Algebra Universalis, 21 (1985) 54–61 0002–5240/85/010054–08$01.50+0.20/0

Iterated discriminator varieties have undecidable theories

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One of the landmarks in the study of decidable varieties is Werner's proof that finitely generated discriminator varieties (with a finite language) have a decidable theory. However, little is known about discriminator varieties which are not finitely generated except that some have decidable theories, and some fairly innocent looking ones have undecidable theories. The most important example of the latter situation is the variety CA_1 of monadic algebras, analyzed by Rubin [4].

In this paper we give a fairly wide class of discriminator varieties with undecidable theories. The basic definitions and terminology follow Burris and Sankappanavar [1].

Let BA be the class of Boolean algebras, with $\mathbf{2}$ the *two-element Boolean algebra*. For $\mathbf{B}$ a Boolean algebra let $\mathbf{B}^*$ be the associated Boolean space. For $\mathbf{A}$ any algebra and $\mathbf{B}$ a Boolean algebra let $\mathbf{A}[\mathbf{B}]^*$ be the *bounded Boolean power* (whose universe is the set of continuous functions from $\mathbf{B}^*$ to A). $P_{\mathbf{B}}(\mathbf{A})$ denotes the class of all bounded Boolean powers of $\mathbf{A}$.

Let K be a class of algebras, and let $\varepsilon(x, y, u, v)$ be a formula in the language of K. $\varepsilon(x, y, u, v)$ is said to be an *encoding formula* for K if it is a primitive positive formula which is equivalent, modulo K, to the formula $x \approx y \rightarrow u \approx v$. (Encoding formulas were introduced in [2]). An expansion K' of a class K is a *coding expansion* if there is an encoding formula for K'.

For K a class of algebra, $\Gamma^a(K)$ is the class of *Boolean products* of K. $\mathbf{A} \leq_{bp} \prod_{x \in X} \mathbf{A}_x$ means $\mathbf{A}$ is a Boolean product of the $\mathbf{A}_x$'s.

THEOREM. *Let* $\mathbf{A}$ *be a nontrivial algebra with an encoding formula. Then, for any coding expansion* $P_{\mathbf{B}}(\mathbf{A})'$ *of* $P_{\mathbf{B}}(\mathbf{A})$, *the theory of* $\Gamma^a(P_{\mathbf{B}}(\mathbf{A})')$ *is hereditarily undecidable.*

Proof. To show that the theory of $\Gamma^a(P_{\mathbf{B}}(\mathrm{A})')$ is hereditarily undecidable we

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Presented by W. Taylor. Received August 20, 1983. Accepted for publication in final form November 2, 1984.
Research supported by NSERC Grant No. A7256

will semantically embed CA_1 into it. For $\mathbf{B}$ a Boolean algebra let $\mathbf{B}^c$ be the expansion of $\mathbf{B}$ by the unary function c where $c(0)=0$, $c(b)=1$ for $b\neq 0$. From Comer [3] we know that any monadic algebra is isomorphic to a Boolean product of $\mathbf{B}^c$'s.

So let $\mathbf{M}=\langle M, \vee, \wedge, ', 0, 1, c\rangle$ be a monadic algebra; without loss of generality suppose $\mathbf{M}\leq_{bp}\prod_{x\in X}(\mathbf{2}[\mathbf{B}_x]^*)^c$. For $x\in X$ let $\mathbf{C}_x=(\mathbf{A}[\mathbf{B}_x]^*)'$, an expansion of $\mathbf{A}[\mathbf{B}_x]^*$ which is in $P_B(\mathbf{A})'$. For $f\in\prod_{x\in X}C_x$ and $a\in A$ let $\mu_a^f\in\prod_{x\in X}2[\mathbf{B}_x]^*$ be defined by $\mu_a^f(x)^{-1}(1)=f(x)^{-1}(a)$. Now let $C\subseteq\prod_{x\in X}C_x$ be given by

$$C=\left\{f\in\prod_{x\in X}C_x:\mu_a^f\in M \text{ for all } a\in A\right\}.$$

We want to show that C is a subuniverse of $\prod_{x\in X}\mathbf{C}_x$, that indeed $\mathbf{C}$ is a Boolean product of the $\mathbf{C}_x$'s, and finally that $\mathbf{M}$ can be semantically embedded in $\mathbf{C}$.

