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Remarks on reducts of varieties

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Stanley Burris

There are a number of interesting classes of algebras which, although not themselves varieties, can be advantageously studied as reducts of varieties. In § 1 we look at strict Horn classes and reducts of varieties, with special attention to the possibility of finding constructive methods of answering some fundamental questions. This leads to § 2 where we examine when a quasi-variety is actually a variety. Finally in § 3 certain strict Horn classes are shown to be homomorphism-preserving reducts of varieties. The examples presented in this paper are not new, but the approach via Skolemization, etc. offers a unifying perspective.

§ 1. Strict Horn Classes

A class **K** of algebras is a *reduct* of a class **K′** of algebras (or **K′** is an *expansion* of **K**) if **K** is obtained from **K′** by forgetting some of the fundamental operations. For example Abelian groups form a reduct of the class of rings. Unfortunately a reduct of a variety may not even be an elementary class (Kogalovskiĭ [10]). Indeed, the situation is even worse, as we shall see in Theorem 2. The negative results on algorithms (such as this one) are based on Lemma 1. If Π is a finite set of semigroup relations with an unsolvable word problem and $\cdot$ is the semigroup operation let Π_{ab} be the set of equations $\Pi\cup\{x\cdot(y\cdot z)=(x\cdot y)\cdot z,\ x\oplus a=x,\ x\oplus b=y\oplus b\}$ (in the language with two binary operations $\cdot$ and $\oplus$, and with nullary operations consisting of the generators of the semigroup belonging to Π), where a and b are words in the original semigroup.

Lemma 1.[1]) *There is no algorithm to determine if* $\Pi_{ab}\vdash x=y$.

[1]) An excellent survey of such decidability questions is given in McNulty [13].

Proof. It is not difficult to see that $\Pi_{ab} \vdash x=y$ iff $\Pi \vdash a=b$, and the latter is an undecidable question. □

Theorem 2. *There is no algorithm by which one could determine from any finite set of equations Σ and any reduct of the variety defined by Σ whether or not this reduct is an elementary class.*

Proof. Let Σ_R be a finite set of equations defining rings, and form $\Sigma = \Pi_{ab} \cup \Sigma_R$ (we assume that the fundamental operations symbols of Π_{ab} and Σ_R are distinct). Let $\mathbf{K}$ be the reduct of the variety defined by Σ obtained by dropping all the ring operations except multiplication. If $\Sigma \nvdash x=y$ then any infinite multiplication semigroup which is a reduct of a ring will also be a reduct of an algebra in $\mathbf{K}$ as every ring R can be arbitrarily juxtaposed on any model of Π_{ab} of the same cardinality to obtain a model of Σ (because the operation symbols in Π_{ab} and Σ_R are distinct). However, the class of multiplicative semigroups which are reducts of rings do not form an elementary class (see [10]), so there is an infinite semigroup S which is not a reduct of any ring, but it is elementarily equivalent to a reduct of a ring. By Shelah [17] there is an index set I and ultrafilter $\mathscr{U}$ such that $S^I/\mathscr{U}$ is a reduct of a ring (note that the multiplicative semigroup reducts of rings are closed under ultrapowers). Choose a model A of Π_{ab} of the same cardinality as S, and juxtapose S arbitrarily on A to form B. Then $B^I/\mathscr{U} \in \mathbf{K}$. Now $B \notin \mathbf{K}$, hence $\mathbf{K}$ cannot be an elementary class. But then clearly $\mathbf{K}$ will be an elementary class iff $\Sigma \vdash x=y$, and the latter obviously holds iff $\Pi_{ab} \vdash x=y$, and by Lemma 1 there is no algorithm for this last question. □

If, nonetheless, we restrict our attention to those reducts of varieties which do form elementary classes we have a rather satisfying result. First recall (see [5]) that a *strict Horn sentence* (in prenex form) looks like

$$Q_1 x_1 \dots Q_n x_n \{\mathop{\&}_j [(\mathop{\&}_i p_{ij} = q_{ij}) \to p_j = q_j]\},$$

where each Q_i is either $\forall$ or $\exists$. If there is exactly one j-index and all the quantifiers are universal it is called a *quasi-identity* (see [11]).

