphi_n \models_{\mathcal{PSA}} \psi$, or $\varphi_1, \ldots, \varphi_n \Rightarrow_{\mathcal{PSA}} \psi$, read $\mathcal{PSA}$ **(semantically) implies** φ, means $\mathcal{PSA} \models (\varphi_1 \wedge \cdots \wedge \varphi_n \to \psi)$.
- φ and ψ are $\mathcal{PSA}$-**(semantically) equivalent**, written $\varphi \Leftrightarrow_{\mathcal{PSA}} \psi$, if $\mathcal{PSA} \models \big(\varphi \leftrightarrow \psi\big)$.
- Two finite sets (or lists) Φ and Ψ of formulas are $\mathcal{PSA}$-**(semantically) equivalent**, written $\Phi \Leftrightarrow_{\mathcal{PSA}} \Psi$, if $\mathcal{PSA} \models \Big(\bigwedge \Phi \leftrightarrow \bigwedge \Psi\Big)$.
- A finite set (or list) Φ of $\mathcal{PSA}$-formulas is $\mathcal{PSA}$-**satisfiable**, written $\mathsf{SAT}_{\mathcal{PSA}}\big(\Phi\big)$, if there is a $\mathcal{PSA}$-interpretation I such that $I\Big(\bigwedge \Phi\Big) = \mathsf{TRUE}$.
- An argument $\varphi_1, \ldots, \varphi_n \ \therefore\ \psi$ is $\mathcal{PSA}$-**valid**, or **valid in** $\mathcal{PSA}$, also written as $\mathsf{Valid}_{\mathcal{PSA}}\big(\varphi_1, \ldots, \varphi_n \ \therefore\ \psi\big)$, means $\varphi_1, \ldots, \varphi_n \models_{\mathcal{PSA}} \psi$.

Remark 2.1. *Since this paper only deals with algebras from $\mathcal{PSA}$, the prefix and subscript $\mathcal{PSA}$, etc.,* will usually be omitted.

Basic formulas are **equations** $p(\mathbf{x}) = q(\mathbf{x})$ and **negated equations** $p(\mathbf{x}) \neq q(\mathbf{x})$. They suffice to express a variety of propositions about sets, including the famous Aristotelian categorical propositions.[3] For example, the assertion 'All x is y' is expressed by $x = xy$, or equivalently, $xy' = 0$, since for any interpretation I in a power-set algebra $\mathbf{PS}(U)$, one has, setting $A := I(x)$ and $B := I(y)$, 'All A is B' holding iff $A \subseteq B$, and this holds iff $A = AB$, or equivalently, $AB' = \emptyset$. The following table gives a sampler of propositions that can be expressed by a basic formula:

[3]Schröder used $\neq 0$ to translate particular propositions into symbolic form in his *Algebra der Logik* (p. 93 in Vol. II). In this work he also 'proved' that Boole's efforts to translate particular propositions by equations (using the infamous symbol V) must fail (pp. 91-93 in Vol. II). Yet in §5.4 of this paper, the reader will find that a slight variation on Boole's use of V indeed works *in the context of valid arguments.* And in §7.2 we will find that it works *in the context of elimination* as well.

Proposition	Basic Formula	Alternative
x is empty	$x = 0$	
x is not empty	$x \neq 0$	
All x is y	$xy' = 0$	$x = xy$
No x is y	$xy = 0$	$x = xy'$
Some x is y	$xy \neq 0$	
Some x is not y	$xy' \neq 0$	
x and y are empty	$x \cup y = 0$	
x and y are disjoint	$xy = 0$	
x is empty and y is the universe	$x \cup y' = 0$	
x or y is not empty	$x \cup y \neq 0$	
etc.		

However some simple relationships among sets cannot be expressed by basic formulas, for example, 'x is empty or y is empty', 'x is empty implies y is empty', 'there are at least 2 elements in the universe', etc.

Propositions are usually formulated in ordinary language, with a few symbols, like 'All S is P'. It is not so easy to precisely describe all the ordinary language statements that qualify as propositions about sets. To get around this awkward situation we simply define our **domain of propositions** about sets to be all propositions $\pi(\mathbf{x})$ which can be expressed by basic formulas $\beta(\mathbf{x})$. Then it is automatic that a list of propositional premisses

$$\pi_1(\mathbf{x}), \ldots, \pi_n(\mathbf{x})$$

can be expressed by a list of basic formulas

$$\beta_1(\mathbf{x}), \ldots, \beta_n(\mathbf{x});$$

and a propositional argument

$$\pi_1(\mathbf{x}), \ldots, \pi_n(\mathbf{x}) \ \therefore\ \pi(\mathbf{x})$$

can be expressed by a basic-formula argument

$$\beta_1(\mathbf{x}), \ldots, \beta_n(\mathbf{x}) \ \therefore\ \beta(\mathbf{x}).$$

Boole focused on two themes in his algebra of logic, namely given a list $\pi_1(\mathbf{x},\mathbf{y})$,..., $\pi_n(\mathbf{x},\mathbf{y})$ of propositional premisses:

(a) how to find the 'complete' result $\pi(\mathbf{y})$ of eliminating the variables $\mathbf{x}$ from the propositional premisses; and
(b) how to express a variable x_i in terms of the other variables, given the propositional premisses.

In our modern version, the propositional premisses correspond exactly to basic-formula premisses $\beta_1(\mathbf{x},\mathbf{y}),\ldots,\beta_n(\mathbf{x},\mathbf{y})$, and the propositional themes are clearly equivalent to the basic-formula themes:

(a) how to find the 'complete' result $\beta(\mathbf{y})$ of eliminating the variables $\mathbf{x}$ from the basic-formula premisses; and
(b) how to express a variable x_i in terms of the other variables, given the basic-formula premisses.

In 1854 Boole [3] presented a General Method for tackling these questions about propositions, a method that used only equations, thus avoiding the use of negated equations. (For a summary of Boole's General Method in modern notation, see [5].) We can parallel essentially all of Boole's General Method in the modern framework described above, with the advantage that neither the translations nor the methods are suspect.

3. Axioms and Rules of Inference

The *laws* or *axioms* for the three set operations are as follows, where p, q and r are any three terms— these basic formulas are satisfied by $\mathcal{PSA}$:

$p \cup p = p$	$p \cdot p = p$	Idempotent Laws
$p \cup 0 = p$	$p \cdot 0 = 0$	0 Laws
$p \cup 1 = 1$	$p \cdot 1 = p$	1 Laws
$p \cup q = q \cup p$	$p \cdot q = q \cdot p$	Commutative Laws
$p \cup (q \cup r) = (p \cup q) \cup r$	$p \cdot (q \cdot r) = (p \cdot q) \cdot r$	Associative Laws
$p \cup p \cdot q = p$	$p \cdot (p \cup q) = p$	Absorption Laws
$p \cup q \cdot r = (p \cup q) \cdot (p \cup r)$	$p \cdot (q \cup r) = p \cdot q \cup p \cdot r$	Distributive Laws
$p \cup p' = 1$	$p \cdot p' = 0$	Complement Laws
$(p \cup q)' = p' \cdot q'$	$(p \cdot q)' = p' \cup q'$	De Morgan Laws
and there is one inequality		
$1 \neq 0$		Non-empty Universe

(These axioms are somewhat redundant.)

The usual *equational rules of inference*, where p, q, r, s are any four terms, are:

- the reflexive, symmetric and transitive rules for equality
- $\dfrac{p = q}{p' = q'}$ (Complement of equals)
- $\dfrac{p = q,\ r = s}{p \cup r = q \cup s}$ (Union of equals)
- $\dfrac{p = q,\ r = s}{p \cdot r = q \cdot s}$ (Intersection of equals)

[NOTE: The last three rules are equivalent to the *replacement* rule.]

