Algebra Universalis, 10 (1980) 333–344 Birkhäuser Verlag, Basel

Remarks on Boolean products

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In [9] Rosenstein showed that $GL_2(R)$ (the group of invertible 2×2-matrices with entries from R), for R a Boolean ring, is isomorphic to the Boolean power $GL_2(\mathbf{2})[X(R)]$,* $\mathbf{2}$ being the two-element Boolean ring and $X(R)$ being the Boolean space of maximal ideals of R. The proof used a considerable amount of computation, and was subsequently simplified, and generalized to $GL_n(R)$, by Gonshor [6].† In §1 we will prove results which are yet more general, and with methods which are quite straightforward. In §2 we formulate a generalization of Comer's study of the connection between factor congruences and sheaf representations. Using the results of §2 it is possible to show that (up to isomorphism) the Boolean products of an $\forall\exists$ class $\mathcal{K}$ of algebras with an encoding formula form an elementary class, and it is finitely axiomatizable if $\mathcal{K}$ is finitely axiomatizable and there are only finitely many fundamental operations–this improves on results obtained in [4], and gives a routine method for axiomatizing numerous model-complete theories of Boolean products studied by Comer, Macintyre and Weisspfennig (see [4]). Finally we apply this result to some classes of projective special linear groups.

1. Boolean product representations of linear groups

The operator $\mathbf{\Gamma}$ (the formation of *Boolean products*) was introduced in [4] as a simple alternative to sheaf constructions over Boolean spaces. For the convenience of the reader we will repeat the definition below. First, if $\mathcal{K}$ is a class of first-order structures and A is a subdirect product of members $\{A_i\}_{i\in I}$ from $\mathcal{K}$ then, for $\Phi(u_1,\ldots,u_m)$ a first-order formula and $f_1,\ldots,f_m\in A$ let

$$[\![\Phi(f_1,\ldots,f_m)]\!]=\{i\in I\mid A_i\models\Phi(f_1(i),\ldots,f_m(i))\}.$$

† We first learned of the results for $GL_n(R)$ from Sabbagh.

Presented by G. Grätzer. Received January 10, 1977. Accepted for publication in final form October 10, 1978.

We will often abbreviate $\Phi(f_1, \ldots, f_m)$ by $\Phi(\vec{f})$, etc. If $J \subseteq I$ and $f \in A$ let $f|_J$ be the restriction of f to J. For X a Boolean space define X^* to be the Boolean algebra of clopen subsets of X.

For $\mathcal{K}$ a class of first-order structures we say $A \in \Gamma(\mathcal{K})$ if

(**1**) A is a subdirect product of members $\{A_x\}_{x \in X}$ of $\mathcal{K}$, and X can be endowed with a Boolean space topology such that

(**2**) $[\![\Phi(f_1, \ldots, f_m)]\!]$ is an open subset of X for $f_1, \ldots, f_m \in A$ and Φ atomic, and

(**3**) (Patchwork Property) if $f, g \in A$ and $N \in X^*$ then $f|_N \cup g|_{X-N} \in A$.

If we are dealing only with algebras (i.e. no fundamental relations) then (**2**) can be replaced by

($\hat{\mathbf{2}}$) $[\![f = g]\!]$ is open for all $f, g \in A$.

If $A \in \Gamma(\mathcal{K})$ then we use $X(A)$, rather than X, to denote the underlying Boolean space. An embedding $\alpha : B \to \Pi_{x \in X} A_x$ gives a *Boolean product representation* if $\alpha(B)$ is a subdirect product of the $\{A_x\}_{x \in X}$ such that (**2**) and (**3**) hold. The simplest example of a Boolean product is a *bounded Boolean power* $B[X]^*$, the algebra of continuous functions from X to B, where X is a Boolean space and B is given the discrete topology (see [3]). The following elementary lemma is quite useful.

LEMMA 1.1. *If $A \in \Gamma(B)$, say $A \subseteq B^X$, and A contains the constant functions of B^X, then $A \cong B[X]^*$.*

Proof. For $b \in B$ let g_b be the constant function from X to B with value b. Then, for $f \in A$ and $b \in B$, $f^{-1}(b) = [\![f = g_b]\!]$, an open subset of X by (**2**), hence $f \in B[X]^*$. Conversely, if $f \in B[X]^*$ then there are $b_1, \ldots, b_n \in B$ for some $n < \omega$ and $N_1, \ldots, N_n \in X^*$ such that $f = g_{b_1|N_1} \cup \cdots \cup g_{b_n|N_n}$, so by (**3**) $f \in A$.

