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Scott sentences and a problem of Vaught for mono-unary algebras

by

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Abstract. First we show that there is a countable ordinal a such that if $\langle A, \{f\}\rangle$ is a mono-unary algebra then one can find a Scott sentence (which describes $\langle A, \{f\}\rangle$ up to isomorphism) whose rank is less than a. Combining this result with Morley's we see that if a sentence of $\mathfrak{L}_{\omega_1\omega}$ for mono-unary algebras has more than denumerably many isomorphism types of countable models then it must have continuum many of these isomorphism types.

We wish to show that for a given countable mono-unary algebra $\mathfrak{A}$ we can construct a reasonably simple *Scott Sentence* $\varphi_{\mathfrak{A}}$ in $\mathfrak{L}_{\omega_1\omega}$, i.e. for any countable mono-unary algebra $\mathfrak{B}$, $\mathfrak{B} \models \varphi_{\mathfrak{A}}$ iff $\mathfrak{B}$ is isomorphic to $\mathfrak{A}$. Then we apply the methods of Morley [1] to determine the possible number of isomorphism types which can be realized among the countable models of a $\mathfrak{L}_{\omega_1\omega}$ sentence for mono-unary algebras.

1. The Scott Sentence. In what follows we will always assume $\mathfrak{L}_{\omega_1\omega}$ involves one non-logical symbol, a unary operation symbol. Let $\mathfrak{L}$ be a subset of $\mathfrak{L}_{\omega_1\omega}$. Define $C_0(\mathfrak{L})$ to be the closure of $\mathfrak{L}$ under $\wedge$, $\vee$, $\neg$, $\exists$ and $\forall$; define $C_1(\mathfrak{L})$ to be $\mathfrak{L}$ union the set of formulas formed by taking the countable conjunction (or disjunction) of a set $\mathcal{F}$ of formulas in $\mathfrak{L}$, where the set of variables which occur free in members of $\mathcal{F}$ is finite.

Define a transfinite sequence $\mathfrak{L}_0 \subseteq \mathfrak{L}_1 \subseteq \ldots$ by the following inductive procedure:

$\mathfrak{L}_0$ is the usual first-order predicate calculus with one unary operation symbol,

$\mathfrak{L}_\xi = \bigcup_{\eta<\xi} \mathfrak{L}_\eta$ for limit ordinals ξ, $\xi < \omega_1$, and $\mathfrak{L}_{\xi+1} = C_0\, C_1(\mathfrak{L}_\xi)$ for $\xi < \omega_1$.

Then $\mathfrak{L}_{\omega_1\omega} = \bigcup_{\xi<\omega_1} \mathfrak{L}_\xi$.

THEOREM 1. *The isomorphism type of a countable mono-unary algebra $\mathfrak{A} = \langle A, \{f\}\rangle$ can be defined by a single sentence $\varphi_{\mathfrak{A}}$ in $\mathfrak{L}_{\omega+4}$.*

Proof. In the following we will introduce the notations which will be used to construct $\varphi_{\mathfrak{A}}$, and following each definition we will state its meaning as well as an $\mathfrak{L}_\xi$ to which it belongs. In much of what follows it will be helpful to visualize a mono-unary algebra $\mathfrak{A} = \langle A, \{f\}\rangle$ as a *directed graph* $\{\langle a, f(a)\rangle : a \in A\}$ (see Fig. 1). We will freely draw upon graph-theoretical terminology such as *predecessor*, *immediate predecessor*, *successor*, *component* and *loop*. Note that in the directed graph of a mono-unary algebra each component contains at most one loop.