Theorem 3. *For an elementary class* $\mathbf{K}$ *of algebras the following are equivalent:*[2])

(a) $\mathbf{K}$ *is the reduct of some variety,*

(b) $\mathbf{K}$ *can be axiomatized by strict Horn sentences,*

(c) $\mathbf{K}$ *is closed under reduced products (including the empty product).*

Proof. The equivalence of (*b*) and (*c*) is exercise 6.2.8 of [5]. If (*a*) holds then clearly (*c*) holds (we do not need the fact that $\mathbf{K}$ is elementary for this implication). So suppose (*b*) holds, and let Σ be a set of strict Horn sentences axiomatizing $\mathbf{K}$.

[2]) Our proof of $(b) \Rightarrow (a)$ is essentially that of T. Evans [6]. See McKenzie's fascinating paper [12] for techniques based on the discriminator function.

By Skolemizing the sentences in Σ we obtain Σ^*, a set of strict universal Horn sentences such that **K** is a reduct of the models of Σ^*. Without loss of generality we can replace Σ^* by a set Σ^{**} of quasi-identities. Expand the language of Σ^{**} by one new $(2n+1)$-ary function $f_n(x_1, y_1, \ldots, x_n, y_n, u)$ for each $n \geqq 1$, and let Σ^{***} be the set of identities

$$f_n(z_1, z_1, z_2, z_2, \ldots, z_n, z_n, u) = u \quad (n \geqq 1)$$

plus all identities

$$f_n(p_1, q_1, \ldots, p_n, q_n, p) = f_n(p_1, q_1, \ldots, p_n, q_n, q)$$

where $p_1 = q_1 \& \ldots \& p_n = q_n \rightarrow p = q$ is in Σ^{**}. Now one only needs to verify that reducts of algebras in Σ^{***} obtained by dropping the f_n's actually satisfy Σ^{**} (which is rather evident), and that any model of Σ^{**} can be expanded to a model of Σ^{***} (say by defining $f_n(a_1, a_1, \ldots, a_n, a_n, b) = b$, and otherwise $f_n(a, \ldots) = a$). Thus **K** is a reduct of the variety defined by Σ^{***}. (If Σ were finite then we would only require finitely many f_n's, hence Σ^{***} would also be finite.) □

An easy application of this general result is Grätzer's characterization of the spectra of varieties[3]). The *spectrum* of a variety is the set of cardinalities of the finite algebras in the variety.

Corollary 4 (Grätzer [8]). *Let S be a subset of the natural numbers. Then S is the spectrum of a variety iff $1 \in S$ and S is closed under multiplication.*

Proof. ($\Leftarrow$) Let **K** be the class of sets which are either infinite or whose cardinality is in S. Then **K** is an elementary class closed under reduced products, hence **K** is a reduct of some variety **V**. But then S is the spectrum of **V**. □

§ 2. Quasi-Varieties

A quasi-variety (see [11]) is an elementary class which can be axiomatized by quasi-identities (see § 1). If we start with a set Σ of strict Horn sentences and Skolemize we (essentially) obtain a set Σ^{**} of quasi-identities (this is of course the notation used in the proof of Theorem 3). In many cases Σ^{**} already defines a variety – if so this gives a more natural variety of which the class defined by Σ is a reduct. A non-trivial example of this is the class **K** of pseudo-complemented semilattices with zero $(S, \wedge, 0)$ axiomatized by

(S1) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

(S2) $x \wedge y = y \wedge x$

(S3) $x \wedge x = x$

(S4) $\forall x \exists y \forall z [x \wedge y = 0 \ \& \ (x \wedge z = 0 \rightarrow z \wedge y = z)]$.