3.1. **A Standard Form.** Every equation $p = q$ can be put in a **standard form** $r = 0$.

Lemma 3.1 (**Standard Form**). *An equation $p(\mathbf{x}) = q(\mathbf{x})$ is equivalent to an equation in the form $r(\mathbf{x}) = 0$, namely let $r(\mathbf{x}) = p(\mathbf{x}) \triangle q(\mathbf{x})$, the symmetric difference of $p(\mathbf{x})$ and $q(\mathbf{x})$, which is defined by:*

$$p \triangle q \ := \ p \cdot q' \ \cup \ p' \cdot q.$$

4. Constituents

Boole introduced constituents to provide a basis for expanding terms.

Definition 4.1. *Given a list of variables $\mathbf{x} := x_1, \ldots, x_k$, the 2^k* **consitituents** *of $\mathbf{x}$ are the following terms:*

$$\begin{array}{l} x_1 x_2 \cdots x_k \\ x_1' x_2 \cdots x_k \\ \quad \vdots \\ x_1' x_2' \cdots x_k'. \end{array}$$

These are called the **x-constituents**. *A useful notation for referring to them is as follows. For $\sigma \in 2^{\widetilde{k}}$, that is, for σ a mapping from $\{1, \ldots, k\}$ to $\{0, 1\}$, let*

$$\mathcal{C}_\sigma(\mathbf{x}) \ := \ \mathcal{C}_{\sigma_1}(x_1) \cdots \mathcal{C}_{\sigma_k}(x_k),$$

where $\mathcal{C}_1(x_j) := x_j$ and $\mathcal{C}_0(x_j) := x_j'$ and $\sigma_i := \sigma(i)$.

Thus, for example, with $k = 5$ and $\sigma = 01101$, $\mathcal{C}_\sigma(\mathbf{x}) = x_1' x_2 x_3 x_4' x_5$.

Lemma 4.2. *For $i, j \in \{0, 1\}$, $\mathcal{PSA}$ satisfies*

$$\mathcal{C}_i(j) \ = \ 1 \quad \text{if} \ \ j = i \tag{1}$$

$$\mathcal{C}_i(j) \ = \ 0 \quad \text{if} \ \ j \neq i. \tag{2}$$

Proof. From the definition of $\mathcal{C}_i(x)$. □

Lemma 4.3. *For $\sigma, \tau \in 2^{\widetilde{k}}$, and $p(\mathbf{x})$ a term, $\mathcal{PSA}$ satisfies*

$$p(\sigma) = 1 \ \text{or} \ p(\sigma) = 0 \tag{3}$$

$$\mathcal{C}_\sigma(\tau) = 1 \quad \text{if} \ \sigma = \tau \tag{4}$$

$$\mathcal{C}_\sigma(\tau) = 0 \quad \text{if} \ \sigma \neq \tau. \tag{5}$$

Proof. The first item is proved by induction on the term $p(\mathbf{x})$.

For the second item, suppose $\sigma = \tau$. Then $\mathcal{C}_{\sigma_j}(\tau_j) \ = \ 1$ for $j = 1, \ldots, k$, by (1), so

$$\mathcal{C}_\sigma(\tau) \ := \ \mathcal{C}_{\sigma_1}(\tau_1) \cdots \mathcal{C}_{\sigma_k}(\tau_k) \ = \ 1.$$

Finally, suppose that $\sigma \neq \tau$. For some i we have $\sigma_i \neq \tau_i$. Then $\mathcal{C}_{\sigma_i}(\tau_i) \ = \ 0$, by (2), so

$$\mathcal{C}_\sigma(\tau) \ := \ \mathcal{C}_{\sigma_1}(\tau_1) \cdots \mathcal{C}_{\sigma_k}(\tau_k) \ = \ 0.$$

$\square$

Lemma 4.4. *For $\sigma, \tau \in 2^{\widetilde{k}}$, $\mathcal{PSA}$ satisfies*

$$\mathcal{C}_\sigma(\mathbf{x}) \cdot \mathcal{C}_\tau(\mathbf{x}) = \mathcal{C}_\sigma(\mathbf{x}) \qquad \text{if } \sigma = \tau \tag{6}$$

$$\mathcal{C}_\sigma(\mathbf{x}) \cdot \mathcal{C}_\tau(\mathbf{x}) = 0 \qquad \text{if } \sigma \neq \tau \tag{7}$$

$$\bigcup_{\sigma \in 2^{\widetilde{k}}} \mathcal{C}_\sigma(\mathbf{x}) = 1. \tag{8}$$

Proof. For (6) and (7), use the fact that one can derive

$$\begin{aligned} \mathcal{C}_\sigma(\mathbf{x}) \cdot \mathcal{C}_\tau(\mathbf{x}) \ &= \ \Big(\bigcap_{j \in \widetilde{k}} \mathcal{C}_{\sigma_j}(x_j)\Big) \ \cdot \ \Big(\bigcap_{j \in \widetilde{k}} \mathcal{C}_{\tau_j}(x_j)\Big) \\ &= \ \bigcap_{j \in \widetilde{k}} \Big(\mathcal{C}_{\sigma_j}(x_j) \cdot \mathcal{C}_{\tau_j}(x_j)\Big). \end{aligned}$$

If $\sigma \neq \tau$ then for some j one has $\{\sigma_j, \tau_j\} = \{0, 1\}$, so $\big\{\mathcal{C}_{\sigma_j}(x_j), \mathcal{C}_{\tau_j}(x_j)\big\} = \{x_j, x'_j\}$, leading to $\mathcal{C}_\sigma(\mathbf{x}) \cdot \mathcal{C}_\tau(\mathbf{x}) = 0$.

If $\sigma = \tau$ then for each j one has

$$\mathcal{C}_{\sigma_j}(x_j) \cdot \mathcal{C}_{\tau_j}(x_j) \ = \ \mathcal{C}_{\sigma_j}(x_j) \cdot \mathcal{C}_{\sigma_j}(x_j) \ = \ \mathcal{C}_{\sigma_j}(x_j),$$

and thus

$$\mathcal{C}_\sigma(\mathbf{x}) \cdot \mathcal{C}_\tau(\mathbf{x}) \ = \ \bigcap_{j \in \widetilde{k}} \mathcal{C}_{\sigma_j}(x_j) \ := \ \mathcal{C}_\sigma(\mathbf{x}).$$

For (8), use the fact that one has

$$1 \ = \ \bigcap_{j \in \widetilde{k}} \Big(x_j \cup x'_j\Big) \ = \ \bigcap_{j \in \widetilde{k}} \Big(\mathcal{C}_1(x_j) \cup \mathcal{C}_0(x_j)\Big).$$

Expanding the right side gives the desired expression of 1 as the union of all the $\mathbf{x}$-constituents. □

Lemma 4.5. *Given a term $t(\mathbf{x},\mathbf{y})$ and an $\mathbf{x}$-constituent $\mathcal{C}_\sigma(\mathbf{x})$, $\mathcal{PSA}$ satisfies*

$$t(\mathbf{x},\mathbf{y}) \cdot \mathcal{C}_\sigma(\mathbf{x}) \;=\; t(\sigma,\mathbf{y}) \cdot \mathcal{C}_\sigma(\mathbf{x}).$$

Proof. By induction on the term $t(\mathbf{x},\mathbf{y})$. □

4.1. **Reduction Theorem.** Every list of equations can be reduced to a single equation.

