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SUBDIRECT REPRESENTATIONS IN AXIOMATIC CLASSES

BY

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In his paper [1] Birkhoff proved that every algebra in an equationally defined class is isomorphic to a subdirect product of subdirectly irreducible algebras from that class, and hence the subdirectly irreducible algebras are key building blocks. In a recent paper of Sabidussi [3] a detailed proof of the theorem of B. Fawcett that every graph is isomorphic to a subdirect product of subdirectly irreducible graphs is given. Our purpose here* is to give an affirmative answer to a question of Sabidussi as to whether Fawcett's theorem is a special case of a more general formulation of Birkhoff's results.

Most of our notation and definitions are taken from Grätzer [2]. A *type* τ is a pair of sequences $\langle\langle n_\gamma\rangle_{\gamma<\alpha}, \langle m_\gamma\rangle_{\gamma<\beta}\rangle$, and a *structure* $\mathfrak{A}$ of type τ is a triple $\langle S; \mathscr{F}, \mathscr{R}\rangle$, where S is the *universe* of $\mathfrak{A}$, $\mathscr{F}$ is a family of functions f_γ on S, $\gamma<\alpha$, the rank of f_γ being n_γ, and $\mathscr{R}$ is a family of relations r_γ on S, $\gamma<\beta$, the rank of r_γ being m_γ. For $\gamma<\alpha$ we have *fundamental operation symbols* $\mathbf{f}_\gamma$ and, similarly, for $\gamma<\beta$ *fundamental relation symbols* $\mathbf{r}_\gamma$, which are used to construct the first-order language $L(\tau)$. A *substructure* of $\langle S; \mathscr{F}, \mathscr{R}\rangle$ is a structure $\langle S'; \mathscr{F}', \mathscr{R}'\rangle$, where S' is a subset of S closed under the operations of $\mathscr{F}$, $\mathscr{F}'$ is the set of operations in $\mathscr{F}$ relativized to S', and $\mathscr{R}'$ is the set of relations in $\mathscr{R}$ relativized to S'.

The *direct product* of the structures $\langle S_i; \mathscr{F}_i, \mathscr{R}_i\rangle$ of type τ, $i\in I$, is the structure whose universe is $\prod_{i\in I} S_i$, with

$$f_\gamma(a_0, \ldots, a_{n_\gamma-1})(i) = f_\gamma\big(a_0(i), \ldots, a_{n_\gamma-1}(i)\big), \qquad i\in I,$$

and $r_\gamma(a_0, \ldots, a_{m_\gamma-1})$ holding iff $r_\gamma\big(a_0(i), \ldots, a_{m_\gamma-1}(i)\big)$ holds for all $i\in I$. The direct product is denoted by

$$\prod_{i\in I}\langle S_i; \mathscr{F}_i, \mathscr{R}_i\rangle.$$

A *subdirect product* of $\langle S_i; \mathscr{F}_i, \mathscr{R}_i\rangle$, $i\in I$, is a substructure $\langle S'; \mathscr{F}', \mathscr{R}'\rangle$ of the direct product such that $\pi_i(S') = S_i$, where π_i is the projection

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map $\prod_{j\in I} S_j \to S_i$. A mapping $\lambda\colon\ S_0 \to S_1$ is a *homomorphism* from $\langle S_0; \mathscr{F}_0, \mathscr{R}_0\rangle$ to $\langle S_1; \mathscr{F}_1, \mathscr{R}_1\rangle$ if

$$\lambda f_\gamma(a_0, \ldots, a_{n_\gamma-1}) = f_\gamma(\lambda a_0, \ldots, \lambda a_{n_\gamma-1}) \quad \text{for } a_0, \ldots, a_{n_\gamma-1}\in S_0,\ \gamma < \alpha,$$