[3]) The usual proof is based on the study of primal clusters.

Skolemizing we obtain (S1)–(S3) plus

(S4′) $x \wedge x^* = 0$

(S5′) $x \wedge z = 0 \rightarrow z \wedge x^* = z$,

a quasi-variety which is indeed a variety. This is not an obvious fact; it was first pointed out in Balbes and Horn [2]. (See Sankappanavar [16] for a historical discussion.) We now look at the problem of determining when a quasi-variety is a variety. (One of the most interesting results in this direction is Ol'šanskiĭ's [15] characterization of finite groups G such that the quasi-variety generated by G is a variety.)

Theorem 5. *There is no algorithm to determine if a finite set of quasi-identities defines a variety.*[4])

Proof. For Π as before let Σ be $\Pi_{ab} \cup \{x+y=y+x \rightarrow x=y\}$. If $\Sigma \nvdash x=y$ let A be a model of Π_{ab} which is not simple, and choose distinct elements a_0, a_1 of A such that for some congruence θ, $[a_0]_\theta=[a_1]_\theta$ and A/θ has at least two elements. Let $\leqq$ well-order A and define the operation $+$ on A by $x+y=a_0$ if $x \leqq y$, $x+y=a_1$ if $y<x$. The resulting algebra A' is a model of Σ, and θ is a congruence of A'. As $A'/\theta \models x+y=y+x$ but $A'/\theta \not\models x=y$ it follows that Σ does not define a variety. Consequently we see that Σ will define a variety iff $\Sigma \vdash x=y$, hence iff $\Pi_{ab} \vdash x=y$, and again by Lemma 1 this is undecidable. □

In spite of Theorem 5 we can still find some useful positive results. From basic facts in universal algebra (see [9]) we know that any quasi-variety **Q** contains the free algebras of the variety generated by **Q**. If Σ is a finite set of quasi-identities let $\tilde{\Sigma}$ be the subset of quasi-identities which are not already identities, and let $F(\tilde{\Sigma})$ be the free algebra in the quasi-variety defined by Σ which is freely generated by $n(\tilde{\Sigma})$ elements, $n(\tilde{\Sigma})$ being the maximum number of variables in a quasi-identity from $\tilde{\Sigma}$.

Theorem 6. *There is an algorithm from which one can determine from any finite set Σ of quasi-identities for which $F(\tilde{\Sigma})$ is a finite algebra if Σ determines a variety. This algoritm does not give an erroneous result when starting from a Σ such that $F(\tilde{\Sigma})$ is infinite, but maybe it does not terminate.*

Proof. A universal sentence Φ with n quantifiers will hold in a variety **V** iff it holds in the n-generated algebras of the variety. Thus to see if Σ defines a variety it suffices to construct $F(\tilde{\Sigma})$ and see if all the homomorphic images of $F(\tilde{\Sigma})$ satisfy $\tilde{\Sigma}$. If so then Σ defines a variety, otherwise not. [We should point out that if we know $F(\tilde{\Sigma})$ is finite, then, without even being given a finite bound on the size of $F(\tilde{\Sigma})$, we can still construct $F(\tilde{\Sigma})$ by enumerating the equational consequences $p(x_1, \ldots, x_n)=q(x_1, \ldots, x_n)$ of Σ, where $n=n(\tilde{\Sigma})$, and checking

[4]) Actually a deeper result is proved in McNulty [14] for universal Horn theories.

down the list until we find a collection of polynomials $p_1, \ldots, p_l$ containing $x_1, \ldots, x_n$ and such that, modulo Σ, it is closed under the fundamental operations. This will give a finite algebra A which, after factoring by the smallest congruence θ such that $A/\theta \models \Sigma$, yields $F(\tilde{\Sigma})$.] □