If R is a commutative ring with unity (we assume that our language has a constant symbol 1) let $M_n(R)$ be the *multiplicative monoid* of $n \times n$-*matrices* (r_{ij}) with entries from R. Then let $GL_n(R)$ be the *group of invertible matrices* in $M_n(R)$, let $SL_n(R)$ be the subgroup of $GL_n(R)$ consisting of those *matrices with determinant* 1, and if $Z(SL_n(R))$ denotes the center of $SL_n(R)$ *define* $PSL_n(R)$ *to be* $SL_n(R)/Z(SL_n(R))$. For $(r_{ij}) \in SL_n(R)$ we use $[(r_{ij})]$ to denote the coset $(r_{ij})/Z(SL_n(R))$ in $PSL_n(R)$.

THEOREM 1.2. *Let $\mathcal{K}$ be a class of commutative rings with unity and suppose*

$R\in\Gamma(\mathcal{K})$. *Then the following maps give Boolean product representations, for* $1\le n<\omega$:

$$\alpha: M_n(R)\to\prod_{x\in X(R)} M_n(R_x),\quad \textit{where}\quad \alpha(r_{ij})(x)=(r_{ij}(x)),$$

$$\beta: GL_n(R)\to\prod_{x\in X(R)} GL_n(R_x),\quad \textit{where}\quad \beta(r_{ij})(x)=(r_{ij}(x)),$$

$$\gamma: SL_n(R)\to\prod_{x\in X(R)} SL_n(R_x),\quad \textit{where}\quad \gamma(r_{ij})(x)=(r_{ij}(x))$$

and

$$\delta: PSL_n(R)\to\prod_{x\in X(R)} PSL_n(R_x),\quad \textit{where}\quad \delta[(r_{ij})](x)=[(r_{ij}(x))].$$

Proof. We will consider only the cases $GL_n(R)$ and $PSL_n(R)$, the others being handled in a similar fashion.

If $(r_{ij})\in M_n(R)$ then (r_{ij}) is invertible in $M_n(R)$ implies $(r_{ij}(x))$ is invertible in $M_n(R_x)$ for $x\in X(R)$. Hence $\beta_x: GL_n(R)\to GL_n(R_x)$ defined by $\beta_x(r_{ij})=(r_{ij}(x))$ is a (well-defined) homomorphism. We need to prove that each β_x is surjective, so suppose x is given and $(s_{ij}(x))\in GL_n(R_x)$ for some $(s_{ij})\in M_n(R)$. Let I be the identity matrix of $M_n(R)$, I_x the identity matrix of $M_n(R_x)$. Choose $(t_{ij})\in M_n(R)$ such that $(s_{ij}(x))(t_{ij}(x))=I_x$. Then

$$U=[\![(s_{ij})(t_{ij})=I]\!]=\bigcap_{1\le i,\,j\le n}\left[\!\!\left[\sum_{1\le k\le n} s_{ik}t_{kj}=\Delta_{ij}\right]\!\!\right],$$

an open subset of $X(R)$, where Δ_{ij} is the Kronecker delta function on R. Choose $N\in X(R)^*$ such that $x\in N\subseteq U$, and let $(\hat{s}_{ij})=(s_{ij})|_N\cup I|_{X(R)-N}$, and $(\hat{t}_{ij})=(t_{ij})_N\cup I|_{X(R)-N}$. Clearly $(\hat{s}_{ij})\cdot(\hat{t}_{ij})=I$, hence $(\hat{s}_{ij})\in GL_n(R)$ and $\beta_x(\hat{s}_{ij})=(s_{ij}(x))$. This gives property (**1**) for $\beta(GL_n(R))$. For (**2**) suppose (r_{ij}) and $(s_{ij})\in GL_n(R)$. Then

$$[\![\beta(r_{ij})=\beta(s_{ij})]\!]=\bigcap_{1\le i,\,j\le n}[\![r_{ij}=s_{ij}]\!],$$

an open subset of $X(R)$. And for (**3**), if $N\in X(R)^*$ then

$$\beta(r_{ij})|_N\cup\beta(s_{ij})|_{X(R)-N}=\beta(r_{ij}|_N\cup s_{ij}|_{X(R)-N})\in\beta(GL_n(R)).$$