and $r_\gamma(a_0, \ldots, a_{m_\gamma-1})$ holds implies $r_\gamma(\lambda a_0, \ldots, \lambda a_{m_\gamma-1})$ holds, where $a_0, \ldots, a_{m_\gamma-1}\in S_0$, and $\gamma < \beta$. The *image* of $\langle S_0; \mathscr{F}_0, \mathscr{R}_0\rangle$ under λ is $\langle \lambda(S_0); \mathscr{F}_2, \mathscr{R}_2\rangle$, where $\mathscr{F}_2$ is the set of restrictions of members of $\mathscr{F}_1$ to λS_0 and, for $b_0, \ldots, b_{m_\gamma-1}\in\lambda(S_0)$, $r_\gamma(b_0, \ldots, b_{m_\gamma-1})$ holds iff $r_\gamma(a_0, \ldots, a_{m_\gamma-1})$ holds for some $a_i\in\lambda^{-1}(b_i)$, $0\leqslant i\leqslant m_\gamma-1$, with $\gamma<\beta$. Note that the image need not be a substructure of $\langle S_1; \mathscr{F}_1, \mathscr{R}_1\rangle$. A structure $\langle S_1; \mathscr{F}_1, \mathscr{R}_1\rangle$ is a *homomorphic image* of $\langle S_0; \mathscr{F}_0, \mathscr{R}_0\rangle$ if $\langle S_1; \mathscr{F}_1, \mathscr{R}_1\rangle$ is the image of $\langle S_0; \mathscr{F}_0, \mathscr{R}_0\rangle$ under some homomorphism. A *congruence* of $\langle S; \mathscr{F}, \mathscr{R}\rangle$ is an equivalence relation θ on S such that if $\langle a_i, b_i\rangle\in\theta$, $0\leqslant i\leqslant n_\gamma-1$, then

$$\langle f_\gamma(a_0, \ldots, a_{n_\gamma-1}), f_\gamma(b_0, \ldots, b_{n_\gamma-1})\rangle\in\theta.$$

If θ is a congruence of $\mathfrak{A} = \langle A; \mathscr{F}, \mathscr{R}\rangle$, then $\mathfrak{A}/\theta$ will denote the *quotient* whose universe is A/θ and where

$$f_\gamma([a_0]_\theta, \ldots, [a_{n_\gamma-1}]_\theta) = [f_\gamma(a_0, \ldots, a_{n_\gamma-1})]_\theta,$$

$[a]_\theta$ being the equivalence class of a modulo θ, and $r_\gamma([a_0]_\theta, \ldots, [a_{m_\gamma-1}]_\theta)$ iff $r_\gamma(b_0, \ldots, b_{m_\gamma-1})$ for some $b_i\in[a_i]_\theta$, $0\leqslant i< m_\gamma$.

Let $\boldsymbol{K}$ be a class of structures of type τ. We relativize our concepts to $\boldsymbol{K}$ as follows. A homomorphism λ from $\mathfrak{A}_0$ to $\mathfrak{A}_1$, where $\mathfrak{A}_0, \mathfrak{A}_1\in\boldsymbol{K}$, is a ***K**-homomorphism* if the image of $\mathfrak{A}_0$ under λ is in $\boldsymbol{K}$. If λ is also one-one, then we speak simply of an *isomorphism*. A congruence θ of $\mathfrak{A}\in\boldsymbol{K}$ is a ***K**-congruence* if it is the kernel of a $\boldsymbol{K}$-homomorphism. $\Delta(\mathfrak{A})$ is the *diagonal relation* on S; $\Delta(\mathfrak{A})$ is always a $\boldsymbol{K}$-congruence.

A subdirect product $\langle S'; \mathscr{F}', \mathscr{R}'\rangle$ of $\langle S_i; \mathscr{F}_i, \mathscr{R}_i\rangle$, $i\in I$, is *full* (1) if the image of $\langle S'; \mathscr{F}', \mathscr{R}'\rangle$ under π_i is $\langle S_i; \mathscr{F}_i, \mathscr{R}_i\rangle$ for each i. If $\mathfrak{A}\in\boldsymbol{K}$ has a non-empty universe, and, for every isomorphism

$$\varepsilon\colon\ \mathfrak{A}\to\prod_{i\in I}\mathfrak{A}_i, \qquad \mathfrak{A}_i\in\boldsymbol{K},$$