An easy calculation shows that for $s_1, s_2\in\prod_{x\in X}2[\mathbf{B}_x]^*$ and $x\in X$ we have

$$(s_1\wedge s_2)(x)^{-1}(1)=[s_1(x)^{-1}(1)]\cap[s_2(x)^{-1}(1)]$$
$$(s_1\vee s_2)(x)^{-1}(1)=[s_1(x)^{-1}(1)]\cup[s_2(x)^{-1}(1)].$$

This will be quite useful in the proof.

LEMMA 1. *If* $f\in C$ *then* $\bigcup\{\text{Range}\, f(x):x\in X\}$ *is finite.*

Proof. For a given $x\in X$ we know Range $f(x)$ is finite as $f(x)\in A[\mathbf{B}_x]^*$. So let Range $f(x)=\{a_1,\ldots,a_n\}$. Then

$$\begin{aligned}[\mu_{a_1}^f\vee\cdots\vee\mu_{a_n}^f](x)^{-1}(1)&=[\mu_{a_1}^f(x)^{-1}(1)]\cup\cdots\cup[\mu_{a_n}^f(x)^{-1}(1)]\\&=f(x)^{-1}(a_1)\cup\cdots\cup f(x)^{-1}(a_n)\\&=\mathbf{B}_x^*,\end{aligned}$$

so $[\mu_{a_1}^f\vee\cdots\vee\mu_{a_n}^f](x)=1$. As the $\mu_{a_i}^f\in M$ it follows that for some clopen neighborhood N of x we have $[\mu_{a_1}^f\vee\cdots\vee\mu_{a_n}^f](y)=1$ for $y\in N$. Thus Range $f(y)\subseteq\{a_1,\ldots,a_n\}$ for $y\in N$. By compactness we see that $\bigcup\{\text{Range}\, f(x):x\in X\}$ is finite. $\square$

In the following $\mathbb{R}(f)$ will mean $\bigcup\{\text{Range}\, f(x):x\in X\}$.

LEMMA 2. *C is a subuniverse of* $\prod_{x\in X}\mathbf{C}_x$.

Proof. Let F be an n-ary function symbol in the language of $P_{\mathbf{B}}(\mathbf{A})'$, and let $f_1, \ldots, f_n \in C$.

Then, for $a \in A$, $x \in X$,

$$\begin{aligned}
\mu_a^{F(f_1,\ldots,f_n)}(x)^{-1}(1) &= [F(f_1, \ldots, f_n)(x)]^{-1}(a) \\
&= F(f_1(x), \ldots, f_n(x))^{-1}(a) \\
&= \bigcup_{\substack{F(a_1,\ldots,a_n)=a \\ a_i \in \mathbb{R}(f_i)}} f_1(x)^{-1}(a_1) \cap \cdots \cap f_n(x)^{-1}(a_n) \\
&= \bigcup_{\substack{F(a_1,\ldots,a_n)=a \\ a_i \in \mathbb{R}(f_i)}} \mu_{a_1}^{f_1}(x)^{-1}(1) \cap \cdots \cap \mu_{a_n}^{f_n}(x)^{-1}(1) \\
&= \Bigg[\bigvee_{\substack{F(a_1,\ldots,a_n)=a \\ a_i \in \mathbb{R}(f_i)}} \mu_{a_1}^{f_1} \wedge \cdots \wedge \mu_{a_n}^{f_n} \Bigg](x)^{-1}(1)
\end{aligned}$$

since the union above is finite. Thus

$$\mu_a^{F(f_1,\ldots,f_n)} = \bigvee_{\substack{F(a_1,\ldots,a_n)=a \\ a_i \in \mathbb{R}(f_i)}} \mu_{a_1}^{f_1} \wedge \cdots \wedge \mu_{a_n}^{f_n},$$

so $\mu_a^{F(f_1,\ldots,f_n)} \in M$. This proves $F(f_1, \ldots, f_n) \in C$ (as it is clearly in $\prod_{x \in X} C_x$). □

Let $\mathbf{C}$ be the subalgebra of $\prod_{x \in X} \mathbf{C}_x$ with universe C.