Next, assuming we have proved the result for $SL_n(R)$, we have (r_{ij}) is in the center of $SL_n(R)$ iff each $(r_{ij}(x))$ is in the center of $SL_n(R_x)$ (this follows solely from the subdirect representation using γ). Hence δ_x from $PSL_n(R)$ to $PSL_n(R_x)$ defined by $\delta_x[(r_{ij})]=[(r_{ij}(x))]$ is a surjective homomorphism as γ_x is surjective and $Z(SL_n(R))\subseteq \gamma_x^{-1}Z(SL_n(R_x))$, so we have property (**1**) for $\delta(PSL_n(R))$. For (**2**) suppose $(r_{ij}), (s_{ij})\in SL_n(R)$. Then, in $\delta(PSL_n(R))$,

$$[\![\delta[(r_{ij})]=\delta[(s_{ij})]]\!]=\{x\in X(R)\mid (r_{ij}(x))(s_{ij}(x))^{-1}\in Z(SL_n(R_x))\}.$$

Now since for any $(t_{ij})\in SL_n(R)$, $(t_{ij}(x))\in Z(SL_n(R_x))$ iff $(t_{ij}(x))=a\cdot I_x$ for some $a\in R$, hence

$$\{x\in X(R)\mid (t_{ij}(x))\in Z(SL_n(R_x))\}=\bigcap_{i\neq j}[\![t_{ij}=0]\!]\cap[\![t_{ii}=t_{jj}]\!],$$

an open subset of $X(R)$. For (**3**) let $N\in X(R)^*$ and $(r_{ij}), (s_{ij})\in SL_n(R)$. Then

$$\delta[(r_{ij})]|_N\cup\delta[(s_{ij})]|_{X(R)-N}=\delta[(r_{ij})|_N\cup(s_{ij})|_{X(R)-N}].$$

COROLLARY 1.3. *For R any commutative ring with unity and X any Boolean space we have, for $1\leq n<\omega$,*

(a) $M_n(R[X]^*)\cong M_n(R)[X]^*$

(b) $GL_n(R[X]^*)\cong GL_n(R)[X]^*$,

(c) $SL_n(R[X]^*)\cong SL_n(R)[X]^*$

and

(d) $PSL_n(R[X]^*)\cong PSL_n(R)[X]^*$.

Proof. All are easy consequences of Lemma 1.1 and Theorem 1.2.

Remark 1. Letting **2** be the two-element ring we have from Corollary 1.3(b) $GL_n(\mathbf{2}[X]^*)\cong GL_n(\mathbf{2})[X]^*$, which is Gonshor's main theorem, and for $n=2$, Rosenstein's theorem, as mentioned in the introduction.

Remark 2. For p a prime number let $\mathbf{p}$ be the Galois field of order p. Then, from [1], if R is a commutative ring with unity satisfying $x^p=x$ one has, for some X, $R\cong\mathbf{p}[X]^*$, and thus $M_n(R)\cong M_n(\mathbf{p})[X]^*,\ldots, PSL_n(R)\cong PSL_n(\mathbf{p})[X]^*$.

For model-theoretic work we are usually more interested in the Boolean product operator $\mathbf{\Gamma}^a$ which is the same as $\mathbf{\Gamma}$ except (**2**) is replaced by

($\mathbf{2}^a$) $[\![\Phi(\vec{f})]\!]\in X^*$for $\vec{f}$ from A and Φ atomic.

Remark 3. If we replace Γ by Γ^a in Theorem 2.1 then the maps α, β, γ and δ give Boolean product representations satisfying (2^a).

2. Congruences and Boolean products

In [5] Comer showed the connection between Boolean lattices of factor congruences and Boolean product representations. However, for the purpose of giving first-order descriptions (see §3) it is important to be able to use only certain sublattices of this Boolean lattice.

Let A be a given algebra and $\mathrm{Con}(A)$ its lattice of congruences, with Δ being the least and ∇ the greatest members of $\mathrm{Con}(A)$. Suppose we are given a sublattice L of $\mathrm{Con}(A)$ such that

(i) $\Delta \in L$,
(ii) $\bigcup L = \nabla$,
(iii) The congruences of L permute (i.e. $\theta_1, \theta_2 \in L \Rightarrow \theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$)
and
(iv) L is a relatively complemented distributive lattice.

Let $X_L = \{\mathfrak{m} \mid \mathfrak{m}$ is a maximal ideal of $L\} \cup \{L\}$, and let T be the topology on X_L generated by $\mathfrak{B} = \{N_\theta \mid \theta \in L\} \cup \{D_\theta \mid \theta \in L\}$, where $N_\theta = \{\mathfrak{m} \in X_L \mid \theta \in \mathfrak{m}\}$ and $D_\theta = X_L - N_\theta$.

For $\mathfrak{m} \in X_L$ let $A_\mathfrak{m} = A/\bigcup \mathfrak{m}$.

