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MODEL COMPANIONS FOR FINITELY GENERATED UNIVERSAL HORN CLASSES

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Abstract. In an earlier paper we proved that a universal Horn class generated by finitely many finite structures has a model companion. If the language has only finitely many fundamental operations then the theory of the model companion admits a primitive recursive elimination of quantifiers and is primitive recursive. The theory of the model companion is $\aleph_0$-categorical iff it is complete iff the universal Horn class has the joint embedding property iff the universal Horn class is generated by a single finite structure. In the last section we look at structure theorems for the model companions of universal Horn classes generated by functionally complete algebras, in particular for the cases of rings and groups.

In a recent paper [6] Wheeler proved that the class of N-colorable graphs is the universal Horn class generated by a single finite graph. Then he proceeds to show that the class of N-colorable graphs has a model companion which is $\aleph_0$-categorical, has a primitive recursive elimination of quantifiers, and is decidable. The existence and $\aleph_0$-categoricity of the model companion can be easily seen from general results in Burris and Werner [2]. We will continue our application of sheaf-theoretic constructions in [2] to give a general setting to the other results of Wheeler on model companions mentioned above.

Our notation will follow that of [2]. We will restate the pertinent definitions and preliminary results here. If A is a subdirect product of structures A_i, $i \in I$, and $\Phi(\vec{u})$ is a first-order formula, and $\vec{f}$ is a sequence of elements from A, then $[\![\Phi(\vec{f})]\!] = \{i \in I \mid A_i \models \Phi(f_1(i), \dots)\}$. If $\mathscr{K}$ is a class of structures then $A \in \Gamma_0^e(\mathscr{K})$ means that there is a Boolean space $X(A)$ without isolated points and structures $A_x \in \mathscr{K}$ for $x \in X(A)$ such that

(1) A is a subdirect product of the A_x,

(2^e) $[\![\Phi(\vec{f})]\!]$ is a clopen subset of $X(A)$ for all first-order formulas $\Phi(\vec{u})$ and parameters $\vec{f}$ from A, and

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(3) for $f, g \in A$ and N a clopen subset of $X(A)$ the function $h = f\restriction_N \cup g\restriction_{X-N}$ is in A.

In addition to the class operator Γ_0^e we will need **I** (closure under isomorphism), **S** (closure under substructure), and **P** (closure under direct product).

The logical symbols we need are $\forall, \exists, \neg, \&, \vee, \rightarrow, \leftrightarrow$ and $\bigvee$, the latter referring to a (possibly) infinite disjunction. The universal Horn class generated by a finite set $\mathscr{K}$ of finite structures is $\mathbf{ISP}(\mathscr{K})$ (this is the smallest class containing $\mathscr{K}$ which can be axiomatized by universal Horn sentences).

A theory T has a model companion T^* if (i) T^* is model-complete and (ii) every model of T can be embedded in a model of T^*, and vice-versa. If T^* exists (given T) then it is unique. We will occasionally refer to the model companion as the models of T^* (this should cause no confusion), and speak of the model companion of an elementary class. The model companion of a universal Horn class is always contained in the class.

§1. Finitely generated universal Horn classes. A class $\mathscr{K}'$ is a finitely generated universal Horn class if $\mathscr{K}' = \mathbf{ISP}(\mathscr{K})$ for some finite collection of finite algebras $\mathscr{K}$. The first result is proved in §8 of [2].

THEOREM 1.1. *Every finitely generated universal Horn class has a model companion.*

A class $\mathscr{K}$ of algebras has the joint embedding property if given $A, B \in \mathscr{K}$ there is a $C \in \mathscr{K}$ such that A and B can be embedded in C.

THEOREM 1.2. *If $\mathscr{K}$ is a finite collection of finite structures and the language is finite then the following are equivalent for the model companion T^* of the theory of* $\mathbf{ISP}(\mathscr{K})$:

(i) *T^* is $\aleph_0$-categorical,*
(ii) *T^* is complete,*
(iii) $\mathbf{ISP}(\mathscr{K})$ *has the joint embedding property,*
(iv) *there is a single finite structure A such that* $\mathbf{ISP}(\mathscr{K}) = \mathbf{ISP}(A)$.