LEMMA 3. *$\mathbf{C}$ is a subdirect product of the $\mathbf{C}_x$'s.*

Proof. Given $x_0 \in X$ and $c \in C_{x_0}$, let Range $c = \{a_1, \ldots, a_n\}$, and let $N_i = c^{-1}(a_i)$, $1 \le i \le n$. As $\mathbf{M}$ is a subdirect product of the algebras $(\mathbf{2}[\mathbf{B}_x]^*)^c$ it follows that we can find $m_i \in M$ such that $m_i(x)^{-1}(1) = N_i$, $1 \le i \le n$. Then $m_1(x_0) \vee \cdots \vee m_n(x_0) = 1$, $m_i(x_0) \wedge m_j(x_0) = 0$ if $i \ne j$. As $\mathbf{M}$ is a Boolean product we can use the patchwork property to show that we can also assume the m_i's above satisfy $m_1 \vee \cdots \vee m_n = 1$, $m_i \wedge m_j = 0$ for $i \ne j$. Now construct $f \in \prod_{x \in X} C_x$ by letting $f(x)^{-1}(a_i) = m_i(x)^{-1}(1)$ for $x \in X$, $1 \le i \le n$. Then $\mu_{a_i}^f = m_i$, and $\mu_a^f = 0$ for $a \notin \{a_1, \ldots, a_n\}$. Thus $f \in C$, and $f(x_0) = c$, so indeed $\mathbf{C}$ is a subdirect product of the $\mathbf{C}_x$'s. □

LEMMA 4. *Equalizers are clopen in* $\mathbf{C}$.

Proof. Let $f, g \in C$. Then, with $A_0 = \mathbb{R}(f) \cup \mathbb{R}(g)$,

$$\begin{aligned}[\![f = g]\!] &= \{x \in X : f(x) = g(x)\}\\ &= \{x \in X : f(x)^{-1}(a) = g(x)^{-1}(a) \text{ for } a \in A_0\}\\ &= \bigcap_{a \in A_0} \{x \in X : \mu_a^f(x)^{-1}(1) = \mu_a^g(x)^{-1}(1)\}.\\ &= \bigcap_{a \in A_0} [\![\mu_a^f = \mu_a^g]\!],\end{aligned}$$

a clopen subset of X. □

LEMMA 5. **C** *has the patchwork property.*

Proof. Let $f, g \in C$ and let N be a clopen subset of X. Let $h = f \restriction_N \cup g \restriction_{X-N}$. Then we have, for $a \in A$, $x \in X$,

$$\mu_a^h(x) = \begin{cases} \mu_a^f(x) & \text{if } x \in N \\ \mu_a^g(x) & \text{if } x \notin N, \end{cases}$$

i.e., $\mu_a^h = \mu_a^f \restriction_N \cup \mu_a^g \restriction_{X-N}$. As **M** has the patchwork property we see that $\mu_a^h \in M$ for $a \in A$, so $h \in C$. □

Combining Lemmas 3–5 we have proved

LEMMA 6. $\mathbf{C} \leq_{bp} \prod_{x \in X} \mathbf{C}_x$.

Now we turn to our main objective, namely to show that CA_1 can be semantically embedded in $P_B(\mathbf{A})'$. Choose an encoding formula $\varepsilon(x, y, u, v)$ for **A** and let $\mathbf{M} \in CA_1$, with **C** as above. Define a relation $\leq^*$ on C^2 by

$$\langle f, g \rangle \leq^* \langle h, k \rangle \quad \text{iff} \quad C \vDash \varepsilon(f, g, h, k).$$

LEMMA 7. *For $f, g \in C$, $\langle f, g \rangle \leq^* \langle h, k \rangle$ holds iff for all $x \in X$ we have $[\![f(x) = g(x)]\!] \subseteq [\![h(x) = k(x)]\!]$.*

Proof. Note that $\mathbf{C} \Vdash \varepsilon(f, g, h, k)$ iff for all $x \in X$ we have $\mathbf{C}_x \Vdash \varepsilon(f(x), g(x), h(x), k(x))$ as ε is primitive positive. Also, since ε is an encoding formula $\mathbf{C}_x \vDash \varepsilon(f(x), g(x), h(x), k(x))$ iff $[\![f(x) = g(x)]\!] \subseteq [\![h(x) = k(x)]\!]$.