THEOREM 2.1. *Given an algebra* A *and sublattice* L *of* $\mathrm{Con}(A)$ *satisfying* (i)–(iv) *above, the canonical map*

$$\alpha : A \to \prod_{\mathfrak{m} \in X_L} A_\mathfrak{m}$$

gives a Boolean product representation of A. $\alpha(A)$ *will satisfy* (2^a) *iff for each* $a, b \in A$ *there is at least* $\theta \in L$ *such that* $\langle a, b\rangle \in \theta$; *and if* ($2^a$) *holds then* $[\![\alpha a = \alpha b]\!] = N_\theta$ *where* θ *is the least congruence of* L *with* $\langle a, b\rangle \in \theta$.

Proof. That α is a Boolean product representation readily follows from Wolf's result using the lattice L' obtained from L by adjoining a unit to L and using X_l as a set of prime ideals of L'. The hull-kernel topology on X_L is the same as the topology generated by $\mathfrak{B}$ since $N_\theta = \bigcup\{D_{\varphi\setminus\theta} \mid \varphi \not\subseteq \theta\}$. For $a, b \in A$ it is clear that

$[\![\alpha a = \alpha b]\!] = \cup\{N_\theta \mid \langle a, b\rangle \in \theta \in L\}$, hence if $[\![\alpha a = \alpha b]\!]$ is clopen then by compactness there is a $\theta \in L$ such that $[\![\alpha a = \alpha b]\!] = N_\theta$, so θ is the smallest member of L containing $\langle a, b\rangle$; and conversely, if θ is the smallest member of L containing $\langle a, b\rangle$ then $[\![\alpha a = \alpha b]\!] = N_\theta$.

3. Encoding formulas and elementary classes of Boolean products

A class $\mathcal{K}$ of algebras is elementary if it is the class of all algebras satisfying some set of first-order sentences, and $\mathcal{K}$ is $\forall\exists$ if these sentences can be chosen as $\forall\exists$ sentences. In this section we will show that for certain $\forall\exists$ classes $\mathcal{K}$ of algebras the class $\mathbf{I\Gamma}^a(\mathcal{K})$ is elementary. The next lemma is a basic tool.

LEMMA 3.1. *Let $\Phi(\vec{w})$ be a formula of the form $\exists\vec{v}\Psi(\vec{v}, \vec{w})$ with Ψ a conjunct of atomic formulas. Then for $B \in \mathbf{\Gamma}^a(\mathcal{K})$ and $\vec{g}$ in B,*

(i) *$[\![\Phi(\vec{g})]\!]$ is an open subset of $X(B)$,*

and

(ii) *$B \models \Phi(\vec{g})$ iff $[\![\Phi(\vec{g})]\!] = X(B)$.*

Proof. (i) is immediate from $[\![\Phi(\vec{g})]\!] = \cup\{[\![\Psi(\vec{f}, \vec{g})]\!] \mid \vec{f}$ is in $B\}$. For (ii), the direction ($\Rightarrow$) is true because Φ is positive. For the converse suppose $[\![\Phi(\vec{g})]\!] = X(B)$. Then one can use the compactness of $X(B)$ along with the Patchwork Property to find a $\vec{f}$ such that $[\![\Psi(\vec{f}, \vec{g})]\!] = X(B)$. But then $B \models \Psi(\vec{f}, \vec{g})$, hence $B \models \Phi(\vec{g})$.

Define P_4^+ to be the set of all primitive positive formulas with 4 free variables, i.e. all formulas of the form $\exists\vec{v}\Phi(\vec{v}, u_1, \ldots, u_4)$ where Φ is a conjunct of atomic formulas. A formula $\varepsilon \in P_4^+$ is an *encoding formula* for a class $\mathcal{K}$ of algebras if $\mathcal{K} \models \varepsilon(u, u', v, v') \leftrightarrow (u = u' \rightarrow v = v')$. Given $\varepsilon \in P_4^+$ let $\mathcal{K}_\varepsilon$ be the largest class of algebras for which ε is an encoding formula (this class is never empty). Our first objective is to show that $\mathbf{I\Gamma}^a(\mathcal{K}_\varepsilon)$ is an elementary class. For $B \in \mathbf{\Gamma}^a(\mathcal{K}_\varepsilon)$ and $h, k \in B$ let $\theta^{hk} = \{\langle f, g\rangle \in B^2 \mid [\![h = k]\!] \subseteq [\![f = g]\!]\}$. One easily sees that θ^{hk} is a congruence on B, and the collection of all θ^{hk} permute and form a relatively complemented distributive sublattice of Con(B), they include Δ, and their union is ∇, so they satisfy **(i)–(iv)** of §2. For $A \in \mathcal{K}_\varepsilon$ one has, by definition, $A \models \varepsilon(u, u', v, v') \leftrightarrow (u = u' \rightarrow v = v')$, and hence, using Lemma 3.1, $B \models \varepsilon(h, k, f, g)$ iff $\langle f, g\rangle \in \theta^{hk}$.