PROOF. (i) $\Rightarrow$ (ii) is clear, and (ii) $\Rightarrow$ (iii) can be found in Robinson [4]. Now (iii) $\Rightarrow$ (iv) as $\mathbf{ISP}(\mathscr{K})$ is locally finite, hence from (iii) there is a finite $A \in \mathbf{ISP}(\mathscr{K})$ such that each member of $\mathscr{K}$ can be embedded in A. But then $\mathbf{ISP}(\mathscr{K}) = \mathbf{ISP}(A)$. Finally (iv) $\Rightarrow$ (i) is stated in §8 of [2]. □

For the remainder of this section we will need to go into some of the details of the results from §8 of [2] which were quoted above. We assume that we are working within a given finite language L (possibly with both relation and function symbols). E_n denotes the collection of existential formulas $\exists\vec{u}\Phi(\vec{u}, v_1, \ldots, v_n)$, where Φ is open. P_n denotes the primitive formulas in E_n, i.e. those for which Φ is a conjunction of atomic and negated atomic formulas. A *basic* atomic formula is one of the form $\mathbf{r}(u_1, \ldots, u_n)$ or $\mathbf{f}(u_1, \ldots, u_n) = u_{n+1}$, where the u_i's are variables, $\mathbf{r}$ is a fundamental relation, and $\mathbf{f}$ is a fundamental operation. Let P_n^* be the conjunctions of members of P_n whose matrics are conjunctions of basic atomic formulas and/or negations of the same. For a class $\mathscr{H}$ of structures in our language we will call a subset R_n of P_n^* a *set of representatives of P_n modulo $\mathscr{H}$* if every member of P_n is equivalent to some member of R_n modulo $\mathscr{H}$.

Also *for the remainder of this section* we will let $\mathscr{K}$ be a fixed finite set of finite structures in the given language. The next result is Lemma 8.2 of [2].

LEMMA 1.3 (MACINTYRE). *If* $A \in \boldsymbol{\Gamma}_0^e(\mathscr{K})$ *and* $\Phi(\vec{v}) \in P_n$, *say* $\Phi(\vec{v})$ *is* $\exists\vec{u}[\Phi_0(\vec{u}, \vec{v})\ \&\ \&_{1 \le i \le k} \neg\, \alpha_i(\vec{u}, \vec{v})]$, *where* $\Phi_0(\vec{u}, \vec{v})$ *is a conjunction of atomic formulas, and each* α_i *is atomic, then, for* $\vec{f}$ *from* A,

$$A \models \Phi(\vec{f})$$

iff

(i)
$$[\![\exists\vec{u}\Phi_0(\vec{u}, \vec{f})]\!] = X(A)$$

and

(ii)
$$[\![\exists\vec{u}(\Phi_0(\vec{u}, \vec{f})\ \&\ \neg\,\alpha_i(\vec{u}, \vec{f}))]\!] \neq \varnothing, \qquad 1 \le i \le k.$$

REMARK. In the above, if there is no positive part Φ_0 of Φ then omit (i), and if there is no negative part omit (ii).

An immediate consequence of this is the following.

COROLLARY 1.4. *Let* $\Phi(\vec{v})$ *be as in the previous lemma. Then*

$$\boldsymbol{\Gamma}_0^e(\mathscr{K}) \models \Phi(\vec{v}) \leftrightarrow \left\{[\exists\vec{u}\Phi_0(\vec{u}, \vec{v})]\ \&\ \underset{1 \le i \le k}{\&}\ [\exists\vec{u}\Phi_0(\vec{u}, \vec{v})\ \&\ \neg\,\alpha_i(\vec{u}, \vec{v})]\right\}.$$

LEMMA 1.5. *There is a primitive recursive procedure to find a finite set* R_n *of representatives of* P_n *modulo* $\boldsymbol{\Gamma}_0^e(\mathscr{K})$, *and there is a primitive recursive procedure, given* $\Phi \in P_n$, *to find a* $\Phi' \in R_n$ *such that* $\boldsymbol{\Gamma}_0^e(\mathscr{K}) \models \Phi \leftrightarrow \Phi'$.

