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Weights of denumerable topological spaces

by

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Abstract. Let $(X, \mathfrak{T})$ be a denumerable topological space — for $|\mathfrak{T}| \leqslant \aleph_\omega$ we show that the weight of $(X, \mathfrak{T})$ equals $|\mathfrak{T}|$; and $|\mathfrak{T}| > \aleph_\omega$ implies the weight of $(X, \mathfrak{T})$ is greater than or equal to $\aleph_\omega$, unless $\mathfrak{T}$ has the power of the continuum.

In this paper we will examine the possible weights of topological spaces $(X, \mathfrak{T})$ where X is denumerable, answering a question of P. Erdös enroute. Throughout we will use lower-case German letters as well as alephs to denote cardinal numbers and lower-case Greek letters to denote ordinals, the letter ω being reserved for the first infinite ordinal. The transfinite sequence $\aleph_0, \aleph_1, \ldots$ denotes the cardinals indexed by ordinals and ordered by size. For α an ordinal, ω_α denotes the least ordinal such that ω_α has $\aleph_\alpha$ predecessors (or members). The cardinal $2^{\aleph_0}$ is also denoted by $\mathfrak{c}$. $|A|$ denotes the cardinal of the set A.

If $(X, \mathfrak{T})$ is a topological space we will let $w(X, \mathfrak{T})$ be the *weight* of $(X, \mathfrak{T})$, that is, the least cardinality of a base of $(X, \mathfrak{T})$. (The reader is referred to Comfort's excellent survey article [2].) For $\mathfrak{n} \leqslant \mathfrak{c}$ define

$$\mathfrak{W}_\mathfrak{n} = \{w(X, \mathfrak{T}) \colon |X| = \aleph_0, |\mathfrak{T}| = \mathfrak{n}\} .$$

Clearly $\mathfrak{l} \in \mathfrak{W}_\mathfrak{n}$ implies $\mathfrak{l} \leqslant \mathfrak{n}$. First we prove $\mathfrak{W}_\mathfrak{n}$ is convex for $\mathfrak{n}$ infinite.

THEOREM 1. *If* $\mathfrak{n}$ *is infinite,* $\mathfrak{l} \in \mathfrak{W}_\mathfrak{n}$ *and* $\mathfrak{l} \leqslant \mathfrak{m} \leqslant \mathfrak{n}$, *then* $\mathfrak{m} \in \mathfrak{W}_\mathfrak{n}$.

Proof. From the hypothesis of Theorem 1 we see that $\mathfrak{l} \in \mathfrak{W}_\mathfrak{n}$ implies $\mathfrak{l}$ is infinite. Let $(X_0, \mathfrak{T}_0)$ be a denumerable topological space such that $|\mathfrak{T}_0| = \mathfrak{n}$, $w(X_0, \mathfrak{T}_0) = \mathfrak{l}$. Let X_1 be a denumerable set disjoint from X_0. Sierpiński [3] shows that it is possible to find a family $\mathcal{F}$ of subsets of X_1 such that (i) $A, B \in \mathcal{F}$ implies $|A \cap B| < \aleph_0$, and (ii) $|\mathcal{F}| = \mathfrak{m}$. Let $\mathcal{S} = \{A \subseteq X_1 \colon X_1 - A \in \mathcal{F}\}$, and let $\mathfrak{T}_1$ denote the topology on X_1 generated by $\mathcal{S}$. It is not difficult to show that $T \in \mathfrak{T}_1$ implies T is a finite intersection of members of $\mathcal{S}$ intersected with a cofinite subset of X_1. Hence $|\mathfrak{T}_1| = |\mathcal{S}| = \mathfrak{m}$. From this we also see that $w(X_1, \mathfrak{T}_1) = \mathfrak{m}$. Now let $X = X_0 \cup X_1$ and let the topology $\mathfrak{T}$ on X be $\{A_0 \cup A_1 \colon A_0 \in \mathfrak{T}_0, A_1 \in \mathfrak{T}_1\}$.

Then

$$|\mathfrak{T}| = \max(|\mathfrak{T}_0|,\ |\mathfrak{T}_1|) = \max(\mathfrak{m}, \mathfrak{n}) = \mathfrak{n},$$

and

$$w(X, \mathfrak{T}) = \operatorname{Max}\big(w(X_0, \mathfrak{T}_0), w(X_1, \mathfrak{T}_1)\big) = \max(\mathfrak{l}, \mathfrak{m}) = \mathfrak{m}.$$

Hence $\mathfrak{m} \in \mathfrak{W}_{\mathfrak{n}}$, and the theorem is proved.

COROLLARY $\mathfrak{W}_{\mathfrak{c}} = \{\mathfrak{m}\colon \aleph_0 \leqslant \mathfrak{m} \leqslant \mathfrak{c}\}$.

Proof. If X is denumerable and $\mathfrak{T}$ is the discrete topology on X then $w(X, \mathfrak{T}) = \aleph_0$, $|\mathfrak{T}| = \mathfrak{c}$. The corollary then follows from Theorem 1.