Now let A be any algebra, and for $a, b \in A$ define the relation $\theta_{ab} = \{\langle c, d\rangle \in A^2 \mid A \models \varepsilon(a, b, c, d)\}$. Let us consider the following statements (it is obvious that the portions in quote marks can be put into first-order form), all of which are true of $\mathbf{I}\Gamma^a(\mathcal{K}_\varepsilon)$:

$\{\Phi_i\}_{i\in I}$ says "for all a, b, θ_{ab} is a congruence containing $\langle a, b\rangle$"

Ψ_1 says "$\langle a, b\rangle \in \theta_{cd} \Rightarrow \theta_{ab} \subseteq \theta_{cd}$"

Ψ_2 says "the set of θ_{ab} permute and form a relatively complemented distributive lattice which includes Δ."

If the language of our algebras has only finitely many fundamental operation symbols then we can replace $\{\Phi_i\}_{i\in I}$ by a single axiom Φ.

LEMMA 3.2. *For* $\varepsilon \in P_4^+$, $\mathbf{I}\Gamma^a(\mathcal{K}_\varepsilon)$ *is an elementary class axiomatized by* $\{\Phi_i\}_{i\in I} \cup \{\Psi_1, \Psi_2\}$.

Proof. We have just noted that $\mathbf{I}\Gamma^a(\mathcal{K}_\varepsilon)$ satisfies $\{\Phi_i\}_{i\in I} \cup \{\Psi_1, \Psi_2\}$. So let A be any algebra satisfying these axioms, and let $L = \{\theta_{ab} \mid a, b \in A\}$. Then from Theorem 2.1 the map

$$\alpha : A \to \prod_{\mathfrak{m}\in X_L} A_{\mathfrak{m}}$$

gives a Boolean product representation of A satisfying $(\mathbf{2}^a)$, so we only need to show that each $A_\mathfrak{m} \in \mathcal{K}_\varepsilon$. For $f, g, h, k \in A$ we have $A \models \varepsilon(f, g, h, k)$ iff $\theta_{hk} \subseteq \theta_{fg}$ (using Ψ_1), hence iff $[\![\alpha f = \alpha g]\!] \subseteq [\![\alpha h = \alpha k]\!]$ (by Theorem 2.1). So for $a, b, c, d \in A_\mathfrak{m}$ with $A_\mathfrak{m} \models a = b \to c = d$ use the Patchwork Property to find $f, g, h, k \in A$ with $[\![\alpha f = \alpha g]\!] \subseteq [\![\alpha h = \alpha k]\!]$, $\alpha f(\mathfrak{m}) = a$, $\alpha g(\mathfrak{m}) = b$, $\alpha h(\mathfrak{m}) = c$ and $\alpha k(\mathfrak{m}) = d$. Then $A \models \varepsilon(f, g, h, k)$, hence as ε is positive, $A_\mathfrak{m} \models \varepsilon(a, b, c, d)$. Conversely, given $a, b, c, d \in A_\mathfrak{m}$ with $A_\mathfrak{m} \models \varepsilon(a, b, c, d)$, first choose $f', g', h', k' \in A$ with $\alpha f'(\mathfrak{m}) = a, \ldots, \alpha k'(\mathfrak{m}) = d$. Then, as $[\![\varepsilon(\alpha f', \alpha g', \alpha h', \alpha k')]\!]$ is an open subset of X_L, choose $N \in X_L^*$ such that $\mathfrak{m} \in N \subseteq [\![\varepsilon(\alpha f', \alpha g', \alpha h', \alpha k')]\!]$ and then use the Patchwork Property to find $f, g, h, k \in A$ with

$$\alpha f|_N = \alpha f'|_N, \ldots, \alpha k|_N = \alpha k'|_N \quad \text{and} \quad \alpha f|_{X-N} = \alpha g|_{X-N} = \alpha h|_{X-N} = \alpha k|_{X-N}.$$

From the $\{\Psi_i\}_{i\in I}$ we know $A \models \varepsilon(f, f, f, f)$, hence $X_L = [\![\varepsilon(\alpha f, \alpha g, \alpha h, \alpha k)]\!]$, and then by Lemma 3.1, $A \models \varepsilon(f, g, h, k)$, so $[\![\alpha f = \alpha g]\!] \subseteq [\![\alpha h = \alpha k]\!]$, and this implies $A_\mathfrak{m} \models a = b \to c = d$.