such that the image of $\mathfrak{A}$ is a full subdirect product of the $\mathfrak{A}_i$, $\mathfrak{A}_i$ is an isomorphic image of $\mathfrak{A}$ under $\pi_i\circ\varepsilon$ for some i, then $\mathfrak{A}$ is said to be ***K**-subdirectly irreducible*. Note that if $\boldsymbol{K}$ is an equationally defined class of algebras, then the $\boldsymbol{K}$-subdirectly irreducible algebras are the subdirectly irreducible algebras in $\boldsymbol{K}$. In [3] Sabidussi gives an explicit description of all $\boldsymbol{K}$-subdirectly irreducible structures where $\boldsymbol{K}$ is the class of graphs.

(1) Sabidussi calls a full subdirect product simply a subdirect product in the case of graphs. However, this does not agree with the conventions we have adopted, namely those of [2].

LEMMA 1. *$\mathfrak{A} = \langle S; \mathscr{F}, \mathscr{R}\rangle$ is $\boldsymbol{K}$-subdirectly irreducible iff $\mathfrak{A} \in \boldsymbol{K}$ and either S has only one element, or, for some $a, b \in S$, $a \neq b$, the only $\boldsymbol{K}$-congruence θ such that $\langle a, b\rangle \notin \theta$ is $\Delta(\mathfrak{A})$, or, for some $r_\gamma \in \mathscr{R}$ and some $a_0, \ldots, a_{m_\gamma - 1} \in S$, the only $\boldsymbol{K}$-congruence θ such that $\neg r_\gamma([a_0]_\theta, \ldots, [a_{m_\gamma-1}]_\theta)$ is $\Delta(\mathfrak{A})$.*

Proof. First suppose $\mathfrak{A} = \langle S; \mathscr{F}, \mathscr{R}\rangle$ is not $\boldsymbol{K}$-subdirectly irreducible. Then, for some isomorphism

$$\varepsilon\colon \mathfrak{A} \to \prod_{i \in I} \mathfrak{A}_i,$$

where the image $\mathfrak{A}$ is a full subdirect product of the $\mathfrak{A}_i$, $\mathfrak{A}_i$ is not, for any i, an isomorphic image of $\mathfrak{A}$ under $\pi_i \circ \varepsilon$. Hence, if S is non-empty, then S has more than one element. Suppose $a, b \in S$ and $a \neq b$. Then, for some i,

$$\pi_i \circ \varepsilon(a) \neq \pi_i \circ \varepsilon(b),$$

whence $\langle a, b\rangle \notin \mathrm{Ker}(\pi_i \circ \varepsilon)$. Note that, since $\varepsilon(\mathfrak{A})$ is full, $\mathrm{Ker}(\pi_i \circ \varepsilon)$ is a $\boldsymbol{K}$-congruence, and it is not $\Delta(\mathfrak{A})$. Also, if $r_\gamma \in \mathscr{R}$ and $a_0, \ldots, a_{m_\gamma-1} \in S$ with $\neg r_\gamma(a_0, \ldots, a_{m_\gamma-1})$, then, for some i, we must have

$$\neg r_\gamma\big(\pi_i \circ \varepsilon(a_0), \ldots, \pi_i \circ \varepsilon(a_{m_\gamma-1})\big),$$

since $\varepsilon(\mathfrak{A})$ is a subdirect product, whence

$$\neg r_\gamma([a_0]_\theta, \ldots, [a_{m_\gamma-1}]_\theta), \qquad \text{where } \theta = \mathrm{Ker}(\pi_i \circ \varepsilon).$$