Now define the relation $\sim$ on C^2 by $\langle f, g \rangle \sim \langle h, k \rangle$ iff $\langle f, g \rangle \leq^* \langle h, k \rangle$ and $\langle h, k \rangle \leq^* \langle f, g \rangle$. Then clearly the following holds.

LEMMA 8. $\sim$ *is the equivalence relation on* C^2 *given by* $\langle f, g\rangle \sim \langle h, k\rangle$ *iff for all* $x \in X$, $[\![f(x) = g(x)]\!] = [\![h(x) = k(x)]\!]$.

Let $B = C^2/\sim$, and let $\leq$ be the partial ordering induced by $\leq^*$ on $C^2/\sim$, i.e., $\langle f, g\rangle/\sim \leq \langle h, k\rangle/\sim$ iff $\langle f, g\rangle \leq^* \langle h, k\rangle$.

Let $\leq_M$ be the usual partial order on $\mathbf{M}$, i.e., $m_1 \leq_M m_2$ iff $m_1 \vee m_2 = m_2$. Our immediate objective is to show $\langle B, \leq\rangle \cong \langle M, \leq_M\rangle$. To this end let us define for $f, g \in C$ the element $m_{f,g} \in \prod_{x\in X} 2[\mathbf{B}_x]^*$ by $m_{f,g}(x)^{-1}(1) = [\![f(x) = g(x)]\!]$.

LEMMA 9. $m_{f,g} \in M$ *for* $f, g \in C$.

Proof. Let $A_0 = \mathbb{R}(f) \cup \mathbb{R}(g)$. Then, for $x \in X$,

$$\begin{aligned} m_{f,g}(x)^{-1}(1) &= \bigcup_{a\in A_0} [\![f(x) = a]\!] \cap [\![g(x) = a]\!] \\ &= \bigcup_{a\in A_0} [f(x)^{-1}(a)] \cap [g(x)^{-1}(a)] \\ &= \bigcup_{a\in A_0} \mu_a^f(x)^{-1}(1) \cap \mu_a^g(x)^{-1}(1) \\ &= \left[\left(\bigvee_{a\in A_0} \mu_a^f \wedge \mu_a^g\right)(x)\right]^{-1}(1), \end{aligned}$$

as A_0 is finite. Thus

$$m_{f,g} = \bigvee_{a\in A_0} \mu_a^f \wedge \mu_a^g;$$

hence $m_{f,g} \in M$. $\square$

Let $\beta : C^2 \to M$ be the map $\beta(\langle f, g\rangle) = m_{f,g}$.

LEMMA 10. $\ker \beta = \sim$.

Proof. We have $\beta(\langle f, g\rangle) = \beta(\langle h, k\rangle)$ iff $m_{f,g} = m_{h,k}$ iff for all $x \in X$, $m_{f,g}(x)^{-1}(1) = m_{h,k}(x)^{-1}(1)$ iff for all $x \in X$, $[\![f(x) = g(x)]\!] = [\![h(x) = k(x)]\!]$ iff $\langle f, g\rangle \sim \langle h, k\rangle$. $\square$

Thus we can define a map $\alpha : B \to M$ by $\alpha(\langle f, g\rangle/\sim) = m_{f,g}$.

LEMMA 11. α *is an embedding of* $\langle B, \leq\rangle$ *into* $\langle M, \leq_M\rangle$.