PROOF. For $\Phi(\vec{v}) \in P_n$ let $\Phi(\vec{v})$ be written in the form

$$\exists\vec{u}\left[\Phi_0(\vec{u}, \vec{v})\ \&\ \underset{1 \le i \le k}{\&}\ \neg\,\alpha_i(\vec{u}, \vec{v})\right]$$

as in the statement of Lemma 1.3. (If $\Phi(\vec{v})$ has no positive part, change the matrix by introducing $v_1 = v_1$; and if there is no negative part we have just $\exists\vec{u}\Phi_0(\vec{u}, \vec{v})$.) As there is a primitive recursive method to transform Φ into an equivalent formula in P_n^* using the schemes

$$p = q \leftrightarrow \exists z[p = z\ \&\ q = z],$$

$$p \neq q \leftrightarrow \exists z \exists w[p = z\ \&\ q = w\ \&\ z \neq w],$$

$$\mathbf{f}(p_1, \ldots, p_n) = v \leftrightarrow \exists z_1 \ldots \exists z_n\left[f(\vec{z}) = v\ \&\ \underset{1 \le i \le n}{\&}\ p_i = z_i\right],$$

$$\mathbf{r}(p_1, \ldots, p_n) \leftrightarrow \exists z_1 \ldots \exists z_n\left[\mathbf{r}(\vec{z})\ \&\ \underset{1 \le i \le n}{\&}\ p_i = z_i\right],$$

we can assume without loss of generality that $\Phi \in P_n^*$. By Corollary 1.4,

$$\boldsymbol{\Gamma}_0^e(\mathscr{K}) \models \Phi(\vec{v}) \leftrightarrow \left\{[\exists\vec{u}\Phi_0(\vec{u}, \vec{v})]\ \&\ \underset{1 \le i \le k}{\&}\ [\exists\vec{u}(\Phi_0(\vec{u}, \vec{v})\ \&\ \neg\,\alpha_i(\vec{u}, \vec{v}))]\right\}.$$

We will show that each of the $k + 1$ primitive formulas on the right-hand side of the bi-implication is equivalent modulo $\boldsymbol{\Gamma}_0^e(\mathscr{K})$ to a formula in P_n^* with a primitive recursive bound $\mu(n)$ on the number of existential quantifiers and on the number of basic atomic formulas appearing in the matrix.

Let M be the maximal size of a structure in $\mathscr{K}$, N the number of structures in $\mathscr{K}$, r the maximum number of variables in a basic atomic formula, and K the number of fundamental operations and relations in the language.

Suppose $\chi(\vec{v}) = \exists u_1 \ldots \exists u_m[\&_{1 \le i \le s} \beta_i(\vec{u}, \vec{v})]$ is in P_n^* and each β_i is basic atomic. Then, for each $B \in \mathscr{K}$ and each $\vec{b}$ from B such that $B \models \exists \vec{u}[\&_{1 \le i \le s} \beta_i(\vec{u}, \vec{b})]$ let $\lambda_{\vec{b}}: \{u_1, \ldots, u_m\} \to B$ be an assignment such that $B \models \&_{1 \le i \le s} \beta_i(\lambda_{\vec{b}}(\vec{u}), \vec{b})$. The equivalence relation ε on $\{u_1, \ldots, u_m\}$ given by $\bigcap\{\mathrm{Ker}(\lambda_{\vec{b}}) \mid \vec{b} \in B \in \mathscr{K}\}$ has at most $m' = M^{N \cdot M^n}$ equivalence classes. Thus, by Lemma 1.3, $\exists \vec{u}[\&_{1 \le i \le s} \beta_i(\vec{u}, \vec{v})]$ is equivalent modulo $\boldsymbol{\Gamma}_0^e(\mathscr{K})$ to

$$\exists \vec{u}\left[\underset{1 \le i \le s}{\&} \beta_i(\vec{u}, \vec{v}) \,\&\, \underset{(u_i, u_j) \in \varepsilon}{\&} u_i = u_j\right],$$

and hence to a member of P_n^* with at most m' existential quantifiers, say $\chi'(\vec{v})$. But then, by cancelling out repeats of basic atomic formulas in the matrix of $\chi'(\vec{v})$ we can assume the matrix of $\chi'(\vec{v})$ has at most $s' = K \cdot (m' + n)^r$ occurrences of basic atomic formulas.

Next suppose $\chi(\vec{v}) = \exists u_1 \ldots \exists u_m[(\&_{1 \le i \le s} \beta_i(\vec{u}, \vec{v})) \,\&\, \neg\alpha(\vec{u}, \vec{v})]$ is in P_n^* with each β_i and α being basic atomic. Then again for each $B \in \mathscr{K}$ and $\vec{b}$ from B such that $B \models \exists \vec{u}[\&_{1 \le i \le s} \beta_i(\vec{u}, \vec{b})]$ choose $\lambda_{\vec{b}}$ as above, and if $B \models \exists \vec{u}[(\&_{1 \le i \le s} \beta_i(\vec{u}, \vec{b})) \& \neg\alpha(\vec{u}, \vec{b})]$ choose $\mu_{\vec{b}}: \{u_1, \ldots, u_m\} \to B$ such that

$$B \models \left(\underset{1 \le i \le s}{\&} \beta_i(\mu_{\vec{b}}(\vec{u}), \vec{b})\right) \& \neg\alpha(\mu_{\vec{b}}(\vec{u}), \vec{b}).$$