Theorem 4.6 (**Reduction).** *A list of equations $p_1(\mathbf{x}) = 0, \ldots, p_n(\mathbf{x}) = 0$ is equivalent to the single equation $p_1(\mathbf{x}) \cup \cdots \cup p_n(\mathbf{x}) = 0$.*

Proof. The direction ($\Rightarrow$) is clear. For the direction ($\Leftarrow$), multiply $p_1(\mathbf{x}) \cup \cdots \cup p_n(\mathbf{x}) = 0$ by any $p_i(\mathbf{x})$ and use an absorption law. □

Remark 4.7. *Reduction was a key step for Boole because his Elimination Theorem only applied to a single equation, not to a list of equations. He had to use a more complicated expression than Theorem 4.6 for his system—he developed several forms for reduction, the main one being $p_1(\mathbf{x})^2 + \cdots + p_n(\mathbf{x})^2 = 0$* (see [3], p. 121)*.*

4.2. **Expansion Theorem.** Any term $t(\mathbf{x},\mathbf{y})$ can be expanded as a 'linear' combination of $\mathbf{x}$-constituents, with coefficients that are terms in the variables $\mathbf{y}$.

Theorem 4.8 (**Boole's Expansion Theorem).** *Given a term $t(\mathbf{x},\mathbf{y})$, $\mathcal{PSA}$ satisfies*

$$t(\mathbf{x},\mathbf{y}) \;=\; \bigcup_{\sigma\in 2^{\widetilde{k}}} t(\sigma,\mathbf{y}) \cdot \mathcal{C}_\sigma(\mathbf{x}).$$

In particular,

$$t(x,\mathbf{y}) \;=\; t(1,\mathbf{y}) \cup t(0,\mathbf{y}).$$

Proof. From the third item of Lemma 4.4, and Lemma 4.5, we have

$$\begin{aligned} t(\mathbf{x},\mathbf{y}) &= \bigcup_{\sigma\in 2^{\widetilde{k}}} t(\mathbf{x},\mathbf{y}) \cdot \mathcal{C}_\sigma(\mathbf{x}) \\ &= \bigcup_{\sigma\in 2^{\widetilde{k}}} t(\sigma,\mathbf{y}) \cdot \mathcal{C}_\sigma(\mathbf{x}). \end{aligned}$$

□

A special case that occurs frequently is when one expands about all the variables in the term—the result, a union of consitituents, is called *the full expansion* of the term; it is also known as *the disjunctive normal form* of the term.

Corollary 4.9. *Given a term $t(\mathbf{x})$, $\mathcal{PSA}$ satisfies*

$$t(\mathbf{x}) \;=\; \bigcup_{\sigma\in 2^{\widetilde{k}}} t(\sigma)\cdot\mathcal{C}_\sigma(\mathbf{x}) \;=\; \begin{cases} 0 & \text{if} \quad \mathcal{PSA}\models t(\mathbf{x})=0 \\ \bigcup_{\substack{\sigma\in 2^{\widetilde{k}} \\ t(\sigma)\neq 0}} \mathcal{C}_\sigma(\mathbf{x}) & \text{otherwise.} \end{cases}$$

Proof. The first equality is from Theorem 4.8. For the second, note that, in the case of a full expansion, each coefficient $t(\sigma)$ is either 0 or 1, by (3). □

4.3. **More on Constituents.**

Definition 4.10. *Given a term $t(\mathbf{x})$, the* **set of constituents of** $t(\mathbf{x})$ *is*

$$\mathcal{C}\big(t(\mathbf{x})\big) \;:=\; \big\{\mathcal{C}_\sigma(\mathbf{x}) : t(\sigma)\neq 0\big\}.$$

With this definition we have an easy consequence of Corollary 4.9.

Corollary 4.11. $\mathcal{PSA}\models s(\mathbf{x})=t(\mathbf{x})$ *iff* $\mathcal{C}\big(s(\mathbf{x})\big)=\mathcal{C}\big(t(\mathbf{x})\big)$.

The next result makes the expressive power of an equation $p(\mathbf{x})=0$ clear—all it says is that the constituents of $p(\mathbf{x})$ are empty.

Corollary 4.12. *An equation $t(\mathbf{x})=0$ is $\mathcal{PSA}$-equivalent to the (conjunction of the) set of equations*

$$\{0=0\} \;\cup\; \big\{\mathcal{C}_\sigma(\mathbf{x})=0 : \mathcal{C}_\sigma(\mathbf{x})\in\mathcal{C}\big(t(\mathbf{x})\big)\big\}.$$

The adjunction of $\{0=0\}$ is needed for the case $\mathcal{PSA}\models t=0$.

The next lemma gives a simple calculus for working with constituents of terms.

Lemma 4.13. *Let $\mathcal{C}(\mathbf{x})$ be the set of $\mathbf{x}$-constituents.*

(a) $\mathcal{C}(0)=\varnothing$ *and* $\mathcal{C}(1)=\mathcal{C}\mathbf{x}$.
(b) $\mathcal{C}\big(t_1\cup\cdots\cup t_n\big) \;=\; \mathcal{C}\big(t_1\big)\cup\cdots\cup\mathcal{C}\big(t_n\big)$
(c) $\mathcal{C}\big(t_1\cdots t_n\big) \;=\; \mathcal{C}\big(t_1\big)\cdots\mathcal{C}\big(t_n\big)$
(d) $\mathcal{C}(t') \;=\; \mathcal{C}(t)'$.

Proof. (Routine.) □

5. Valid Basic-Formula Arguments

The goal of finding general conditions under which arguments $\varphi_1, \ldots, \varphi_n \therefore \varphi$ are valid does not seem to have been part of the algebra or logic of the 1800s. One was not so much interested in devising a test to see if one had found a consequence of a set of premisses; rather the focus was on forging methods for actually finding consequences from the premisses, preferably the most general consequences. The topics that interested Boole, and later Schröder, in the algebra of logic were *elimination* and *solution*—we will discuss those later, in §6.

Definition 5.1. *Given a term* $p(\mathbf{x})$*, let* $C_1(\mathbf{x}), \ldots, C_m(\mathbf{x})$ (where $m = 2^k$) be a listing of the $\mathbf{x}$-constituents. Define the universe U_p and an interpretation I_p into $\mathbf{PS}(U_p)$ by:

$$\begin{aligned} U_p &:= \big\{i \in \{1, \ldots, m\} : C_i(\mathbf{x}) \notin \mathcal{C}(p)\big\} \\ I_p(x_\ell) &:= A_\ell := \big\{i \in U_p : C_i(\mathbf{x}) \in \mathcal{C}(x_\ell)\big\}, \text{ for } \ell = 1, \ldots, k. \end{aligned}$$

Note that $C_i(\mathbf{x}) \in \mathcal{C}(x_\ell)$ means that x_ℓ, and not x'_ℓ, appears in $C_i(\mathbf{x})$.