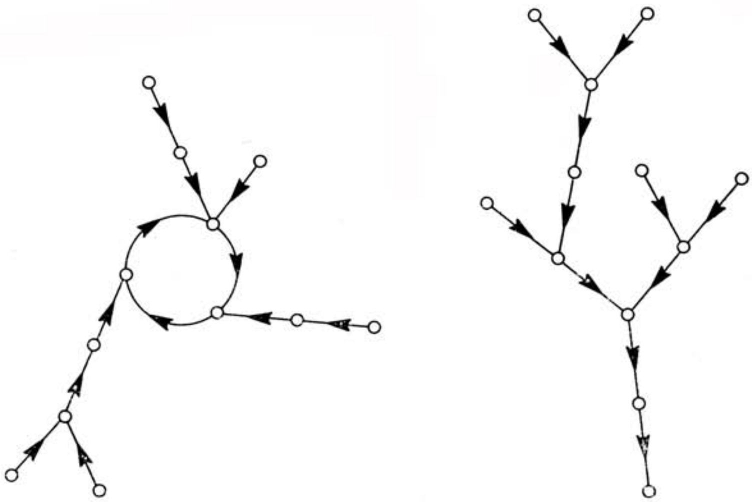

Fig. 1

(1) $$D(x_0, \ldots, x_n) = \bigwedge_{0 \leqslant i < j \leqslant n} (x_i \neq x_j) .$$

(This formula is in $\mathfrak{L}_0$ and expresses the predicate: $x_0, \ldots, x_n$ *are pairwise distinct.*)

(2) $$P(x_0, x_1) = D(x_0, x_1) \wedge (fx_0 = x_1) .$$

(This formula is in $\mathfrak{L}_0$ and says: x_0 *is an immediate predecessor of* x_1.)

(3) If $S(x_0)$ is any formula in $\mathfrak{L}_{\omega_1\omega}$ and $\alpha \leqslant \omega$, let

$$\exists^{\alpha}_{x_0} S(x_0) = \begin{cases} \neg \exists x_0 S(x_0) , & \text{if } \alpha = 0 ; \\ \exists x_0 \ldots \exists x_{\alpha-1} [D(x_0, \ldots, x_{\alpha-1}) \wedge S(x_0) \wedge \ldots \wedge S(x_{\alpha-1})] \wedge & \\ \wedge \neg \exists x_0 \ldots \exists x_{\alpha} [D(x_0, \ldots, x_{\alpha}) \wedge S(x_0) \wedge \ldots \wedge S(x_{\alpha})] , & \text{if } 1 \leqslant \alpha < \omega ; \\ \bigwedge_{1 \leqslant \beta < \omega} \exists x_0 \ldots \exists x_{\beta} [D(x_0, \ldots, x_{\beta}) \wedge S(x_0) \wedge \ldots \wedge S(x_{\beta})] , & \text{if } \alpha = \omega . \end{cases}$$

If $S(x_0)$ contains free variables other than x_0, then a suitable change of variables is employed to prevent them from becoming bound.

(This is in $\mathfrak{L}_{\beta+1}$ if $S(x_0)$ is in $\mathfrak{L}_{\beta}$ and says: *There are exactly* α x_0*'s such that* $S(x_0)$.)

(4) $$L(x_0) = \bigwedge_{1 \leqslant n < \omega} (f^n x_0 = x_0).$$

(This is in $\mathfrak{L}_1$ and says: x_0 *generates a loop.*)

(5) $$P^*(x_0, x_1) = P(x_0, x_1) \wedge \neg L(x_0) .$$

($P^*(x_0, x_1)$ is in $\mathfrak{L}_1$ and expresses: x_0 *immediately precedes* x_1 *and does not generate a loop.*)

(6) For $k < \omega$, $\psi \in \omega+1^k$ define

$$P^\psi(x_0) = \exists^{\psi(0)}_{x_1}[P^*(x_1, x_0) \wedge \exists^{\psi(1)}_{x_2}[P(x_2, x_1) \wedge \ldots \wedge \exists^{\psi(k-1)}_{x_k} P(x_k, x_{k-1})] \ldots] .$$

($P^\psi(x)$ is in $\mathfrak{L}_{k+1}$ and says: *There are exactly* $\psi(0)$ *immediate predecessors of* x_0 *which do not generate a loop, each of which has exactly* $\psi(1)$ *immediate predecessors, each of which ... each of which has exactly* $\psi(k-1)$ *immediate predecessors.*)

(7) Returning to our algebra $\mathfrak{A} = \langle A, \{f\}\rangle$, and focusing our attention on an element a in A, let

$$P_a(x_0) = \bigwedge \{P^\psi(x_0)\colon\ P^\psi(a) \text{ holds, where } \psi \in \omega+1^k,\ k < \omega\} .$$

($P_a(x_0)$ is in $\mathfrak{L}_{\omega+1}$ and *tells the structure of all predecessors of* a *which are not in a loop.*)

(8) Let $S_a(x_0)$ be whichever of the following formulas is true of a:

$$(f^m x_0 = f^{m+n} x_0) \wedge \bigwedge \{\neg(f^i x_0 = f^{i+j} x_0)\colon\ i \leqslant m,\ j \leqslant n,\ i+j < m+n\} ,$$

where $n, j \geqslant 1$, or

$$\bigwedge \{\neg(f^m x_0 = f^n x_0)\colon\ m, n < \omega,\ m \neq n\} .$$

($S_a(x_0) \in \mathfrak{L}_1$ and *describes the structure of the successors of* a.)