For the converse, suppose $\mathfrak{A}$ is $\boldsymbol{K}$-subdirectly irreducible. If S has more than one element and only one $\boldsymbol{K}$-congruence, namely $\Delta(\mathfrak{A})$, then the proof is trivial; so suppose $\mathfrak{A}$ has at least two $\boldsymbol{K}$-congruences and let ν be the canonical homomorphism from $\mathfrak{A}$ into $\prod_{\theta \neq \Delta(\mathfrak{A})} \mathfrak{A}/\theta$, where each θ is a $\boldsymbol{K}$-congruence. Since $\mathfrak{A}$ is subdirectly irreducible, it follows that either ν is not injective or $\nu(\mathfrak{A})$ is not a substructure. If ν is not injective, then, for some $a, b \in S$, $a \neq b$, we have $\langle a, b\rangle \in \theta$ for every $\boldsymbol{K}$-congruence θ except $\Delta(\mathfrak{A})$, and if $\nu(\mathfrak{A})$ is not a substructure, then, for some $r_\gamma \in \mathscr{R}$, $a_0, \ldots, a_{m_\gamma-1} \in S$, we have $r_\gamma([a_0]_\theta, \ldots, [a_{m_\gamma-1}]_\theta)$ for every $\boldsymbol{K}$-congruence $\theta \neq \Delta(\mathfrak{A})$, but $\neg r_\gamma(a_0, \ldots, a_{m_\gamma-1})$.

A family of sets is *inductive* if it is closed under unions of chains.

LEMMA 2. *Let $\mathfrak{A} \in \boldsymbol{K}$ have an inductive set of $\boldsymbol{K}$-congruences. Then $\mathfrak{A}$ is isomorphic to a full subdirect product of $\boldsymbol{K}$-subdirectly irreducible structures.*

Proof. Let $\mathfrak{A} = \langle S; \mathscr{F}, \mathscr{R}\rangle$ and suppose $a, b \in S$, $a \neq b$. Then, by Zorn's Lemma, there is a maximal $\boldsymbol{K}$-congruence θ of $\mathfrak{A}$ with respect to the property that $\langle a, b\rangle \notin \theta$. Using Lemma 1 we see that $\mathfrak{A}/\theta$ is subdirectly irreducible. Also, for each $r_\gamma \in \mathscr{R}$ and $a_0, \ldots, a_{m_\gamma-1} \in S$ with $\neg r_\gamma(a_0, \ldots, a_{m_\gamma-1})$, there is a maximal $\boldsymbol{K}$-congruence θ with respect to the property

$\neg r_\gamma([a_0]_\theta, \ldots, [a_{m_\gamma-1}]_\theta)$, and again $\mathfrak{A}/\theta$ is subdirectly irreducible. Thus the canonical map from $\mathfrak{A}$ to

$$\prod\{\mathfrak{A}/\theta\colon \mathfrak{A}/\theta \text{ is } \boldsymbol{K}\text{-subdirectly irreducible}\}$$

is such that the image of $\mathfrak{A}$ is a full subdirect product which is isomorphic to $\mathfrak{A}$.

A class $\boldsymbol{K}$ of algebras of type τ is *universal* if it is the class of models of a set of universal sentences from $L(\tau)$.

In general, the set of $\boldsymbol{K}$-congruences of a structure $\mathfrak{A}$ in a universal class $\boldsymbol{K}$ form neither a meet semilattice nor a join semilattice as the following example shows:

Let $\boldsymbol{K}$ be the class of structures with three unary predicates P_0, P_1 and P_2 axiomatized by the universal (Horn) sentence

$$\forall \boldsymbol{x}\big(\neg \boldsymbol{P}_0(\boldsymbol{x}) \vee \neg \boldsymbol{P}_1(\boldsymbol{x}) \vee \boldsymbol{P}_2(\boldsymbol{x})\big),$$

and let $\mathfrak{A} = \langle\{0, 1, 2, 3, 4\}, P_0, P_1, P_2\rangle$ with $P_0(1)$, $P_1(2)$, $P_2(3)$, $P_2(4)$, and $\neg P_i(x)$ otherwise. Then $\mathfrak{A} \in \boldsymbol{K}$ and the two $\boldsymbol{K}$-congruences whose equivalence classes are given by $\big\{\{0, 1, 2, 3\}, \{4\}\big\}$ and $\big\{\{0, 1, 2, 4\}, \{3\}\big\}$ do not have a g.l.b. among the $\boldsymbol{K}$-congruences although the $\boldsymbol{K}$-congruences corresponding to the partitions $\big\{\{0, 1\}, \{2\}, \{3\}, \{4\}\big\}$ and $\big\{\{0, 2\}, \{1\}, \{3\}, \{4\}\big\}$ are lower bounds.