Example 5.2. *Let* $(\mathbf{x}) := \mathbf{x_1}, \mathbf{x_2}$ *and* $p(x_1, x_2)$ *be* $x_1x_2 \cup x'_1x'_2$*. List the four* $\mathbf{x}$*-constituents:*

$$\begin{aligned} C_1(x_1, x_2) &:= x_1x_2 & C_2(x_1, x_2) &:= x_1x'_2 \\ C_3(x_1, x_2) &:= x'_1x_2 & C_4(x_1, x_2) &:= x'_1x'_2. \end{aligned}$$

Then $\mathcal{C}(x_1) = \{\mathcal{C}_1, \mathcal{C}_2\}$*,* $\mathcal{C}(x_1) = \{\mathcal{C}_1, \mathcal{C}_3\}$*, and* $\mathcal{C}\big(p(x_1, x_2)\big) = \big\{\mathcal{C}_1, \mathcal{C}_4\big\}$*, so we have* $U_p := \{2, 3\}$*,* $I_p(x_1) = \{2\}$*,* $I_p(x_2) = \{3\}$*,* $I_p(\mathcal{C}_1(\mathbf{x})) = I_p(x_1) \cap I_p(x_2) = \{2\} \cap \{3\} = \emptyset$*,* $I_p(\mathcal{C}_2(\mathbf{x})) = \{2\}$*,* $I_p(\mathcal{C}_3(\mathbf{x})) = \{3\}$*,* $I_p(\mathcal{C}_4(\mathbf{x})) = \emptyset$*. Observe that if* $(\mathcal{C}_i(\mathbf{x})) \in \mathcal{C}\big(p(\mathbf{x})\big)$*, then* $I_p(\mathcal{C}_i(\mathbf{x})) = \emptyset$*, and if* $(\mathcal{C}_i(\mathbf{x})) \notin \mathcal{C}\big(p(\mathbf{x})\big)$*, then* $I_p(\mathcal{C}_i(\mathbf{x})) = \{i\}$*.*

The following lemma generalizes the above example.

Lemma 5.3. I_p *interprets the* $C_i(\mathbf{x})$ *in* U_p *as follows:*

$$I_p\big(C_i(\mathbf{x})\big) := C_i(\mathbf{A}) = \begin{cases} \emptyset & \text{if } C_i(\mathbf{x}) \in \mathcal{C}(p) \\ \{i\} & \text{if } C_i(\mathbf{x}) \notin \mathcal{C}(p). \end{cases} \tag{9}$$

Thus $I_p\big(C_i(\mathbf{x})\big) \neq \emptyset$ *iff* $C_i(\mathbf{x})$ *is not a constituent of* $p(\mathbf{x})$*.*

Proof. We have

$$\begin{aligned}
j \in I_p\big(C_i(\mathbf{x})\big) \quad &\Leftrightarrow \quad \bigwedge_{\ell\in\widetilde{k}} \Big[C_i(\mathbf{x}) \in \mathcal{C}(x_\ell) \Leftrightarrow j \in A_\ell\Big]\\
&\Leftrightarrow \quad \bigwedge_{\ell\in\widetilde{k}} \Big[C_i(\mathbf{x}) \in \mathcal{C}(x_\ell) \Leftrightarrow \big(j \in U_p \ \wedge \ C_j(\mathbf{x}) \in \mathcal{C}(x_\ell)\big)\Big]\\
&\Leftrightarrow \quad \big(j \in U_p\big) \ \wedge \ \bigwedge_{\ell\in\widetilde{k}} \Big[C_i(\mathbf{x}) \in \mathcal{C}(x_\ell) \Leftrightarrow C_j(\mathbf{x}) \in \mathcal{C}(x_\ell)\Big]\\
&\Leftrightarrow \quad \big(j \in U_p\big) \ \wedge \ \big(C_i(\mathbf{x}) = C_j(\mathbf{x})\big)\\
&\Leftrightarrow \quad \big(j \in U_p\big) \ \wedge \ \big(i = j\big)\\
&\Leftrightarrow \quad \Big(C_i(\mathbf{x}) \notin \mathcal{C}\big(p(\mathbf{x})\big)\Big) \ \wedge \ \big(j = i\big)
\end{aligned}$$

□

Lemma 5.4. *Given terms $p(\mathbf{x})$ and $q(\mathbf{x})$, one has*

(a) $I_p\big(\mathcal{C}_\sigma(\mathbf{x})\big) = \emptyset$ *iff* $\mathcal{C}_\sigma(\mathbf{x}) \in \mathcal{C}\big(p(\mathbf{x})\big)$, *for* $\sigma \in 2^{\widetilde{k}}$.

(b) $\mathcal{C}\big(q(\mathbf{x})\big) \subseteq \mathcal{C}\big(p(\mathbf{x})\big)$ *iff* $I_p(q) = \emptyset$.

(c) *The equational argument*

$$p(\mathbf{x}) = 0 \ \therefore \ q(\mathbf{x}) = 0$$

is valid iff

$$\mathcal{C}\big(q(\mathbf{x})\big) \subseteq \mathcal{C}\big(p(\mathbf{x})\big).$$

Proof. (a) follows from (9), and (b) from (a) and Corollary 3.9. The direction ($\Leftarrow$) of (c) follows from Corollary 4.12, For the direction ($\Rightarrow$) of (c), assume $\mathsf{Valid}\big[p(\mathbf{x}) = 0 \ \therefore \ q(\mathbf{x}) = 0\big]$. By Lemma 5.4(b), $I_p(p) = 0$, thus we must have $I_p(q) = 0$. This gives $\mathcal{C}\big(q(\mathbf{x})\big) \subseteq \mathcal{C}\big(p(\mathbf{x})\big)$, again by Lemma 5.4(b). □

5.1. **Equational Arguments.** The next result says that an equational argument is valid iff the constituents of the conclusion are among the constituents of the premisses.

Theorem 5.5 (Equational Arguments). *The equational argument*

$$p_1(\mathbf{x}) = 0, \ \ldots, \ p_m(\mathbf{x}) = 0 \ \therefore \ p(\mathbf{x}) = 0$$

is valid iff

$$\mathcal{C}\big(p(\mathbf{x})\big) \;\subseteq\; \mathcal{C}\big(p_1(\mathbf{x})\big) \cup \cdots \cup \mathcal{C}\big(p_m(\mathbf{x})\big).$$

Proof. This follows from Lemma 4.13(b) and Lemma 5.4(c), since the premisses can be reduced to the single equation $p_1(\mathbf{x}) \cup \cdots \cup p_m(\mathbf{x}) = 0$, by Theorem 4.6. □

5.2. **Equational Conclusion.** Some basic-formula arguments are rather trivially valid because it is not possible to make all the premisses true, under any interpretation. A simple example would be $x = 0, x \neq 0 \;\therefore\; \beta$. This argument is valid, but not very interesting. The next theorem says that positive conclusions only require positive premisses, provided the premisses are satisfiable.

Theorem 5.6 (**Equational Conclusion**)**.** *Suppose the list*

$$p_1(\mathbf{x}) = 0,\ \ldots,\ p_m(\mathbf{x}) = 0,\ q_1(\mathbf{x}) \neq 0,\ \ldots,\ q_n(\mathbf{x}) \neq 0 \tag{10}$$

of basic formulas is satisfiable. Then the basic-formula argument

$$p_1(\mathbf{x}) = 0,\ \ldots,\ p_m(\mathbf{x}) = 0,\ q_1(\mathbf{x}) \neq 0,\ \ldots,\ q_n(\mathbf{x}) \neq 0 \;\therefore\; p(\mathbf{x}) = 0 \tag{11}$$

is valid iff the equational argument

$$p_1(\mathbf{x}) = 0,\ \ldots,\ p_m(\mathbf{x}) = 0 \;\therefore\; p(\mathbf{x}) = 0 \tag{12}$$

is valid.