Proof. From the previous lemma it is clear that α is $1-1$. Also

$$\begin{aligned}\langle f, g\rangle/{\sim} \leq \langle h, k\rangle/{\sim} \quad &\text{iff} \quad \langle f, g\rangle \leq^* \langle h, k\rangle \\ &\text{iff} \quad [\![f(x)=g(x)]\!] \subseteq [\![h(x)=k(x)]\!] \quad \text{for} \quad x\in X \\ &\text{iff} \quad m_{f,g}(x)^{-1}(1) \subseteq m_{h,k}(x)^{-1}(1) \quad \text{for} \quad x \in X \\ &\text{iff} \quad m_{f,g} \leq m_{h,k}. \quad \square\end{aligned}$$

LEMMA 12. *α is an isomorphism from $\langle B, \leq\rangle$ to $\langle M, \leq_M\rangle$.*

Proof. We only need to show α is onto. So let $m\in M$, and choose $a_1, a_2 \in A$, $a_1 \neq a_2$. Choose f to be the element of $\prod_{x\in X} C_x$ which satisfies $f(x)^{-1}(a_1)=\mathbf{B}_x^*$ for $x\in X$. Clearly $f\in C$. Then choose the $g\in\prod_{x\in X} C_x$ which satisfies (i) $\mathbb{R}(g)\subseteq\{a_1, a_2\}$, and (ii) $g(x)^{-1}(a_1)=m(x)^{-1}(1)$ for $x\in X$. As $\mu^g_{a_1}=m$, $\mu^g_{a_2}=m'$, and $\mu^g_a=0$ otherwise, it is clear that $g\in C$. Now, for $x\in X$,

$$\begin{aligned}m_{f,g}(x)^{-1}(1) &= [\![f(x)=g(x)]\!] \\ &= g(x)^{-1}(a_1) \\ &= m(x)^{-1}(1),\end{aligned}$$

so $m_{f,g}=m$. Thus $\alpha(\langle f, g\rangle/{\sim})=m$, so α is onto. $\square$

Let M_0 be the set of closed elements of $\mathbf{M}$, i.e., $M_0=\{m\in M: c(m)=m\}$, and let $B_0=\alpha^{-1}(M_0)$.

LEMMA 13. *For $m\in M$, $m\in M_0$ iff $m(x)^{-1}(1)\in\{\phi, B_x^*\}$ for $x\in X$.*

Proof. $m\in M_0$ iff $c(m)=m$. This holds iff $c(m(x))=m(x)$ for $x\in X$. Thus $m\in M_0 \Leftrightarrow m(x)^{-1}(1)\in\{\phi, \mathbf{B}_x^*\}$, for $x\in X$. $\square$

Our final objective is to show that there is a formula $\Delta_0(u, v)$ such that $\mathbf{C}\models\Delta_0(f, g)$ iff $\langle f, g\rangle/{\sim}\in B_0$. To this end choose an encoding formula $\varepsilon_0(x, y, u, v)$ for $P_{\mathbf{B}}(\mathbf{A})'$, and define the relation $\leq_0$ on C^2 by

$$\langle f, g\rangle \leq_0 \langle h, k\rangle \quad \text{iff} \quad \mathbf{C}\models\varepsilon_0(f, g, h, k).$$

LEMMA 14. *For $\langle f, g\rangle\in C$, $\langle f, g\rangle/{\sim}\in B_0$ iff $[\![f(x)=g(x)]\!]\in\{\phi, \mathbf{B}_x^*\}$ for $x\in X$.*

Proof.

$$\begin{aligned}\langle f, g\rangle/\!\sim\, \in B_0 \quad &\text{iff} \quad m_{f,g} \in M_0\\ &\text{iff} \quad m_{f,g}(x)^{-1}(1) \in \{\phi, \mathbf{B}_x^*\}\\ &\text{iff} \quad [\![f(x) = g(x)]\!] \in \{\phi, \mathbf{B}_x^*\}.\end{aligned}$$

LEMMA 15. *For $f, g \in C$ we have*

$$\langle f, g\rangle \leq_0 \langle h, k\rangle \quad \textit{iff} \quad [\![f = g]\!] \subseteq [\![h = k]\!].$$