In this case let δ be the equivalence relation on $\{u_1, \ldots, u_m\}$ defined by

$$\bigcap\{\mathrm{Ker}(\lambda_{\vec{b}}) \mid \vec{b} \in B \in \mathscr{K}\} \cap \bigcap\{\mathrm{Ker}(\mu_{\vec{b}}) \mid \vec{b} \in B \in \mathscr{K}\},$$

and note that the number of equivalence classes of δ is no more than $m'' = m^{2N \cdot M^n}$. Again by Lemma 1.3, $\exists \vec{u}[(\&_{1 \le i \le s} \beta_i(\vec{u}, \vec{v})) \,\&\, \neg\alpha(\vec{u}, \vec{v})]$ is equivalent modulo $\boldsymbol{\Gamma}_0^e(\mathscr{K})$ to

$$\exists \vec{u}\left[\left(\underset{1 \le i \le s}{\&} \beta_i(\vec{u}, \vec{v})\right) \& \neg\alpha(\vec{u}, \vec{v}) \,\&\, \underset{(u_i, u_j) \in \delta}{\&} u_i = u_j\right],$$

and hence to a member χ'' of P_n^* with at most m'' existential quantifiers and (cancelling out repeats in the matrix) with at most $s'' = K \cdot [m'' + n]^r$ basic atomic formulas in the matrix, and $\neg\alpha$ (with suitable variables) as the only negated basic atomic formula.

There is clearly a primitive recursive procedure to list, given n, all the finitely many formulas of the form $\chi'(\vec{v})$ and $\chi''(\vec{v})$ described above, where the quantified variables are restricted to $u_1, \ldots, u_{m''}$; and there is a primitive recursive bound $\upsilon(n)$ on the number of such formulas. Let R_n be the set of $\chi'(\vec{v})$ as well as all formulas of the form $\chi'(\vec{v}) \,\&\, \&_{1 \le i \le t} \chi_i''(\vec{v})$ where $t \le \upsilon(n)$. Then, returning to our original formula $\Phi(\vec{v})$ we know $\exists \vec{u} \Phi_0(\vec{u}, \vec{v})$ and each $\exists \vec{u}[\Phi_0(\vec{u}, \vec{v}) \,\&\, \neg\alpha_i(\vec{u}, \vec{v})]$ is equivalent modulo $\boldsymbol{\Gamma}_0^e(\mathscr{K})$ to some $\chi'(\vec{v})$ respectively to some $\chi''(\vec{v})$ from R_n, so $\Phi(\vec{v})$ is equivalent modulo $\boldsymbol{\Gamma}_0^e(\mathscr{K})$

to some member of R_n (again by Lemma 1.3), and the reduction to a member of R_n is primitive recursive. □

Given $\Phi \in P_n$ let Φ^* be the finite disjunction $\bigvee\{\Psi \in R_n \mid \Gamma_0^e(\mathscr{K}) \models \Psi \to \neg\Phi\}$.

LEMMA 1.6. *For $\Phi \in P_n$ there is a primitive recursive procedure to find Φ^* from Φ.*

PROOF. Let T_0 be the theory of atomless Boolean algebras. Then there is primitive recursive translation of sentences χ in our given language to sentences χ^* in the language of Boolean algebras (based on Comer's Feferman-Vaught theorem for Γ_0^e) such that $\Gamma_0^e(\mathscr{K}) \models \chi$ iff $T_0 \models \chi^*$ (see [2], Lemma 4.4). As there is a well-known primitive recursive decision procedure for T_0 (using elimination of quantifiers), it follows that we have a primitive recursive procedure to determine if $\Gamma_0^e(\mathscr{K}) \models \Psi \to \neg\Phi$, for $\Psi \in R_n$. □

Let T'' be the universal Horn part of the theory of $\mathbf{ISP}(\mathscr{K})$, and let $\mathscr{M}$ be the model companion of $\mathbf{ISP}(\mathscr{K})$.