Proof. The direction (12) ⇒ (11) is trivial. So suppose (11) is valid. First let us replace the equational premisses in (11) with a single equation $p_0(\mathbf{x}) = 0$, giving the argument

$$p_0(\mathbf{x}) = 0,\ q_1(\mathbf{x}) \neq 0,\ \ldots,\ q_n(\mathbf{x}) \neq 0 \;\therefore\; p(\mathbf{x}) = 0, \tag{13}$$

where $p_0(\mathbf{x}) := p_1(\mathbf{x}) \cup \cdots \cup p_m(\mathbf{x})$. The arguments (11) and (13) are both valid or both invalid.

By Corollary 4.11 and Lemma 5.4(a), the interpretation I_{p_0} makes $p_0(\mathbf{x}) = 0$ true. From the satisfiability of (10), it follows that

$$p_0(\mathbf{x}) = 0,\ q_1(\mathbf{x}) \neq 0,\ \ldots,\ q_n(\mathbf{x}) \neq 0$$

is satisfiable. Then by Lemma 5.4(b), each $\mathcal{C}(q_j)$ has a constituent that is not in $\mathcal{C}(p_0)$. Thus I_{p_0} makes some constituent in each $\mathcal{C}(q_j)$ non-empty, and thus it makes each $q_j(\mathbf{x}) \neq 0$ true. From this it follows that the interpretation I_{p_0} makes all the premisses of (11) true. Since we have assumed (11) is a valid argument, it follows that I_{p_0} makes $p(\mathbf{x}) = 0$ true, thus $I_{p_0}\big(p(\mathbf{x})\big) = \emptyset$. By Lemma 5.4 (b), it follows that $\mathcal{C}(p) \subseteq \mathcal{C}(p_0)$; consequently, by Lemma 5.4(c), the argument $p_0(\mathbf{x}) = 0 \therefore p(\mathbf{x}) = 0$ is valid. Thus (12) is valid.

□

5.3. **Negated-Equation Conclusion.** Now we turn to the case when the conclusion is a negated equation. Perhaps surprisingly, such an argument reduces in a simple manner to a disjunction of equational arguments. We assume the equational premisses have already been reduced to a single equation $p_0(\mathbf{x}) = 0$.

Theorem 5.7. *Consider the following assertions:*

$$\mathsf{Valid}\big(p_0(\mathbf{x}) = 0,\ q_1(\mathbf{x}) \neq 0,\ \ldots,\ q_n(\mathbf{x}) \neq 0\ \therefore\ q(\mathbf{x}) \neq 0\big) \tag{14}$$

$$\mathsf{Valid}\big(p_0(\mathbf{x}) = 0,\ q_j(\mathbf{x}) \neq 0\ \therefore\ q(\mathbf{x}) \neq 0\big) \tag{15}$$

$$\mathsf{Valid}\big(p_0(\mathbf{x}) = 0,\ q(\mathbf{x}) = 0\ \therefore\ q_j(\mathbf{x}) = 0\big) \tag{16}$$

$$\mathcal{C}\big(q_j(\mathbf{x})\big)\ \subseteq\ \mathcal{C}\big(p_0(\mathbf{x})\big)\ \cup\ \mathcal{C}\big(q(\mathbf{x})\big). \tag{17}$$

Then

(a) *For each j,* (15) *holds iff* (16) *holds.*
(b) *For each j,* (16) *holds iff* (17) *holds.*
(c) (14) *holds iff for some j,* (15) *holds.*

Proof. Item (a) follows from simple propositional logic, namely the propositional formula $(P \wedge \neg Q) \rightarrow \neg R$ is equivalent to $(P \wedge R) \rightarrow Q$.

Item (b) follows from Theorem 5.5.

The direction ($\Leftarrow$) of (c) clearly holds. So it only remains to show that if (14) holds one has (15) holding for some j—we will show the contrapositive.

Suppose for every j, (15) is false. Then for each j, (17) is false, by (a) and (b). Let $\widehat{p} := p_0 \cup q$. From the failure of (17) for each j, one has $\mathcal{C}(q_j) \nsubseteq \mathcal{C}(\widehat{p})$ for each j. Consequently, by Lemma 5.4(b), the interpretation $I_{\widehat{p}}$ makes $\widehat{p} = 0$ true, but $q_j = 0$ false, for each j. This means $I_{\widehat{p}}$ makes the premisses of the argument in (14) true, but the conclusion false. Thus (14) does not hold.

□

Combining the above theorems, we see that the study of valid basic-formula arguments reduces in a simple manner to the study of valid equational arguments, which in turn reduces to comparing constituents of the terms involved in the arguments.

5.4. **Using Boole's V in Valid Arguments.** So far we have adopted Schröder's translation of particular propositions, using $\neq 0$. Boole did not do this, but rather tried to use an equational translation. Consider the proposition 'Some x is y'. In 1847 he used $V = xy$ as his primary translation (see [1], p. 20), where V is a new idempotent symbol. In 1854 (see [3], pp. 61-64) he changed the translation to $V \cdot x = V \cdot y$. Neither translation seems fully capable of doing what Boole claimed, although the first seems closer to achieving his goals. There is a simple intermediate translation, namely $V = V \cdot xy$, or equivalently, $V \cdot (xy)' = 0$, that works much better. We say it is an intermediate translation because $V = xy \;\Rightarrow\; V = V \cdot xy \;\Rightarrow\; V \cdot x = V \cdot y$.

Let us call an equation of the form $V \cdot p(\mathbf{x}) = 0$ a **V-equation**. Next we show how the two-way translation

$$p \neq 0 \quad \rightleftharpoons \quad V \cdot p' = 0$$

between negated equations and V-equations can be used in the study of valid arguments, to fulfill Boole's goal of a viable *equational* translation of particular statements.

First we look at the case of a single negative premiss, where we use a simple fact about sets, namely

$$A \subseteq B \cup C \quad \text{iff} \quad C' \subseteq B \cup A'. \tag{18}$$

This follows from noting that both sides are equivalent to $AB'C' = \emptyset$.

Theorem 5.8. *The following are equivalent:*

(a)

$$\mathsf{Valid}\big(p = 0,\, q_0 \neq 0 \;\therefore\, q \neq 0\big)$$

(b)

$$\mathsf{Valid}\big(p = 0,\, V \cdot q_0' = 0 \;\therefore\, V \cdot q' = 0\big).$$

Proof. By simple propositional logic reasoning, (a) is equivalent to

$$\mathsf{Valid}\big(p = 0,\, q = 0 \ \therefore\ q_0 = 0\big). \tag{19}$$

By Theorem 5.5, assertion (19) is equivalent to

$$\mathcal{C}(q_0) \ \subseteq\ \mathcal{C}(p) \cup \mathcal{C}(q), \tag{20}$$

which, in view of (18) and Lemma 4.13(d), we can rewrite as

$$\mathcal{C}(q') \ \subseteq\ \mathcal{C}(p) \cup \mathcal{C}(q_0'). \tag{21}$$

Looking at $(V, \mathbf{x})$-constituents, (21) is equivalent to

$$\mathcal{C}(V \cdot q') \ \subseteq\ \mathcal{C}(p) \cup \mathcal{C}(V \cdot q_0'), \tag{22}$$

which, by Theorem 5.5, is equivalent to

$$\mathsf{Valid}\big(p = 0,\, V \cdot q_0' = 0 \ \therefore V \cdot q' = 0\big).$$

$\square$

When looking at the equivalence of (a) and (b) in this theorem it is easy to jump to the conclusion that somehow one has been able to replace each negated equation with an *equivalent* equation involving a new symbol V; that, for example, the assertion "Some x is y" is fully expressed by the equation $V = V \cdot (xy)$, or equivalently, $V \cdot (xy)' = 0$. If we think of V as standing for 'something', then these two equations can be viewed as saying "Something is in both x and y". That would be useful as a mnemonic device, but the reality is that $xy \neq 0$ and $V \cdot (xy)' = 0$ do not express the same thing, that is, they are not equivalent.