Let us apply this method to pseudo-complemented semilattices (with zero). Letting Σ be (S1), (S2), (S3), (S4′), (S5′) we have $n(\tilde{\Sigma})=2$. As every meet semilattice S with zero can be embedded in the distributive lattice with zero of non-empty lower segments of S (a lower segment of S is a subset L such that $x \in S, y \in L$ imply $x \wedge y \in L$) we can apply the prime ideal theorem of Stone–Birkhoff (see [9]) to obtain, for any given $a, b \in S$ with $a \neq b$, a homomorphism of S onto the two-element meet semilattice with zero. This is easily checked to preserve the Skolem operation * of (S4′), (S5′). Hence the quasi-variety defined by Σ is generated by a two-element algebra A_0. But then $F(\tilde{\Sigma})$ must be a subalgebra of A_0^4, so from this point it is routine to find $F(\tilde{\Sigma})$ and verify that all of its quotients satisfy $\tilde{\Sigma}$.

One aspect of the previous paragraph does not appear to be constructive, namely knowing to use the Stone–Birkhoff result to show that $F(\tilde{\Sigma})$ is finite. The next result shows that one must resort to special methods to show that a given $F(\tilde{\Sigma})$ is finite.

Theorem 7. *There is no algorithm by which one could determine from a finite set of quasi-identities Σ whether or not $F(\tilde{\Sigma})$ is finite.*

Proof. Let Π be as previously defined and let Σ be the quasivariety defined by $\Pi \cup \{a=b \to x=y\}$, where we assume the language of Σ has an additional binary operation $+$ (which does not appear in Σ). Then $F(\tilde{\Sigma})$ is finite iff $\Pi \vdash a=b$, and this is undecidable.

§ 3. Homomorphism-Preserving Reducts of Varieties

Let us look at two examples of strict Horn classes where Skolemizing immediately yields a variety. **RCDL** is the class of *relatively complemented distributive lattices* $(L, \vee, \wedge)$. **RCDL** is axiomatized by the usual equations Σ_D for distributive lattices plus the strict Horn sentence

$$\forall x \forall y \forall z \exists w [p(x, y, z) \vee w = x \vee z \;\;\&\;\; p(x, y, z) \wedge w = x \wedge z]$$

where $p(x, y, z) = (y \vee (x \wedge z)) \wedge (x \vee z)$. (This sentence says that, for L a distributive lattice and elements $a, b, c \in L$, the element b projected into the interval $[a \wedge c, a \vee c]$ has a relative complement in the interval.) Using a new ternary operation symbol t we can Skolemize this sentence to obtain the variety Σ^{**} defined by Σ_D plus the equations $p(x, y, z) \vee t(x, y, z) = x \vee z$, $p(x, y, z) \wedge t(x, y, z) = x \wedge z$. (A closely related system is given in Balbes and Dwinger [1]; the one above is referred to in Burris and Werner [4].) The second example is the class **CBR** of *commutative* *bi-regular rings* with unity $(R, +, \cdot, -, 0, 1)$ axiomatized by the usual commutative ring equations Σ_{CR} plus the strict Horn sentence $\forall x \exists y\, [y^2=y \& x=xy \& x(1-y)=0]$ (which says that the principal ideal generated by x is also generated by an idempotent element y). Skolemizing we obtain a variety $\mathbf{CBR}^{**}$ of algebras $(R, +, \cdot, -, 0, 1, ')$ axiomatized by Σ_{CR} plus $(x')^2=x'$, $x=xx'$, and $x(1-x')=0$.