DEFINITION. If $\mathcal{B}$ is a family of sets, let

$$\mathcal{B}^* = \{A\colon A \text{ is the union of a subset of } \mathcal{B}\}.$$

LEMMA [1]. *If $\mathcal{B}$ is a family of subsets of a denumerable set X and $\aleph_0 \leqslant |\mathcal{B}| < \min(\aleph_\omega, \mathfrak{c})$, then $|\mathcal{B}^*| = |\mathcal{B}|$ or $|\mathcal{B}^*| = \mathfrak{c}$.*

Proof. Let $|\mathcal{B}| = \aleph_n$, $n < \omega$. Then we can index members of $\mathcal{B}$ by ω_n, that is, $\mathcal{B} = \{\mathcal{B}_a\colon a < \omega_n\}$. We can conveniently express $\mathcal{B}^*$ by

$$\mathcal{B}^* = \bigcap_{x \in X} \bigcup_{a < \omega_n} \bigcap_{y \in X} \big\{A\colon x \in A \rightarrow \big(x \in B_a \mathbin{\&} (y \in B_a \rightarrow y \in A)\big)\big\}.$$

For $x \in X$, $a < \omega_n$ let

$$\mathcal{F}_{x,a]} = \bigcap_{y \in X} \big\{A\colon x \in A \rightarrow \big(x \in B_a \mathbin{\&} (y \in B_a \rightarrow y \in A)\big)\big\}.$$

Then $\mathcal{F}_{x,a}$ can be thought of, in a natural way, as a Borel subset of 2^X (see [1]), and hence $|\mathcal{F}_{x,a}|$ is either countable or equals $\mathfrak{c}$. If $n = 0$ it also follows that $\mathcal{B}^*$ is a Borel subset of 2^X and thus for $n = 0$ the lemma is valid. So assume $n \geqslant 1$. We have

$$\begin{aligned}
\mathcal{B}^* &= \bigcap_{x \in X} \bigcup_{a < \omega_n} \mathcal{F}_{x,a} \\
&= \bigcap_{x \in X} \bigcup_{a < \omega_n} \Big(\bigcup_{\beta \leqslant a} \mathcal{F}_{x,\beta}\Big) \\
&= \bigcup_{f \in \omega_n} X \bigcap_{x \in X} \Big(\bigcup_{\beta \leqslant f(x)} \mathcal{F}_{x,\beta}\Big) \\
&= \bigcup_{a < \omega_n} \bigcap_{x \in X} \Big(\bigcup_{\beta \leqslant a} \mathcal{F}_{x,\beta}\Big),
\end{aligned}$$

the last equality following from the fact that ω_n has uncountable cofinality. By induction on $|a|$ we can easily show that

$$\Big|\bigcap_{x \in X} \Big(\bigcup_{\beta \leqslant a} \mathcal{F}_{x,\beta}\Big)\Big| \leqslant |a| \quad \text{or equals } \mathfrak{c}.$$

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(1) The author is indebted to P. Erdös for posing the problem of the cardinality of $\mathcal{B}^*$.

If the latter possibility is the case for any $\alpha < \omega_n$, then $|\mathfrak{B}^*| = \mathfrak{c}$. If the latter is not the case for every $\alpha < \omega_n$, then $|\mathfrak{B}^*| \leqslant \sum_{\alpha<\omega_n} |\alpha| = \omega_n$. But since $|\mathfrak{B}^*| \geqslant |\mathfrak{B}| = \omega_n$ we must have $|\mathfrak{B}^*| = |\mathfrak{B}|$.

The next theorem shows that $\mathfrak{W}_\mathfrak{n}$ is quite different from $\mathfrak{W}_\mathfrak{c}$ for $\mathfrak{n} < \mathfrak{c}$.

THEOREM 2. (i) *For* $\mathfrak{n} < \mathfrak{c}$, *if* $\aleph_0 \leqslant \mathfrak{n} \leqslant \aleph_\omega$ *then* $\mathfrak{W}_\mathfrak{n} = \{\mathfrak{n}\}$.

(ii) *For* $\aleph_\omega < \mathfrak{n} < \mathfrak{c}$, $\mathfrak{m} \in \mathfrak{W}_\mathfrak{n}$ *implies* $\mathfrak{m} \geqslant \aleph_\omega$.

Proof. Let $\mathfrak{n}$ be as in (i). If $\aleph_0 \leqslant w(X, \mathfrak{T}) < \mathfrak{n}$, then by the lemma, $|\mathfrak{T}| = w(X, \mathfrak{T})$, hence $w(X, \mathfrak{T}) \notin \mathfrak{W}_\mathfrak{n}$. Part (ii) is also an immediate consequence of the lemma.

The results presented here are far from complete, so we state two closely related questions which seem to require other methods.

PROBLEM 1. For $\aleph_0 \leqslant \mathfrak{n} \leqslant \mathfrak{c}$, does $|\mathfrak{W}_\mathfrak{n}| \geqslant 2$ imply $\mathfrak{n} = \mathfrak{c}$?

PROBLEM 2. Is $\mathfrak{W}\aleph_{\omega+1} = \{\aleph_{\omega+1}\}$?

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Added in proof. R. McKenzie has just informed the author that the answer to Problem 1 is "yes".

References

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