THEOREM 1.7. *$\mathscr{M}$ is primitive recursively axiomatized by $T'' \cup \{\Phi \vee \Phi^* \mid \Phi \in P_n, n < \omega\}$.*

PROOF. Since a universal Horn sentence χ holds in $\mathbf{ISP}(\mathscr{K})$ iff it holds in $\mathscr{K}$, it is clear that T'' is primitive recursive, and then from Lemma 1.6 the entire proposed set of axioms is primitive recursive. Now $\Gamma_0^e(\mathscr{K}) \models \Phi^* \leftrightarrow \bigvee\{\Psi \in P_n \mid \Gamma_0^e(\mathscr{K}) \models \Psi \to \neg\Phi\}$, so by §8 of [2] the above indeed axiomatizes $\mathscr{M}$. □

LEMMA 1.8. *For $\Phi \in P_n$, $\mathscr{M} \models \Phi^* \leftrightarrow \neg\Phi$.*

PROOF. Clearly $\mathscr{M} \models \Phi^* \to \neg\Phi$, and as $\mathscr{M} \models \Phi^* \vee \Phi$, it follows that $\mathscr{M} \models \Phi^* \leftrightarrow \neg\Phi$.

THEOREM 1.9. *The theory of $\mathscr{M}$ admits a primitive recursive elimination of (existential) quantifiers.*

PROOF. Just combine Lemmas 1.6 and 1.8 to obtain a primitive recursive transition from existential to universal formulas, and vice-versa. □

THEOREM 1.10. *The theory of $\mathscr{M}$ is primitive recursive, hence decidable.*

PROOF. Given a sentence Φ, first transform it into an equivalent universal sentence Φ' (modulo $\mathscr{M}$) by Theorem 1.9. Now $\mathscr{M} \models \Phi'$ iff $\mathbf{ISP}(\mathscr{K}) \models \Phi'$, and the latter holds iff $\mathbf{P}(\mathscr{K}) \models \Phi'$. The techniques of Feferman and Vaught (see §4 of [2]) give a primitive recursive method of deciding if $\mathbf{P}(\mathscr{K}) \models \Phi'$. □

§2. Model companions for universal Horn class generated by functionally complete algebras. Although we have a primitive recursive axiomatization of the model companion for a finitely generated universal Horn class, there are no general structure theorems known for this model companion. In this section we will look at the situation when the universal Horn class is generated by a single functionally complete algebra. (The general background for this section can be found in Chapter IV of [1].) A finite algebra A is said to be *functionally complete* if for every map $\lambda: A^n \to A$, $0 < n < \omega$, there exist a term $p(\vec{u}, \vec{v})$ and parameters $\vec{a}$ from A such that $A \models \lambda(\vec{u}) = p(\vec{u}, \vec{a})$, i.e., if every such λ is representable by a term with parameters. Throughout this section we assume our functionally complete algebras to have a universe of at least two elements, to avoid an obviously trivial case.

A *principal congruence formula* in a given language is a formula $\pi(u, u', v, v')$ of the form

$$\exists\vec{w}\left[u = p_1(\sigma_1(1), \vec{w}) \;\&\; u' = p^n(\sigma_n(2), \vec{w}) \;\&\; \mathop{\&}_{1 \le i \le n-1} p_i(\sigma_i(2), \vec{w}) = p_{i+1}(\sigma_{i+1}(1), \vec{w})\right],$$

where the p_i's are terms and $\{\sigma_i(1), \sigma_i(2)\} = \{v, v'\}$ for $1 \le i \le n$. Let Π denote the collection of all principal congruence formulas in the given language. For A an algebra and $a, b \in A$ let $\theta_A(a, b)$ be the principal congruence of A generated by (a, b). A well-known result of Mal'cev (see [1], p. 221) says $(c, d) \in \theta_A(a, b)$ iff $A \models \bigvee\!\!\!\bigvee \{\pi(c, d, a, b) \mid \pi \in \Pi\}$.

LEMMA 2.1. *If A is a functionally complete algebra in a congruence permutatable variety, then, for some $\pi \in \Pi$, $(c, d) \in \theta_A(a, b)$ iff $A \models \pi(c, d, a, b)$.*

PROOF. Let m be the number of elements in A, and in $B = A^{m^4}$ choose f, g, h, k such that $\{(c, d, a, b) \in A^4 \mid a = b \to c = d\} = \{(f(i), g(i), h(i), k(i)) \mid i < m^4\}$. Then (see [5]) $(f, g) \in \theta_B(h, k)$, so let $\pi \in \Pi$ be such that $B \models \pi(f, g, h, k)$. □

The *ternary discriminator* t on a set A is the function defined by

$$t(u, v, w) = \begin{cases} u & \text{if } u \neq v, \\ w & \text{if } u = v. \end{cases}$$

(This function played a major role in [2].)