In both of these examples the variety created by Skolemizing the strict Horn sentence has the same homomorphisms as the original class. Before stating a general theorem which will prove this assertion let us look at a closely related example where Skolemizing destroys some of the homomorphisms. Let Σ_{BL} be a set of equations defining bounded lattices $(L, \vee, \wedge, 0, 1)$, and let Σ be Σ_{BL} plus the sentence $\forall x \exists y (x \wedge y=0 \& x \vee y=1)$. Thus Σ axiomatizes the (strict Horn) class of complemented bounded lattices **CBL**. Suppose **CBL** is a reduct of a variety **V** (not necessarily the obvious one obtained by Skolemizing). Then we will show that there are algebras $A, B \in \mathbf{CBL}$ such that for any expansions A', B' into **V** there will be a homomorphism $\alpha\colon A \to B$ such that $\alpha\colon A' \to B'$ is not a homomorphism. Namely let A be the four-element bounded lattice $(\{0, a, b, 1\}, \vee, \wedge, 0, 1)$ with $a \vee b=1$, $a \wedge b=0$, and let B be the non-modular five-element bounded lattice $(\{0, a, b_1, b_2, 1\}, \vee, \wedge, 0, 1)$ where $b_1 < b_2$. Then $\alpha_1\colon A \to B$ defined by $\alpha_1(0)=0$, $\alpha_1(1)=1$, $\alpha_1(a)=a$ and $\alpha_1(b)=b_1$ is an embedding, as well as $\alpha_2\colon A \to B$ defined by $\alpha_2(x)=\alpha_1(x)$ if $x \neq b$, $\alpha_2(b)=b_2$. As $\{0, a, 1\}$ is not a subuniverse of A' (note that $(\{0, a, 1\}, \vee, \wedge, 0, 1) \notin \mathbf{CBL}$) it follows that there is a polynomial $p(x, y, z)$ such that $A' \models p(0, a, 1)=b$. Now $p(\alpha_1 0, \alpha_1 a, \alpha_1 1)=p(\alpha_2 0, \alpha_2 a, \alpha_2 1)$, but $\alpha_1 b \neq \alpha_2 b$, hence either $\alpha_1\colon A' \to B'$ or $\alpha_2\colon A' \to B'$ is not a homomorphism. With this example in mind perhaps the hypotheses of the following theorem will seem rather natural. The symbol $\exists !$ denotes "there exists a unique".

Theorem 8. *Let* **K** *be a class of algebras defined by a set* Σ *of positive first-order sentences in prenex form, i.e. by sentences of the form* $\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \Phi(\vec{x}, \vec{y})$, *where* $\Phi(\vec{x}, \vec{y})$ *is positive and quantifier-free. Let* **U** *be the universal class defined by the set* Σ^* *of universal sentences obtained by Skolemizing* Σ. *Then the following are equivalent:*

(i) *every algebra in* **K** *is the reduct of a unique algebra in* **U**;

(ii) *if* $A, B \in \mathbf{K}$ *and* $\alpha\colon A \to B$ *is a homomorphism then* $\alpha\colon A' \to B'$, *where* A', B' *are expansions of* A *into* **U**, *is also a homomorphism*;

(iii) *for each sentence* $\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \Phi(\vec{x}, \vec{y})$ *in* Σ *we have* $\Sigma \vdash \forall x_1 \exists ! y_1 \ldots \ldots \forall x_n \exists ! y_n \Phi(\vec{x}, \vec{y})$.

Proof. *(ii)⇒(iii)*. Suppose that for some sentence $\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \Phi(\vec{x}, \vec{y})$ in Σ we have $\Sigma \nvdash \forall x_1 \exists ! y_1 \ldots \forall x_n \exists ! y_n \Phi(\vec{x}, \vec{y})$. Then there is a model A of Σ such that $A \not\models \forall x_1 \exists ! y_1 \ldots \forall x_n \exists ! y_n \Phi(\vec{x}, \vec{y})$, hence there are two distinct choices $\vec{f}$ and $\vec{g}$ of Skolem functions of $\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \Phi(\vec{x}, \vec{y})$ for A. Thus $A' \models \forall \vec{x} \Phi(\vec{x}, \vec{f}(\vec{x}))$ and $A'' \models \forall \vec{x} \Phi(\vec{x}, \vec{g}(\vec{x}))$, where A' is the expansion of A by $\vec{f}$ and A'' is the expansion of A by $\vec{g}$. Since $f_i \neq g_i$ for some i, choose $\vec{a}$ from

A such that $f_i(\vec{a}) \neq g_i(\vec{a})$. Then with α the identity map on A we have $\alpha f_i(\vec{a}) \neq$ $\neq g_i(\alpha\vec{a})$, so $\alpha\colon A' \to A''$ is not a homomorphism, so (ii) fails if (iii) fails.