LEMMA 3.3. *Suppose* $\varepsilon \in P_4^+$. *For every* $\forall\exists$ *sentence* Φ *one can (effectively) find an* $\forall\exists$ *positive sentence* $\hat{\Phi}$ *such that for* $A \in \mathcal{K}_\varepsilon$, $A \models \hat{\Phi}$ *iff* $A \models \Phi$ *or* $\#A = 1$.

LEMMA 3.5. *For $\varepsilon \in P_4^+$ and $A \in \Gamma^a(\mathcal{K}_\varepsilon)$ we have, for $h_i, k_i \in A$, $i \in I$,*

(a) $A \models \varepsilon(a, b, c, d)$ *iff* $[\![a = b]\!] \subseteq [\![c = d]\!]$

and

(b) $A \models \forall w \forall w' [\mathop{\&}_{i \in I} \varepsilon(h_i, k_i, w, w') \to w = w']$ *iff*

$$X(A) = \bigcup_{i \in I} [\![h_i = k_i]\!].$$

Proof. (a) is immediate from Lemma 3.1, and then (b) is an easy exercise with the Patchwork Property.

If Ψ is an $\forall\exists$ sentence in the form

$$\forall \vec{u} \exists \vec{v} \bigvee_{1 \le i \le n} \mathop{\&}_{j \in J_i} (p_{ij} = q_{ij})$$

and $\varepsilon \in P_4^+$ let Ψ^ε be:

$$\forall \vec{u} \exists z_1 \exists z_1' \cdots \exists z_n \exists z_n' \exists \vec{g}_1 \cdots \exists \vec{g}_n \{ \mathop{\&}_{1 \le i \le n} \mathop{\&}_{j \in J_i} \varepsilon(z_i, z_i', p_{ij}(u, \vec{g}_i), q_{ij}(u, \vec{g}_i))$$

$$\& \forall w \forall w' [\mathop{\&}_{1 \le i \le n} \varepsilon(z_i, z_i', w, w') \to w = w'] \}.$$

LEMMA 3.6. *For $\varepsilon \in P_4^+$ and Ψ an $\forall\exists$ positive sentence (in the above form) we have, for $A \in \Gamma^a(\mathcal{K}_\varepsilon)$, $[\![\Psi]\!] = X(A)$ iff $A \models \Psi^\varepsilon$.*

Proof. This is an obvious combination of Lemmas 3.4 and 3.5.

For $\varepsilon \in P_4^+$ let Λ be the sentence

$$\exists u \exists u' \forall v \forall v' \varepsilon(u, u', v, v').$$

LEMMA 3.7. *For $\varepsilon \in P_4^+$ and $A \in \Gamma^a(\mathcal{K}_\varepsilon)$, $A \models \Lambda$ iff $Z(A) \in X(A)^*$.*

Proof. ($\Rightarrow$) Choose $f, f' \in A$ such that $A \models \forall v \forall v' \varepsilon(f, f', v, v')$, hence $[\![\forall v \forall v' \varepsilon(f, f', v, v')]\!] = X(A)$, thus $[\![f = f']\!] = Z(A)$.

($\Leftarrow$) Choose f, f' such that $Z(A) = [\![f = f']\!]$. Then $X(A) = [\![\forall v \forall v' \varepsilon(f, f', v, v')]\!]$, so $A \models \Lambda$.

THEOREM 3.8. *Let $\mathcal{K}$ be an $\forall\exists$ class axiomatized by $\{\Sigma_j\}_{j\in J}$ and suppose ε is an encoding formula for $\mathcal{K}$. Then* $\mathbf{I\Gamma}^a(\mathcal{K})$ *is an elementary class axiomatized by*

(a) $\{\Phi_i\}_{i\in I} \cup \{\Psi_1, \Psi_2\} \cup \{(\hat{\Sigma}_j)^\varepsilon\}_{j\in J}$ *if there is a one-element algebra in $\mathcal{K}$,*

or

(b) $\{\Phi_i\}_{i\in I} \cup \{\Psi_1, \Psi_2, \Lambda\} \cup \{(\hat{\Sigma}_j)^\varepsilon\}_{j\in J}$ *otherwise.*

If the language of $\mathcal{K}$ has finitely many operation symbols then $\mathbf{I\Gamma}^a(\mathcal{K})$ *is finitely axiomatizable iff $\mathcal{K}$ is finitely axiomatizable.*

Proof. Just apply Lemmas 3.2, 3.3, 3.6 and 3.7 for (a), (b). For the last claim about finite axiomatizability note that $\mathcal{K}$ is definable in $\mathbf{I\Gamma}^a(\mathcal{K})$ by the statement

$$\varepsilon(u, u', v, v') \leftrightarrow (u = u' \rightarrow v = v').$$

Now let us look at some classes $\mathcal{K}$ which have an encoding formula.