One heuristic behind the use of a new constant V is that one can express $t \neq 0$ with a formula

$$(\exists V)\big(V \neq 0 \text{ and } V \subseteq t\big)$$

which is equivalent to

$$(\exists V)\big(V \neq 0 \text{ and } V = V \cdot t\big)$$

as well as

$$(\exists V)\big(V \neq 0 \text{ and } V \cdot t' = 0\big).$$

When reasoning with such a formula it would be usual to say "Choose such a V", giving

$$V \neq 0 \text{ and } V \cdot t' = 0.$$

This formula is, of course, not equivalent to the original $t \neq 0$, but it implies the latter. When one drops the formula $V \neq 0$, then also this implication fails, that is, neither of $t \neq 0$ and $V \cdot t' = 0$ implies the other. Thus the equivalence of (a) and (b) in Theorem 5.8 is certainly not due to a simple replacement of basic formulas by equivalent formulas. The fact that (a) and (b) are equivalent is based on the magic of the global interaction of formulas in an argument—it is something that one does not expect to be true, but it might be true, and through the curiosity of exploration one discovers a proof that indeed the two are equivalent.

Theorem 5.9. *The following are equivalent:*

(a)
$$\mathsf{Valid}\big(p = 0,\ q_0 \neq 0,\ \ldots,\ q_{n-1} \neq 0\ \therefore q \neq 0\big).$$

(b) *For some j,*
$$\mathsf{Valid}\big(p = 0,\ q_j \neq 0\ \therefore q \neq 0\big).$$

(c) *For some j,*
$$\mathsf{Valid}\big(p = 0,\ V_j \cdot q_j' = 0\ \therefore V_j \cdot q' = 0\big).$$

(d) *For some j,*
$$\mathsf{Valid}\big(p = 0,\ V_0 \cdot q_0' = 0,\ \ldots,\ V_{n-1} \cdot q_{n-1}' = 0\ \therefore V_j \cdot q' = 0\big).$$

Proof. By Theorem 5.7(c), item (a) is equivalent to (b); and by Theorem 5.8, (b) is equivalent to (c).

Clearly (c) implies (d). Now suppose (d) holds, and choose a j such that the indicated argument is valid. By setting all V_i, $i \neq j$, equal to 0, we have (d) implies (c).

□

6. Elimination and Solution

6.1. **A Single Equation.** The following gives Schröder's version of Boole's results on elimination and solution (p. 447 in [10] Vol. I). The elimination condition is the same as Boole's, but the solution is much simpler, thanks to working with power-set algebras instead of Boole's system.

Theorem 6.1 (Elimination and Solution Theorem). *Given a term $p(x, \mathbf{y})$, the equation $p(x, \mathbf{y}) = 0$ is $\mathcal{PSA}$-equivalent to*

$$p(1, \mathbf{y}) \cdot p(0, \mathbf{y}) \;=\; 0 \;\wedge\; (\exists z)\big[x \;=\; z' \cdot p(0, \mathbf{y}) \;\cup\; z \cdot p(1, \mathbf{y})'\big]. \tag{23}$$

$(\exists x)\big[p(x, \mathbf{y}) = 0\big]$ *is $\mathcal{PSA}$-equivalent to*

$$p(1, \mathbf{y}) \cdot p(0, \mathbf{y}) \;=\; 0. \tag{24}$$

Thus (24) *is the complete result of eliminating x from the equation $p(x, \mathbf{y}) = 0$; if this condition holds, then*

$$x \;=\; z' \cdot p(0, \mathbf{y}) \;\cup\; z \cdot p(1, \mathbf{y})' \tag{25}$$

gives the general solution of $p(x, \mathbf{y}) = 0$ for x.

Proof. The equation $p(x, \mathbf{y}) = 0$ can be written as

$$p(1, \mathbf{y}) \cdot x \;\cup\; p(0, \mathbf{y}) \cdot x' \;=\; 0,$$

which is equivalent to

$$p(1, \mathbf{y}) \cdot x = 0 \;\wedge\; p(0, \mathbf{y}) \cdot x' = 0,$$

which can be written as

$$p(0, \mathbf{y}) \subseteq x \subseteq p(1, \mathbf{y})'. \tag{26}$$

There is an x which makes (26) true iff $p(0, \mathbf{y}) \subseteq p(1, \mathbf{y})'$, that is, iff $p(1, \mathbf{y}) \cdot p(0, \mathbf{y}) = 0$. If (26) is fulfilled, then
(a) $x = x' \cdot p(0, \mathbf{y}) \cup x \cdot p(1, \mathbf{y})'$, so there is a z as required by (23); and
(b) if $x = z' \cdot p(0, \mathbf{y}) \cup z \cdot p(1, \mathbf{y})'$ for some z, then clearly (26) holds. □

Schröder (p. 460 of [10], Vol. I) credits Boole with the previous elimination theorem, calling it Boole's Main Theorem, even though Boole claimed this result for his own algebra of logic, not the modern one presented here. Likewise we credit Boole with the next result.

Corollary 6.2 (Boole's Elimination Theorem). $(\exists \mathbf{x})\big[p(\mathbf{x}, \mathbf{y}) \;=\; 0\big]$ *is $\mathcal{PSA}$-equivalent to*

$$0 \;=\; \bigcap_{\sigma \in 2^{\widetilde{k}}} p(\sigma, \mathbf{y}), \tag{27}$$

the complete result of eliminating $\mathbf{x}$ *from* $p(\mathbf{x}, \mathbf{y}) = 0$.

Schröder does not give a general formula to find the solution for $\mathbf{x}$ in $p(\mathbf{x}, \mathbf{y}) = 0$ as a function of $\mathbf{y}$ when $\mathbf{x}$ is a list of more than one variable.

6.2. **Schröder's Elimination Program.** Schröder goes on to set up an ambitious program to tackle quantifier elimination for arbitrary open (i.e., quantifier-free) formulas $\omega(\mathbf{x}, \mathbf{y})$. Every open formula is equivalent to a disjunction of conjunctions of basic formulas. Since

$$(\exists \mathbf{x})\big[\varphi_1(\mathbf{x}, \mathbf{y}) \vee \ \ldots \ \vee \varphi_m(\mathbf{x}, \mathbf{y})\big]$$

is equivalent to

$$(\exists \mathbf{x})\big[\varphi_1(\mathbf{x}, \mathbf{y})\big] \vee \ \ldots \ \vee (\exists \mathbf{x})\big[\varphi_m(\mathbf{x}, \mathbf{y})\big],$$

quantifier elimination for open formulas reduces to quantifier elimination for conjunctions of basic formulas, that is, to formulas of the form

$$(\exists \mathbf{x})\big[p(\mathbf{x}, \mathbf{y}) = 0 \ \wedge \ q_1(\mathbf{x}, \mathbf{y}) \neq 0 \ \wedge \ \cdots \ \wedge \ q_n(\mathbf{x}, \mathbf{y}) \neq 0\big].$$

(Only one equation is needed in view of the Reduction Theorem.)

7. Elimination and Solution for Basic Formulas

Schröder extended his Theorem 6.1 to the case of one equation and one negated equation (see Corollary 7.2 below). Our next result extends the parametric solution portion of Schröder's result to include any number of negated equations.