LEMMA 3. *Let $\boldsymbol{K}$ be a universal class of structures of type τ. Then, for $\mathfrak{A} \in \boldsymbol{K}$, the set of $\boldsymbol{K}$-congruences of $\mathfrak{A}$ is inductive.*

Proof. Let θ_i, $i \in I$, be a chain of $\boldsymbol{K}$-congruences of $\mathfrak{A} = \langle S; \mathscr{F}, \mathscr{R}\rangle$, and let Γ be a set of universal sentences defining $\boldsymbol{K}$. By a well-known reduction we can assume that every sentence in Γ is of the form

$$\forall x_0 \ldots \forall x_n \bigvee_{i=0}^{k} \sigma_i,$$

where each σ_i is either an atomic formula or the negation of an atomic formula in $L(\tau)$. Let

$$\theta = \bigvee_{i \in I} \theta_i.$$

Then θ is a congruence of $\mathfrak{A}$; we will show that $\mathfrak{A}/\theta \in \boldsymbol{K}$, whence θ is a $\boldsymbol{K}$-congruence. So let

$$\sigma = \forall x_0 \ldots \forall x_n \bigvee_{i=0}^{k} \sigma_i,$$

a member of Γ in prenex form with each σ_i either an atomic formula or the negation of an atomic formula in $L(\tau)$. If this sentence fails to be true in $\mathfrak{A}/\theta$, then, for some $a_0, \ldots, a_n \in S$,

$$\bigvee_{i=0}^{k} \sigma_i([a_0]_\theta, \ldots, [a_n]_\theta)$$

is false in $\mathfrak{A}/\theta$, whence $\sigma_i([a_0]_\theta, \ldots, [a_n]_\theta)$ is false in $\mathfrak{A}/\theta$ for all i. If σ_i is atomic for a given i, this would imply $\sigma_i([a_0]_{\theta_k}, \ldots, [a_n]_{\theta_k})$ is false in $\mathfrak{A}/\theta_k$ for all $k \in I$, and if σ_i is the negation of an atomic formula, then $\sigma_i([a_0]_{\theta_k}, \ldots, [a_n]_{\theta_k})$ would be false for some $k \in I$. Among the latter cases only finitely many i are involved, and hence there is a $k_0 \in I$ such that

$$\sigma_i([a_0]_{\theta_{k_0}}, \ldots, [a_n]_{\theta_{k_0}})$$

is false in $\mathfrak{A}/\theta_{k_0}$ for all i; but then σ fails to hold in $\mathfrak{A}/\theta_{k_0}$, a contradiction.

Combining the lemmas we have proved the following

THEOREM 1. *Let **K** be a universal class of structures. Then every structure in **K** is isomorphic to a full subdirect product of **K**-subdirectly irreducible structures.*

Of course, if we want a universal class $\boldsymbol{K}$ to be closed under subdirect products, then we need a universal Horn class (for example, the class of graphs). To indicate that we have a nearly best possible result for axiomatic theories we will consider two examples, the first being the class $\boldsymbol{K}_D$ of dense linear orders without end points $\langle S, < \rangle$ axiomatized by

$$\forall x \forall y (x < y \text{ or } y < x \text{ or } x = y), \qquad \forall x \forall y \forall z (x < y \;\&\; y < z \to x < z),$$

$$\forall x \forall y (x < y \to \neg y < x \;\&\; \neg x = y),$$

$$\forall x \forall y \exists z [x \neq y \to (x < z < y \text{ or } y < z < x)],$$

$$\forall x \exists y \exists z (x < y \;\&\; z < x).$$

The only countable model (up to isomorphism) is the rationals $\mathfrak{Q} = \langle Q, < \rangle$, and it is easy to check that, given $q_0, q_1 \in Q$ with $q_0 \neq q_1$, there is a $\boldsymbol{K}_D$-congruence $\theta \neq \Delta(\mathfrak{Q})$ such that $\langle q_0, q_1 \rangle \notin \theta$ and, for $q_0, q_1 \in Q$ with $\neg(q_0 < q_1)$, there is a $\boldsymbol{K}_D$-congruence $\theta \neq \Delta(\mathfrak{Q})$ with $\neg([q_0]_\theta < [q_1]_\theta)$. Hence $\mathfrak{Q}$ is not $\boldsymbol{K}_D$-subdirectly irreducible, and so we do not have a generalization of Birkhoff's theorem for $\boldsymbol{K}_D$. Note that this is a finitely axiomatized $\forall\exists$-theory of relational structures.