(a) A formula $\tau \in P_4^+$ is a *discriminator formula* for $\mathcal{K}$ if $\mathcal{K} \models \tau(u, u', v, v') \leftrightarrow (u = u' \;\&\; v = v')$ or $(u \neq u' \;\&\; u = v')$. In [4] we proved that if τ is a discriminator formula for $\mathcal{K}$ then $\mathbf{I\Gamma}^a(\mathcal{K})$ is an elementary class, provided $\mathcal{K}$ is an elementary class. Our axiomatization of $\mathbf{I\Gamma}^a(\mathcal{K})$ included axioms for $\mathbf{ISP_R}(\mathcal{K})$, hence we were unable to conclude that $\mathbf{I\Gamma}^a(\mathcal{K})$ is finitely axiomatizable if there are only finitely many fundamental operations and $\mathcal{K}$ is finitely axiomatizable. However now we are in a position to make this conclusion, for letting $\varepsilon(u, u', v, v')$ be $\exists w[\tau(u, u', v, w') \;\&\; \tau(u, u', v', w)]$ we have an encoding formula for $\mathcal{K}$, so Theorem 3.8 applies. In our study of model companions in [4] we were particularly interested in showing that $\mathbf{I\Gamma}_0^a(\mathcal{K}) = \mathbf{I}\{A \in \mathbf{\Gamma}^a(\mathcal{K}) \mid X(A)$ has no isolated points$\}$ is an elementary class. Since $\mathbf{I\Gamma}_0^a(\mathcal{K})$ is definable in $\mathbf{I\Gamma}^a(\mathcal{K})$ by "there are no atoms θ_{ab} in L," it is clearly also an elementary class, and finitely axiomatizable if there are only finitely many fundamental operations and $\mathcal{K}$ is finitely axiomatizable.

(b) A formula $\pi \in P_4^+$ is a *principal congruence formula* if it is of the form

$$\exists \vec{v}[u_1 = p_0(u_{\sigma_0(0)}, \vec{v}) \;\&\; u_2 = p_n(u_{\sigma_n(1)}, \vec{v}) \;\&\; \mathop{\&}_{0\le i\le n-1} p_i(u_{\sigma_i(1)}, \vec{v})$$
$$= p_{i+1}(u_{\sigma_{i+1}(0)}, \vec{v})]$$

where $\{\sigma_i(0), \sigma_i(1)\} = \{3, 4\}$ for $0 \le i \le n$, and $p_0, \ldots, p_n$ are polynomials. π is a *principal congruence formula for $\mathcal{K}$* if, in addition, for $A \in \mathcal{K}$ and $a, b, c, d \in A$,

$A \models \pi(a, b, c, d)$ iff $\langle a, b\rangle \in \theta(c, d)$. If π is a principal congruence formula for $\mathcal{K}$ and $\mathcal{K}$ is a class of simple algebras then trivially $\varepsilon(u, u', v, v') = \pi(v, v', u, u')$ is an encoding formula for $\mathcal{K}$. The following lemma gives a useful test for the existence of a principal congruence formula.

LEMMA 3.9. *Let $\mathcal{S}$ be a finite set of finite simple algebras such that*

(i) $\#\operatorname{Con}(S \times S') = 4$ for $S, S' \in \mathcal{S}$,

and

(ii) $\operatorname{Con}(S_1 \times \cdots \times S_n)$ *is a modular lattice for*

$$S_1, \ldots, S_n \in \mathcal{S}, \qquad n < \omega.$$

Then there is a principal congruence formula π for $\mathcal{S}$.

Proof. Let us suppose that $\mathcal{S} = \{S_1, \ldots, S_k\}$. Then choose elements $f, g, h, k \in (\prod_{1\le i\le k} S_i)^m$, where $m = (\#S_1 \times \cdots \times \#S_k)^4$, such that for $1 \le i \le k$, $\{\langle f(j)(i), g(j)(i), h(j)(i), k(j)(i)\rangle \mid 0 \le j \le m\}$ is precisely the set $\{\langle a, b, c, d\rangle \in S_i^4 \mid c = d \to a = b\}$. The assumptions (i) and (ii) guarantee that $\#\operatorname{Con}[(\prod_{1\le i\le k} S_i)^m] = 2^{km}$ (see [2]), hence, as $h(j)(i) = k(j)(i) \to f(j)(i) = g(j)(i)$, $\langle f, g\rangle \in \theta(h, k)$, so there is a principal congruence formula π such that $\pi(f, g, h, k)$ holds, and this must be a principal congruence formula for $\mathcal{S}$.