7.1. **Parametric solutions to systems of basic formulas.**

Theorem 7.1. *Given a system*

$$p(x, \mathbf{y}) = 0, \ q_1(x, \mathbf{y}) \neq 0, \ \ldots, \ q_n(x, \mathbf{y}) \neq 0 \tag{28}$$

of basic formulas, write them in the form (by setting $a(y) = p(1, y), b(y) = p(0, y)$, etc.)

$$\begin{aligned} a(\mathbf{y}) \cdot x \ \cup \ b(\mathbf{y}) \cdot x' &= 0 \\ c_1(\mathbf{y}) \cdot x \ \cup \ d_1(\mathbf{y}) \cdot x' &\neq 0 \\ &\vdots \end{aligned}$$

$$c_n(\mathbf{y}) \cdot x \ \cup \ d_n(\mathbf{y}) \cdot x' \neq 0.$$

Let $\varphi(\mathbf{y}, \mathbf{v})$ *be the conjuction of the formulas:*

$$\begin{aligned} & a(\mathbf{y}) \cdot b(\mathbf{y}) = 0 \\ & \quad 0 \neq v_i \subseteq c_i(\mathbf{y}) \cdot a'(\mathbf{y}) \ \cup \ d_i(\mathbf{y}) \cdot b'(\mathbf{y}) \\ & \quad v_i \cap v_j \subseteq c_i(\mathbf{y}) \ \cup \ d_j(\mathbf{y}) \cdot c_j(\mathbf{y}) \ \cup \ d_i(\mathbf{y}) \quad \textit{for } i \neq j \end{aligned}$$

Then the system (28) *is* $\mathcal{PSA}$*-equivalent to*

$$(29)\quad (\exists w)(\exists v_1) \cdots (\exists v_k)\Big[\ \varphi \ \wedge \ \Big(x = w \cdot \Big(a \cup \bigcup_{i \in \widetilde{k}} v_i {c_i}'\Big)' \ \cup \ w' \cdot \Big(b \cup \bigcup_{i \in \widetilde{k}} v_i {d_i}'\Big)\Big)\Big].$$

Proof. The original system (28) is clearly $\mathcal{PSA}$-equivalent to

$$(\exists v_1) \cdots (\exists v_k)\Big[\bigwedge_{j \in \widetilde{k}} (v_j \neq 0) \ \wedge \ (p = 0) \ \wedge \ \bigwedge_{j \in \widetilde{k}} (v_j \subseteq q_j)\Big].$$

The formula

$$(p = 0) \ \wedge \ \bigwedge_{j \in \widetilde{k}} \big(v_j \subseteq q_j\big)$$

is $\mathcal{PSA}$-equivalent to the conjunction of equations

$$(p = 0) \ \wedge \ \bigwedge_{j \in \widetilde{k}} \big(v_j q_j' = 0\big).$$

Reduce this, by Theorem 4.6, to a single equation

$$\big(ax \cup bx'\big) \ \cup \ \Big(\bigcup_{j \in \widetilde{k}} v_j \cdot \big(c_j x \cup d_j x'\big)'\Big) \ = \ 0,$$

and expand it about x to obtain

$$\Big(a \cup \bigcup_{j \in \widetilde{k}} v_j {c_j}'\Big) \cdot x \ \cup \ \Big(b \cup \bigcup_{j \in \widetilde{k}} v_j {d_j}'\Big) \cdot x' \ = \ 0.$$

This is $\mathcal{PSA}$-equivalent, by Theorem 6.1, to the conjunction of the two formulas

$$(30)\qquad 0 \ = \ \Big(a \cup \bigcup_{j \in \widetilde{k}} v_j {c_j}'\Big) \cdot \Big(b \cup \bigcup_{j \in \widetilde{k}} v_j {d_j}'\Big)$$

$$(31)\qquad (\exists w)\Big[x \ = \ w \cdot \Big(a \cup \bigcup_{j \in \widetilde{k}} v_j {c_j}'\Big)' \ \cup \ w' \cdot \Big(b \cup \bigcup_{j \in \widetilde{k}} v_j {d_j}'\Big)\Big].$$

Expanding (30) as a polynomial in the v_j transforms it into the conjunction of the equations

$$\begin{array}{rcll} 0 & = & ab & \\ 0 & = & v_j \cdot \big(a{d_j}' \cup b{c_j}' \cup {c_j}'{d_j}'\big) & \text{for } j \in \widetilde{k} \\ 0 & = & v_i v_j \cdot \big({c_i}'{d_j}' \cup {c_j}'{d_i}'\big) & \text{for } i,j \in \widetilde{k}, i \neq j, \end{array}$$

which can be rewritten as

$$\begin{array}{rcll} 0 & = & ab & \\ 0 & = & v_j \cdot \big(c_j a' \cup d_j b'\big)' & \text{for } j \in \widetilde{k} \\ 0 & = & v_i v_j \cdot \big((c_i \cup d_j) \cdot (c_j \cup d_i)\big)' & \text{for } i,j \in \widetilde{k}, i \neq j. \end{array}$$

Thus (28) is $\mathcal{PSA}$-equivalent to the existence of $\mathbf{v}$ and w such that:

$$\begin{array}{rcll} 0 & = & ab & \\ 0 & \neq & v_j \ \subseteq \ c_j a' \cup d_j b' & \text{for } j \in \widetilde{k} \\ v_i v_j & \subseteq & (c_i \cup d_j) \cdot (c_j \cup d_i) & \text{for } i,j \in \widetilde{k}, i \neq j \\ x & = & \Big[w \cdot \Big(a \cup \bigcup_{j\in\widetilde{k}} v_j {c_j}'\Big)'\Big] \ \cup \ \Big[w' \cdot \Big(b \cup \bigcup_{j\in\widetilde{k}} v_j {d_j}'\Big)\Big]. & \end{array}$$

□

Note that in (29), the restrictions on the parameters v_i are very simple, and there is no restriction on the parameter w.

If one restricts the above to the case where there is exactly one negated equation, then one has a full elimination result, extending Theorem 6.1. (See p. 205-209 of [10] Vol. II)

Corollary 7.2 (Schröder). *The formula*

$$(\exists x)\big[p(x,\mathbf{y}) = 0 \ \wedge \ q(x,\mathbf{y}) \neq 0\big] \tag{32}$$

is $\mathcal{PSA}$-equivalent to

$$\big[p(1,\mathbf{y}) \cdot p(0,\mathbf{y}) \ = \ 0\big] \ \wedge \ \big[q(1,\mathbf{y}) \cdot p(1,\mathbf{y})' \ \cup \ q(0,\mathbf{y}) \cdot p(0,\mathbf{y})' \ \neq \ 0\big]. \tag{33}$$

Proof. By Theorem 7.1,

$$(\exists x)\big[p(x,\mathbf{y}) = 0 \ \wedge \ q(x,\mathbf{y}) \neq 0\big] \tag{34}$$

is equivalent to the following, where mention of the $\mathbf{y}$'s has been suppressed:

$$(\exists z)\Big[\big(p(1)\cdot p(0) = 0\big) \ \wedge \ \big(0 \neq z \subseteq q(1)\cdot p(1)' \ \cup \ q(0)\cdot b(0)'\big)\Big],$$

which in turn is equivalent to (33). □

The difficulties with quantifier elimination for a conjunction of basic formulas starts with two negated equations. The simplest example to illustrate this is try to eliminate x from

$$(\exists x)\Big(xy \neq 0 \ \wedge x'y \neq 0\Big). \tag{35}$$

Clearly $y \neq 0$ follows from (35). However this is not equivalent to (35) because there will be an x as in (35) iff y has at least 2 elements. The formula (35) is equivalent to the expression

$$|y| \geq 2. \tag{36}$$

Unfortunately there is no way to express $|y| \geq 2$ by an open formula in the language of power-set algebra that we are using (with fundamental operations $\cup, \cap, '$ and constants $0, 1$). In modern terminology, power-set algebra does not admit elimination of quantifiers.