We remark that if $\mathfrak{B} = \langle B, \{f\}\rangle$ is a countable mono-unary algebra and $b \in B$, then $P_a(b)$ implies b has the same predecessor structure as a, discarding those points which generate a loop. Likewise $S_a(b)$ implies a and b have the same successor structure.

(9) $$K_a(x_0) = S_a(x_0) \wedge \bigwedge_{n<\omega} \{P_{f^n(a)}(f^n x_0)\} .$$

($K_a(x_0) \in \mathfrak{L}_{\omega+2}$ and *tells the structure of the component of* a *in* $\mathfrak{A}$.)

(10) $$DK(x_0, x_1) = \bigwedge_{m,n<\omega} \neg(f^m x_0 = f^n x_0) .$$

($DK(x_0, x_1) \in \mathfrak{L}_1$ and says: x_0 and x_1 *belong to distinct components.*)

(11) Let I_a be whichever of the following is true of a:

$$\exists x_0 \ldots \exists x_\alpha \big[\big(\bigwedge_{i\leqslant\alpha} K_a(x_i)\big) \wedge \big(\bigwedge_{0\leqslant i<j\leqslant\alpha} DK(x_i, x_j)\big)\big] \wedge$$
$$\wedge \neg \exists x_0 \ldots \exists x_{\alpha+1}\big[\big(\bigwedge_{i\leqslant\alpha+1} K_a(x_i)\big) \wedge \big(\bigwedge_{0\leqslant i<j\leqslant\alpha+1} DK(x_i, x_j)\big)\big] ,$$

where $\alpha < \omega$ or

$$\bigwedge_{\alpha<\omega} \exists x_0 \ldots \exists x_\alpha \big[\big(\bigwedge_{i\leqslant\alpha} K_a(x_i)\big) \wedge \big(\bigwedge_{0\leqslant i<j\leqslant\alpha} DK(x_i, x_j)\big)\big] .$$

($I_a \in \mathfrak{L}_{\omega+3}$. From I_a we can determine the number of components isomorphic to the component of a, as well as the structure of the component of a.)

Finally, to describe the Scott Sentence, let $\{a_\lambda\colon \lambda \in \Lambda\}$ be a subset of A such that it contains exactly one element from each component of $\mathfrak{A}$. Then the sentence:

$$\varphi_{\mathfrak{A}} = (\textstyle\bigwedge_{\lambda\in\Lambda} I_{a_\lambda}) \wedge \forall x_0 \exists x_1 [\neg DK(x_0, x_1) \wedge \textstyle\bigvee_{\lambda\in\Lambda} K_{a_\lambda}(x_1)] \tag{12}$$

is readily seen to completely describe the isomorphism type of $\mathfrak{A}$, and is in $\mathfrak{L}_{\omega+4}$.

2. The number of isomorphism types. The remainder of the paper is an adaptation of Morley [1]. Let $\mathfrak{L}$ be a subset of $\mathfrak{L}_{\omega_1\omega}$. Suppose $\mathfrak{L}$ is closed under $\boldsymbol{C}_0$, substitution of one variable for another (with a suitable renaming of bound variables to prevent a clash), and contains all subformulas of its members. Then if $\mathfrak{L}$ is countable we will say that it is *regular*. If T is a theory of mono-unary algebras consisting of a sentence from $\mathfrak{L}_{\omega_1\omega}$, and $\boldsymbol{K}$ is the class of models of T which are countable, then we will say T is *scattered* if, for every regular $\mathfrak{L} \subseteq \mathfrak{L}_{\omega_1\omega}$ and $n < \omega$, $S_n(\mathfrak{L}, \boldsymbol{K})$ is countable, where $S_n(\mathfrak{L}, K)$ denotes the set of n-types in $\mathfrak{L}$ realized by models in $\boldsymbol{K}$.