Second consider the class $\boldsymbol{K}_P$ of structures $\langle S, P \rangle$, where P is a unary predicate satisfying $\{x \in S\colon P(x)\}$ is infinite and $\{x \in S\colon \neg P(x)\}$ is also infinite. Again we can argue that there is only one countable model and it is not $\boldsymbol{K}_P$-subdirectly irreducible. This example is an infinitely axiomatized $\exists$-theory of relational structures.

We remark that for any class $\boldsymbol{K}$ the finite structures in $\boldsymbol{K}$ are isomorphic to full subdirect products of $\boldsymbol{K}$-subdirectly irreducible structures by Lemma 2.

A class $\boldsymbol{K}$ of structures of type τ is *existential* if it is the class of models of some set of existential sentences in $L(\tau)$.

THEOREM 2. *If **K** is a finitely axiomatizable existential class of relational structures, then every $\mathfrak{A} \in$ **K** is isomorphic to a full subdirect product of **K**-subdirectly irreducible structures.*

Proof. Let $\mathfrak{A} = \langle S, \mathscr{R}\rangle \in \boldsymbol{K}$ and suppose $a, b \in S$, $a \neq b$. Then, since only finitely many existential sentences are needed to axiomatize $\boldsymbol{K}$, it follows that there is a $\boldsymbol{K}$-congruence θ of finite index such that $[a]_\theta \neq [b]_\theta$, and hence there is a maximal $\boldsymbol{K}$-congruence θ of $\mathfrak{A}$ such that $[a]_\theta \neq [b]_\theta$. Likewise, for $r_\gamma \in \mathscr{R}$ and $a_0, \ldots, a_{m_\gamma - 1} \in S$ with $\neg r_\gamma(a_0, \ldots, a_{m_\gamma-1})$, there is a maximal $\boldsymbol{K}$-congruence θ with respect to the property $\neg r_\gamma([a_0]_\theta, \ldots, [a_{m_\gamma-1}]_\theta)$. Hence the canonical map ν from $\mathfrak{A}$ to

$$\prod \{\mathfrak{A}/\theta\colon \mathfrak{A}/\theta \text{ is } \boldsymbol{K}\text{-subdirectly irreducible}\}$$

suffices to prove the theorem.

Theorem 2 cannot be extended to cover finitely axiomatizable existential classes of algebras as the following example shows:

Let $\boldsymbol{K}$ be the class of algebras $\langle A, \vee, \wedge, \pi, \sigma, f\rangle$ axiomatized by $\exists \boldsymbol{x}\big(\pi(\boldsymbol{x}) \neq \boldsymbol{x}\big)$ and $\exists \boldsymbol{x}\big(\sigma(\boldsymbol{x}) \neq \boldsymbol{x}\big)$, and consider the algebra

$$\mathfrak{A} = \langle (Z - \{0\}) \cup \{-\infty, +\infty\}, \vee, \wedge, \pi, \sigma, f\rangle,$$