LEMMA 2.2. *If A is a functionally complete algebra in a congruence permutable variety, then there is a positive formula $\tau(v_1, v_2, v_3, v_4) \in P_4$ such that $A \models \tau(a_1, a_2, a_3, a_4)$ iff $t(a_1, a_2, a_3) = a_4$, t being the discriminator function.*

PROOF. Let $t^*(v_1, v_2, v_3, \vec{e})$ be a term with parameters which represents t. If the universe of A is $\{a_1, \ldots, a_n\}$ let Δ_A^+ be the collection of formulas

$$\{\mathbf{f}(u_{i_1}, \ldots, u_{i_k}) = u_{i_{k+1}} \mid \mathbf{f}(a_{i_1}, \ldots, a_{i_k}) = a_{i_{k+1}}\},$$

where $\mathbf{f}$ is any fundamental operation on A. Now let $\tau(v_1, v_2, v_3, v_4)$ be

$$\exists \vec{r} \left[\Delta_A^+(\vec{r}) \,\&\, \mathop{\&}_{1 \le i \le 4} \pi(v_i, r_1, r_1, r_2) \,\&\, t^*(v_1, v_2, v_3, \vec{s}) = v_4 \right],$$

where π is as described in Lemma 2.1, and $\vec{s}$ is a sequence of variables selected from the r's which correspond to the $\vec{e}$ in the definition of t^* above. If $\tau(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4)$ holds in A, then choose a sequence $\vec{b}$ from A to witness $\vec{r}$. As A is simple and $A \models \Delta_A^+(\vec{b})$, either $b_1, \ldots, b_n$ are all equal or they are the universe of A. In the former case $A \models \pi(\hat{a}_i, b_1, b_1, b_2)$ implies $\hat{a}_1 = \cdots = \hat{a}_4$; hence $t(\hat{a}_1, \hat{a}_2, \hat{a}_3) = \hat{a}_4$. In the latter case we have $t^*(\hat{a}_1, \hat{a}_2, \hat{a}_3, \vec{c}) = \hat{a}_4$, where $\vec{c}$ is the appropriate sequence of b_i's; hence $t(\hat{a}_1, \hat{a}_2, \hat{a}_3) = \hat{a}_4$, as $\vec{c}$ is the image of $\vec{e}$ under an automorphism of A (namely the automorphism $a_i \to b_i$). Conversely, if $A \models t(\hat{a}_1, \hat{a}_2, \hat{a}_3) = \hat{a}_4$ then the interpretation $r_i \to a_i$ leads to $A \models \tau(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4)$.

To state our main result in this section we need one more definition, namely that of the class operator $\boldsymbol{\Gamma}_0^a$. It is defined the same as $\boldsymbol{\Gamma}_0^e$ except condition ($\mathbf{2}^e$) is replaced by

($\mathbf{2}^a$) $[\![\Phi(\vec{f})]\!]$ is a clopen subset of $X(A)$ for all atomic formulas $\Phi(\vec{u})$ and parameters $\vec{f}$ from A.

THEOREM 2.3. *Suppose A is a functionally complete algebra in a congruence permutable variety and $\mathcal{M}$ is the model companion of* $\mathbf{ISP}(A)$.

(a) *If A has exactly no one-element subalgebra then $\mathcal{M} = \mathbf{I}\boldsymbol{\Gamma}_0^e(A)$.*

(b) *If A has exactly one one-element subalgebra then*

$$\mathcal{M} = \mathbf{I}\{B \in \boldsymbol{\Gamma}_0^a(A_+) \mid [\![\forall x \forall y(x = y)]\!] \text{ is a singleton}\},$$

A_+ being $\{A\}$ union the one-element algebra.

PROOF. In view of Lemma 2.2 this is just an application of Theorem 10.7 of [2]. □

EXAMPLES. (1) A finite simple ring (in the language $\{+, \cdot, -, 0, 1\}$) R is functionally complete (Werner [5]); hence the model companion of $\mathbf{ISP}(R)$ is $\mathbf{I}\Gamma_0^e(R)$.

(2) A finite nonabelian simple group G is functionally complete (Maurer-Rhodes [3]); hence the model companion of $\mathbf{ISP}(G)$ is $\mathbf{I}\{H \in \Gamma_0^a(G_+) \mid [\![\forall u \forall v(u = v)]\!]$ is a singleton$\}$.

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