Assume that T is a scattered theory of mono-unary algebras, and $\boldsymbol{K}$ its class of countable models. Let $\mathfrak{L}_0^*$ be a regular language containing $\mathfrak{L}_0, \mathfrak{L}(x_0)$, $DK(x_0, x_1)$, $P^*(x_0, x_1)$, $P^\varphi(x_0)$ for all $\psi \in \omega+1^k$, $k < \omega$, and all possible $S_a(x_0)$ as described in (8).

Let $\mathfrak{A} = \langle A, \{f\}\rangle$ and $\mathfrak{B} = \langle B, \{f\}\rangle$ be two algebras in K, and let $a \in A$, $b \in B$. Returning to (7) one sees that either $P_a(x_0)$ is identical to $P_b(x_0)$, or $P_a(x_0) \wedge P_b(x_0)$ is always false. Since $P_a(x_0)$ is a conjunction of formulas in $\mathfrak{L}_0^*$, it follows that for some $\psi \in S_1(\mathfrak{L}_0^*, K)$, $\bigwedge \psi \to P_a(x_0)$, and if $P_a(x_0)$ is not identical to $P_b(x_0)$, then $\neg(\bigwedge \psi \to P_a(x_0))$. Since T is scattered it follows that there are only countably many formulas of the form $P_a(x_0)$, where $\mathfrak{A} = \langle A, \{f\}\rangle \in \boldsymbol{K}$ and $a \in A$. Let $\mathfrak{L}_1^*$ be a regular language containing $\mathfrak{L}_0^*$ and formulas of the form $P_a(x_0)$.

By an argument of the above style we can also conclude that there are only countably many formulas of the form $K_a(x_0)$. Let us denote them by $K_n(x_0)$, $n < a$, for a suitable $a \leqslant \omega$. Referring to (11) it is immediate that there are only countably many sentences of the form I_a. Let us introduce the notation $I_{i,j}$, $i < a$, $j \leqslant \omega$, where i refers to the isomorphism type described by $K_i(x_0)$, and j tells the number of components of this type. Let $\mathfrak{L}_2^*$ be a regular language containing $\mathfrak{L}_1^*$ and the $I_{i,j}$.

Let $\theta, \psi \in \omega+1^a$. If $\theta \neq \psi$ it is easy to verify that $\bigwedge_{i<a} I_{i,\theta(i)}$ and $\bigwedge_{i<\omega} I_{i,\psi(i)}$ are contradictory. Since $S_0(\mathfrak{L}_2^*, \boldsymbol{K})$ is countable, it will follow that there are only countably many $\theta \in \omega+1^a$ such that $\bigwedge_{i<a} I_{i,\theta(i)}$ is true of some model of T. Since the sentence $\bigwedge_{i<a} I_{i,\theta(i)}$ completely describes

the isomorphism type of a model in $\boldsymbol{K}$ which satisfies it, $\boldsymbol{K}$ has only countably many different isomorphism types.

THEOREM 2. *The number of isomorphism types of countable mono-unary algebras which satisfy a sentence of* $\mathfrak{L}_{\omega_1\omega}$ *is either countable or* 2^{ω}. (*This answers a problem of Vaught — in the case of mono-unary algebras* (see [3]).)

Proof. In [1] Morley proved everything stated except he allowed the possibility of ω_1 isomorphism types in a scattered theory, and we have just finished excluding this.

In conclusion we remark that all of the possible numbers of isomorphism types can be realized by a suitable theory of mono-unary algebras. Also, by some obvious modifications Theorem 2 is still true if we add a finite number of constants to our language (which already involves one unary operation).

References

[1] M. Morley, *The number of countable models*, Journ. Symb. Logic 35 (1970), pp. 14–18.

[2] D. Scott, *Logic with denumerably long formulas and finite strings of quantifiers*, Theory of Models, Amsterdam (1965), pp. 329–341.

[3] R. L. Vaught, *Denumerable Models of Complete Theories*, Proceedings of the Symposium in Foundations of Mathematics, Infinitistic Methods, New York 1961, pp. 303–321.

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