where $\vee$ and $\wedge$ are just the usual lattice-theoretic join and meet, respectively, on the extended integers without zero, $\pi(x) = x-1$ if $1 < x < +\infty$, $\pi(x) = x$ otherwise, $\sigma(x) = x+1$ if $-\infty < x < -1$, $\sigma(x) = x$ otherwise, $f(x) = +\infty$ if $x \geqslant 1$, and $f(x) = -\infty$ if $x \leqslant -1$. Then $\mathfrak{A}$ is in $\boldsymbol{K}$, and the only $\boldsymbol{K}$-congruences of $\mathfrak{A}$ are of the form $\theta_{m,n}$, $1 \leqslant m, n < +\infty$, where $\langle x, y\rangle \in \theta_{m,n}$ iff $x = y$ or $1 \leqslant x, y \leqslant m$ or $-n \leqslant x, y \leqslant -1$. Note that $\mathfrak{A}$ is not $\boldsymbol{K}$-subdirectly irreducible, and $\mathfrak{A}/\theta_{m,n} \cong \mathfrak{A}$ for all $\theta_{m,n}$. Thus $\mathfrak{A}$ cannot be expressed as a full subdirect product of $\boldsymbol{K}$-subdirectly irreducibles.

With this we can also show that we cannot generalize Theorem 2 to finitely axiomatizable $(\forall \cup \exists)$-theories of relational structures, for if we replace each of the operation symbols $\vee, \wedge, \boldsymbol{\pi}, \boldsymbol{\sigma}, \boldsymbol{f}$ by a relation symbol $\boldsymbol{r}_\vee(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}), \ldots, \boldsymbol{r}_f(\boldsymbol{x}, \boldsymbol{y})$ and consider the class $\boldsymbol{K}$ axiomatized by

$$\exists \boldsymbol{x}\big(\neg \boldsymbol{r}_\pi(\boldsymbol{x}, \boldsymbol{x})\big),$$

$$\exists \boldsymbol{x}\big(\neg \boldsymbol{r}_\sigma(\boldsymbol{x}, \boldsymbol{x})\big),$$

$$\forall \boldsymbol{x} \forall \boldsymbol{y} \forall \boldsymbol{z} \forall \boldsymbol{w}\big[\big(\boldsymbol{r}_\vee(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \;\&\; \boldsymbol{r}_\vee(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w})\big) \to \boldsymbol{z} = \boldsymbol{w}\big],$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\forall \boldsymbol{x} \forall \boldsymbol{y} \forall \boldsymbol{z}\big[\big(\boldsymbol{r}_f(\boldsymbol{x}, \boldsymbol{y}) \;\&\; \boldsymbol{r}_f(\boldsymbol{x}, \boldsymbol{z})\big) \to \boldsymbol{y} = \boldsymbol{z}\big],$$

where the universal axioms assert that the relations are functions, then we can use the same example above.

In [4] Taylor defined the concept of "pure-irreducible". There is a striking similarity between Lemma 1 of this paper and his Lemma 3.4. Furthermore, as Taylor points out, let us expand the language of a structure $\mathfrak{A}$ to include predicate symbols $\boldsymbol{s}(\vec{\boldsymbol{y}})$ for each $(\exists, \wedge)$-formula

$\exists\vec{x}\varphi(\vec{x}, \vec{y})$. Then in the universal class K axiomatized by

$$\{\forall\vec{y}(\exists\vec{x}\varphi(\vec{x}, \vec{y}) \to s(\vec{y}))\}$$

the expansion of $\mathfrak{A}$ is K-subdirectly irreducible iff $\mathfrak{A}$ is pure-irreducible, and Taylor's Theorem 3.6 is a consequence of our Theorem 1.

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Added in proof. Mal'cev has some generalizations of Birkhoff's Theorem to arbitrary structures in *The metamathematics of algebraic systems,* North Holland, 1971. He uses subdirect products, whereas we use full subdirect products, so in many cases our results are stronger (see, for example, his Theorem 4 in *Subdirect products of models*). On the other hand, he obtains results for $\forall\exists$-classes (see the remark to his Theorem 3).

REFERENCES

[1] G. Birkhoff, *Subdirect unions in universal algebra,* Bulletin of the American Mathematical Society 50 (1944), p. 764-768.

[2] G. Grätzer, *Universal algebra,* van Nostrand, 1968.

[3] G. Sabidussi, *Subdirect representations of graphs,* Centre de Recherches Mathématiques, Université de Montreal, 1973.

[4] W. Taylor, *Residually small varieties,* Algebra Universalis 2 (1972), p. 33-53.

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