(i)⇒(iii). Use the A' and A'' above to show (i) fails if (iii) fails.

(iii)⇒(i). If (iii) holds then the Skolem functions for each model of Σ are unique for each $A \in \mathbf{K}$, so (i) holds.

(iii)⇒(ii). Suppose $A, B \in \mathbf{K}$ and $\alpha\colon A \to B$ is a homomorphism. Let $\forall x_1 \exists y_1 \dots$ $\dots \forall x_n \exists y_n \Phi(\vec{x}, \vec{y}) \in \Sigma$ and let $\vec{f}_A$, respectively $\vec{f}_B$, be the unique Skolemizing functions of this sentence on A, respectively B. For $a_1, \dots, a_n \in A$, $A \models \Phi(\vec{a}, \vec{f}_A(\vec{a}))$, hence $B \models \Phi(\alpha\vec{a}, \alpha\vec{f}_A(\vec{a}))$ as positive formulas are preserved by homomorphisms. But $B \models \Phi(\alpha\vec{a}, \vec{f}_B(\alpha\vec{a}))$, so from the uniqueness properties, $\alpha\vec{f}_A(\vec{a}) = \vec{f}_B(\alpha\vec{a})$, hence α preserves the Skolem functions. □

Remark. Conditions (i) and (ii) together state that the forgetful functor from $\mathbf{U}$ to $\mathbf{K}$ is an isomorphism.

Remark. In Theorem 8 the class $\mathbf{U}$ is a variety if the sentences in Σ are of the form $\forall x_1 \exists y_1 \dots \forall x_n \exists y_n \Phi(\vec{x}, \vec{y})$ where Φ is a conjunction of atomic formulas. In this case $\mathbf{U}$ is defined by equations Σ^{**}.

It is easy to prove that our axiom system for **RCDL** has the desired uniqueness properties. Hence the only subdirectly irreducible member of **RCDL**** is the two-element algebra $(2, \vee, \wedge, t)$, where $2 = \{0, 1\}$, and this algebra satisfies

$$t(x, y, z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{if } x = y. \end{cases}$$

This defines the remarkable *ternary discriminator function* which, by the theorem of Keimel and Werner (see [4]), guarantees that every algebra in **RCDL**** is isomorphic to a *Boolean product* (see [4]) of one- and two-element algebras. Results in Burris [3] furthermore say that every algebra in **RCDL**** is isomorphic to the algebra of all continuous functions f from a Boolean space X to $(2, \vee, \wedge, t)$ such that for some fixed pair of elements $x_i \in X$, $f(x_i) = i$, $i = 0, 1$. (This gives Feinstein's [7] characterization of members of **RCDL** as the sublattices of Boolean lattices obtained by intersecting one maximal ideal with one maximal filter.)

Likewise we can apply Theorem 8 to the axioms for **CBR** to show that the reducts of subdirectly irreducible members of **CBR**** are precisely the fields. Hence for R' a subdirectly irreducible member of **CBR**** the function $(x-y)' \cdot x +$ $+ [1-(x-y)'] \cdot z$ is the ternary discriminator on R', so applying the Bulman–Fleming and Werner (see [4]) results on discriminator varieties it follows that every commutative biregular ring is a Boolean product of fields.

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Department of Pure Mathematics,
Faculty of Mathematics,
University of Waterloo,
Waterloo, Ontario, Canada,
N2L 3G1

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