THEOREM 3.10. *If $\mathcal{S}$ is a finite set of finite simple algebras satisfying* (i)–(ii) *in Lemma* 3.9 *then $\mathbf{I\Gamma}^a(\mathcal{S})$ is an elementary class which is finitely axiomatizable if there are only finitely many fundamental operations.*

Proof. Just combine Theorem 3.8 with Lemma 3.9.

LEMMA 3.11. *If A is a finite algebra then $\mathbf{I\Gamma}^a(A) = \mathbf{I}\{A[X]^* \mid X$ is a Boolean space$\}$.*

Proof. Suppose A has universe $\{a_1 \ldots, a_n\}$. Select finitely many fundamental operations $\mathbf{f}_0, \ldots, \mathbf{f}_k$ such that every bijection α of A perserving $\mathbf{f}_0, \ldots, \mathbf{f}_k$ is actually an automorphism of A. Then let $\Delta(u_1, \ldots, u_n)$ be the set of those atomic or negated atomic formulas $\Phi(u_1, \ldots, u_n)$, each involving at most one extra-logical constant selected from $\mathbf{f}_0, \ldots, \mathbf{f}_k$, and such that $A \models \Phi(a_1, \ldots, a_n)$. Then, for $B \in \mathbf{\Gamma}^a(A)$ we have $X(B) = [\![\exists \vec{u}\ \&\ \Delta(\vec{u})]\!]$, so by Lemma 3.1 there are $f_1, \ldots, f_n \in B$ with $B \models \&\ \Delta(f_1, \ldots, f_n)$. Let g_i be the constant map in $A[X(B)]^*$ with value a_i, $1 \le i \le n$. Then for $N_1, \ldots, N_n \in X(B)^*$ with $X(B) = \bigcup_{1\le i\le n} N_i$ and $N_i \cap N_j = \emptyset$ for $i < j$ define $\alpha : (f_1\restriction_{N_1} \cup \cdots \cup f_n\restriction_{N_n}) \to (g_1\restriction_{N_1} \cup \cdots \cup g_n\restriction_{N_n})$. α is the desired isomorphism from B to $A[X(B)]^*$.

COROLLARY 3.12. *Let $\mathfrak{R}$ be an elementary class of rings with unity satisfying $x^p = x$ for p a prime. Then, for $\langle n, p\rangle \notin \{\langle 2, 2\rangle, \langle 2, 3\rangle\}$, $\mathbf{I}(\{PSL_n(R) \mid R \in \mathfrak{R}\})$ is an elementary class of groups which is finitely axiomatizable if $\mathfrak{R}$ is finitely axiomatizable.*

Proof. For $\langle n, p\rangle \notin \{\langle 2, 2\rangle, \langle 2, 3\rangle\}$ the group $PSL_n(\mathbf{p})$ is a simple non-abelian group (see [7]). By Remark 2 of §1 and Corollary 1.3 $\mathbf{I}(\{PSL_n(R) \mid R \models x^p = x\}) = \mathbf{I}(\{PSL_n(\mathbf{p})[X]^* \mid X$ is a Boolean space$\})$, and by Lemma 3.11 this is just $\mathbf{I}\Gamma^a(PSL_n(\mathbf{p}))$, a finitely axiomatizable elementary class by Theorem 3.10. Using an encoding formula ε for $PSL_n(\mathbf{p})$ one can interpret X^* in $PSL_n(\mathbf{p})[X]^*$ (using equivalence classes of ordered pairs). So if Φ is an axiom of $\mathfrak{R}$ first apply an Ershov translation (see [3] and [4]) to obtain a sentence Φ^* in the language of Boolean algebras such that $\mathbf{p}[X]^* \models \Phi$ iff $X^* \models \Phi^*$. Then, relativizing Φ^* to the interpretation of X^* in $PSL_n(\mathbf{p})[X]^*$ one obtains a sentence Φ_ε such that $PSL_n(\mathbf{p})[X]^* \models \Phi_\varepsilon$ iff $\mathbf{p}[X]^* \models \Phi$. Thus $\mathbf{I}(\{PSL_n(R) \mid R \in \mathfrak{R}\})$ is definable in $\mathbf{I}\Gamma^a(PSL_n(\mathbf{p}))$.

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