Nonetheless, Schröder struggled on, showing how admitting symbols for *elements* of sets would allow him to carry out elimination for the case of eliminating one variable. Then he says that the result for eliminating two variables would follow similar reasoning, but would be much more complicated, etc. (See §49 in [10] Vol. II.)

In 1919 Skolem [11] gave an elegant improvement on Schröder's approach to elimination by showing that if one adds the predicates $|\ | \geq n$, for $n \geq 0$, then this augmented version of power-set algebra has a straightforward procedure for the elimination of quantifiers.

7.2. **Using Boole's V in Elimination.** In §5.4 we saw that one could replace negated equations by V-equations, in the spirit of Boole, when studying the validity of arguments. Now we ask if one can do the same when working with elimination. Since Schröder's elimination, in the original language of power-set algebras, halts with one negated equation, we will restrict our attention to considering a V-version of Theorem 7.2. We will use a V-translation to convert the negated equation in the premiss into an equation, apply Boole' elimination result to the two premisses, split this into an equation and a V-equation, and then convert the V-equation back into a negated equation. It will be noted that this method gives the correct answer found by Schröder. Thus, for example, one can apply the V-method to derive the valid syllogisms which have one particular premiss and one universal premiss.

Theorem 7.3. *The V-translation of*

$$(\exists x)\big[p(x,\mathbf{y}) = 0 \ \wedge\ q(x,\mathbf{y}) \neq 0\big] \tag{37}$$

is

$$(\exists x)\big[p(x,\mathbf{y}) = 0 \ \wedge\ V \cdot q'(x,\mathbf{y}) = 0\big]. \tag{38}$$

Eliminating x using Theorem 6.1 gives

$$p(1,\mathbf{y}) \cdot p(0,\mathbf{y}) \ \cup\ V \cdot \Big(q(1,\mathbf{y}) \cdot p(1,\mathbf{y})' \ \cup\ q(0,\mathbf{y}) \cdot p(0,\mathbf{y})'\Big)' \ = \ 0, \tag{39}$$

which translates back into

$$\big[p(1,\mathbf{y}) \cdot p(0,\mathbf{y}) \ = \ 0\big] \ \wedge\ \big[q(1,\mathbf{y}) \cdot p(1,\mathbf{y})' \ \cup\ q(0,\mathbf{y}) \cdot p(0,\mathbf{y})' \ \neq\ 0\big]. \tag{40}$$

Proof. The V-translation converts

$$p(x,\mathbf{y}) = 0 \ \wedge\ q(x,\mathbf{y}) \neq 0$$

into

$$p(x,\mathbf{y}) = 0 \ \wedge\ V \cdot q(x,\mathbf{y})' = 0.$$

Reducing this to a single equation gives

$$p(x,\mathbf{y}) \ \cup\ V \cdot q(x,\mathbf{y})' = 0.$$

The complete result of eliminating x is, by Theorem 6.1,

$$\Big(p(1,\mathbf{y}) \ \cup\ V \cdot q(1,\mathbf{y})'\Big) \cdot \Big(p(0,\mathbf{y}) \ \cup\ V \cdot q(0,\mathbf{y})'\Big) \ = \ 0.$$

Multiplying this out gives

$$p(1,\mathbf{y})\cdot p(0,\mathbf{y})\ \cup\ V\cdot\Big(p(1,\mathbf{y})\cdot q(0,\mathbf{y})'\ \cup\ p(0,\mathbf{y})\cdot q(1,\mathbf{y})'\ \cup\ q(0,\mathbf{y})'\cdot q(1,\mathbf{y})'\Big)\ =\ 0,$$

which is equivalent to the two equations

$$p(1,\mathbf{y})\cdot p(0,\mathbf{y}) = 0 \tag{41}$$

$$V\cdot\Big(p(1,\mathbf{y})\cdot q(0,\mathbf{y})'\ \cup\ p(0,\mathbf{y})\cdot q(1,\mathbf{y})'\ \cup\ q(0,\mathbf{y})'\cdot q(1,\mathbf{y})'\Big)\ =\ 0. \tag{42}$$

In view of (41), the equation (42) is equivalent to

$$V\cdot\Big(q(1,\mathbf{y})\cdot p(1,\mathbf{y})'\ \cup\ q(0,\mathbf{y})\cdot p(0,\mathbf{y})'\Big)'\ =\ 0,$$

which translates back to the negated equation

$$q(1,\mathbf{y})\cdot p(1,\mathbf{y})'\ \cup\ q(0,\mathbf{y})\cdot p(0,\mathbf{y})'\ \neq\ 0.$$

This, combined with (41), gives (40), the same result as in Theorem 7.2. □

$$t(\mathbf{x})\ =\ \begin{cases} 0 & \text{if}\quad \mathcal{PSA}\models t(\mathbf{x})=0 \\ \bigcup_{\substack{\sigma\in 2^{\widetilde{k}}\\ t(\sigma)\neq 0}} \mathcal{C}_\sigma(\mathbf{x}) & \text{otherwise.}\end{cases}$$

References

[1] George Boole, *The Mathematical Analysis of Logic, Being an Essay Towards a Calculus of Deductive Reasoning*, Originally published in Cambridge by Macmillan, Barclay, & Macmillan, 1847. Reprinted in Oxford by Basil Blackwell, 1951.

[2] ——, *The Calculus of Logic*, The Cambridge and Dublin Mathematical Journal, **3** (1847), 183-198.

[3] ——, *An Investigation of The Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities.* Originally published by Macmillan, London, 1854. Reprint by Dover, 1958.

[4] Frank Mark Brown, *George Boole's Deductive System.* Notre Dame J. Formal Logic **50** No. 3 (303–330), 2009.

[5] Stanley Burris, *George Boole.* The online Stanford Encyclopedia of Philosophy at `http://plato.stanford.edu/entries/boole/`.

[6] Stanley Burris and H.P. Sankappanavar, *Boole's method II. The original method.*

[7] Stanley Burris and H.P. Sankappanavar, *The Horn Theory of Boole's partial algebras*, The Bulletin of Symbolic Logic **19** (2013), 97-105.

[8] Theodore Hailperin, *Boole's Logic and Probability, (Series: Studies in Logic and the Foundations of Mathematics, 85*, Amsterdam, New York, Oxford: Elsevier North-Holland. 1976. 2nd edition, Revised and enlarged, 1986.

[9] William Stanley Jevons, *Pure Logic, or the Logic of Quality apart from Quantity: with Remarks on Boole's System and on the Relation of Logic and Mathematics.* Edward Stanford, London, 1864. Reprinted 1971 in Pure Logic and Other Minor Works ed. by R. Adamson and H.A. Jevons, Lennox Hill Pub. & Dist. Co., NY.

[10] Ernst Schröder, *Algebra der Logik*, Vols. IIII, 18901910; reprint by Chelsea 1966.

[11] Thoralf Skolem, *Untersuchung über die Axiome des Klassenkalküls und über Produktations– und Summationsprobleme, welche gewisse Klassen von Aussagen betreffen.* Videnskabsakademiet i Kristiania, Skrifter I, No. 3, 1919, pp. 37. Also in "Selected Works in Logic by Th. Skolem", ed. by Jens Erik Fenstak, Scand. Univ. Books, Universitetsforlaget, Oslo, 1970.