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THE MODEL COMPLETION OF THE CLASS OF $\mathscr{L}$-STRUCTURES

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Although some classes of structures, such as groups and lattices, which have alot of complicated models have no model companion, this paper shows that "having a model companion" is not tied to "having a good structure theory for the models", for given any language we show that the class of all $\mathscr{L}$-structures has a model completion whose theory is decidable if $\mathscr{L}$ is recursive.

Let $\mathscr{L}$ be a first-order language. Our key observation is that any existential $\mathscr{L}$-formula $\varphi(v_0, \ldots, v_n)$ is effectively equivalent to a disjunction of primitive formulas, each in the *expanded* form

$$(*) \qquad \exists u_0 \ldots \exists u_m [\bigwedge_{\langle i,j\rangle \in J_0} v_i = v_j \wedge \bigwedge_{\langle i,j\rangle \in J_1} v_i \neq v_j \wedge \bigwedge_i \beta_i(\boldsymbol{v}) \wedge \bigwedge_j \gamma_j(\boldsymbol{u}, \boldsymbol{v})],$$

where $J_0 \subseteqq (n+1) \times (n+1)$, $J_1 = (n+1) \times (n+1) - J_0$, $\boldsymbol{v} = (v_0, \ldots, v_n)$, $\boldsymbol{u} = (u_0, \ldots, u_m)$, and where each $\beta_i(\boldsymbol{v})$ and $\gamma_j(\boldsymbol{u}, \boldsymbol{v})$ is in one of the forms

(1)	$c_1 = c_2$	c_1, c_2 constants
(2)	$c_1 \neq c_2$	c_1, c_2 constants
(3)	$v_i = c$	c a constant
(4)	$v_i \neq c$	c a constant
(5)	$u_i \neq c$	c a constant
(6)	$u_i \neq u_j$	
(7)	$u_i \neq v_j$	
(8)	r(variables)	r a fundamental relation symbol
(9)	$\neg r$(variables)	r a fundamental relation symbol
(10)	f(variables) = variable	f a fundamental operation symbol,

where each $\gamma_j(\boldsymbol{u}, \boldsymbol{v})$ involves at least one variable from $\{u_0, \ldots, u_m\}$.

One can easily check if such an expanded primitive formula is satisfiable in some $\mathscr{L}$-structure $\boldsymbol{A}$ (by some choice of elements $\boldsymbol{a}$) by examining the subformulas of the types listed above occurring in φ to see that one does not have a glaring contradiction in one of the forms

$$w \neq w,$$

$$r(\boldsymbol{w}), \neg r(\boldsymbol{w}),$$

$$w_0 = w_1, \ldots, w_{k-1} = w_k, \quad w_0 \neq w_k,$$

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where for $0 \leqq i \leqq k-1$ either $w_i = w_{i+1}$ or $w_{i+1} = w_i$ occurs as a conjunct of the matrix of φ, and either $w_0 \neq w_k$ or $w_k \neq w_0$ also occurs as a conjunct.

Given a primitive $\mathscr{L}$-formula $\varphi(v_0, \ldots, v_n)$ in the expanded form (*) let $\varphi_0(v_0, \ldots, v_n)$ be the open formula

$$\bigwedge_{\langle i,j\rangle \in J_0} v_i = v_j \wedge \bigwedge_{\langle i,j\rangle \in J_1} v_i \neq v_j \wedge \bigwedge_i \beta_i(\boldsymbol{v}).$$

Theorem. *Let $K_{\mathscr{L}}$ be the class of $\mathscr{L}$-structures. Then $K_{\mathscr{L}}$ has a model completion $K_{\mathscr{L}}^*$ axiomatized by*

$$\Sigma = \{\varphi(\boldsymbol{v}) \leftrightarrow \varphi_0(\boldsymbol{v}) \colon \varphi \textit{ is a satisfiable expanded primitive formula}\}.$$

Proof. Since $K_{\mathscr{L}}$ is a universal class with the amalgamation property, it suffices to show that Σ axiomatizes the class of existentially closed structures in $K_{\mathscr{L}}$. Suppose that $\boldsymbol{A}$ is an existentially closed structure in $K_{\mathscr{L}}$, and let $\varphi(\boldsymbol{v})$ be an expanded primitive formula which is satisfiable. We only need to show $\boldsymbol{A} \models \varphi_0(\boldsymbol{v}) \to \varphi(\boldsymbol{v})$ since every $\mathscr{L}$-structure satisfies $\varphi(\boldsymbol{v}) \to \varphi_0(\boldsymbol{v})$. So suppose $\boldsymbol{A} \models \varphi_0(\boldsymbol{a})$. Then one can extend $\boldsymbol{A}$ to an $\mathscr{L}$-structure $\boldsymbol{B}$ by adding new elements $b_0, \ldots, b_m$ such that $\boldsymbol{B} \models \varphi(\boldsymbol{a})$ with $b_0, \ldots, b_m$ providing witnesses for the existentially quantified variables $u_0, \ldots, u_m$. Since $\boldsymbol{A}$ is existentially closed, it follows that $\boldsymbol{A} \models \varphi(\boldsymbol{a})$. Thus $\boldsymbol{A} \models \varphi_0(\boldsymbol{v}) \to \varphi(\boldsymbol{v})$.

For the converse we need to show that every $\mathscr{L}$-structure satisfying Σ is indeed existentially closed. So let $\boldsymbol{A} \models \Sigma$, and let $\boldsymbol{B}$ be an extension of $\boldsymbol{A}$ with $\boldsymbol{B} \models \varphi(\boldsymbol{a})$, where $\varphi(\boldsymbol{v})$ is a primitive formula and $\boldsymbol{a} \in A^n$. Without loss of generality we assume that $\varphi(\boldsymbol{v})$ is in expanded form. Then $\boldsymbol{B} \models \varphi_0(\boldsymbol{a})$, so $\boldsymbol{A} \models \varphi_0(\boldsymbol{a})$ as $\varphi_0(\boldsymbol{v})$ is an open formula. But then by the axioms Σ, $\boldsymbol{A} \models \varphi(\boldsymbol{a})$. Thus $\boldsymbol{A}$ is indeed existentially closed. □

Corollary. *$K_{\mathscr{L}}^*$ has a decidable theory if $\mathscr{L}$ is recursive.*

Proof. The theorem gives an effective elimination of quantifiers for $K_{\mathscr{L}}^*$. It remains to show that the open formulas true of $K_{\mathscr{L}}^*$ are decidable. However, since every $\mathscr{L}$-structure can be embedded in an existentially closed structure it follows that the open formulas true of $K_{\mathscr{L}}^*$ are precisely those true of $K_{\mathscr{L}}$, and hence decidable (open formulas can be effectively expressed as a disjunction of open expanded primitive formulas, and then one can easily determine if they are true of all $\mathscr{L}$-structures). □

Reference

[1] MACINTYRE, A., Model completeness. In: Handbook of Mathematical Logic. North-Holland Publ. Comp., Amsterdam–New York 1977, pp. 139–180.

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