Proof. Simply observe that (since ε_0 is an encoding formula)

$$\begin{aligned}\langle f, g\rangle \leq_0 \langle h, k\rangle \quad &\text{iff} \quad \mathbf{C} \models \varepsilon(f, g, h, k)\\ &\text{iff} \quad \mathbf{C} \models \varepsilon(f(x), g(x), h(x), k(x)) \quad \text{for} \quad x \in X\\ &\text{iff} \quad f(x) = g(x) \to h(x) = k(x) \quad \text{for} \quad x \in X\\ &\text{iff} \quad [\![f = g]\!] \subseteq [\![h = k]\!]. \quad \square\end{aligned}$$

Thus we have $\leq^* \subseteq \leq_0$. The opposite containment will be used to construct a formula which defines $\beta^{-1}(B_0)$.

LEMMA 16. *For $f, g \in C$*

$$\langle f, g\rangle/\!\sim\, \in B_0 \quad \textit{iff} \quad \langle f, g\rangle \leq_0 \langle h, k\rangle \Rightarrow \langle f, g\rangle \leq^* \langle h, k\rangle \quad \text{for} \quad \langle h, k\rangle \in C^2.$$

Proof. ($\Rightarrow$) Let $\langle f, g\rangle/\!\sim\, \in B_0$. Then $c(m_{f,g}) = m_{f,g}$, so for all $x \in X$, $m_{f,g}(x)^{-1}(1) \in \{\phi, B_x^*\}$. Thus $[\![f(x) = g(x)]\!] \in \{\phi, \mathbf{B}_x^*\}$ for $x \in X$. If $\langle f, g\rangle \leq_0 \langle h, k\rangle$ then, for $x \in X$, $f(x) = g(x)$ implies $h(x) = k(x)$. Hence $f(x) = g(x)$ certainly implies $[\![f(x) = g(x)]\!] \subseteq [\![h(x) = k(x)]\!]$ (as both $= \mathbf{B}_x^*$); and $f(x) \neq g(x)$ also yields $[\![f(x) = g(x)]\!] \subseteq [\![h(x) = k(x)]\!]$ as $[\![f(x) = g(x)]\!] = \emptyset$. Thus $\langle f, g\rangle \leq_0 \langle h, k\rangle$ implies $[\![f(x) = g(x)]\!] \subseteq [\![h(x) = k(x)]\!]$ for all $x \in X$, i.e., $\langle f, g\rangle \leq^* \langle h, k\rangle$.
($\Leftarrow$) Suppose $\langle f, g\rangle \leq_0 \langle h, k\rangle \Rightarrow \langle f, g\rangle \leq^* \langle h, k\rangle$ for $h, k \in C$. Let $h, k \in C$ be such that $m_{h,k} = c(m'_{f,g})'$. Then, for $x \in X$, $m_{h,k}(x)^{-1}(1) = \emptyset$ iff $m_{f,g}(x)^{-1} \neq \mathbf{B}_x^*$, so $[\![h(x) = k(x)]\!] = \emptyset$ iff $[\![f(x) = g(x)]\!] \neq \mathbf{B}_x^*$, i.e., iff $f(x) \neq g(x)$. Thus we have $\langle f, g\rangle \leq_0 \langle h, k\rangle$ since $f(x) = g(x)$ implies $[\![h(x) = k(x)]\!] = B_x^*$ for $x \in X$. Then from $\langle f, g\rangle \leq^* \langle h, k\rangle$ we see that for any $x \in X$ with $f(x) \neq g(x)$ we must have $[\![f(x) = g(x)]\!] \subseteq [\![h(x) = k(x)]\!] = \emptyset$, so indeed $\langle f, g\rangle/\!\sim\, \in B_0$. $\square$

As $\alpha:\langle B, B_0, \leq\rangle \to \langle M, M_0, \leq_M\rangle$ is an isomorphism we have proved that the formulas

$$\Delta(x, y): x \approx x$$
$$\Delta_0(x, y): \forall u\, \forall v[\varepsilon_0(x, y, u, v) \to \varepsilon(x, y, u, v)]$$
$$\rho(x, y, u, v): \varepsilon(x, y, u, v)$$

suffice to semantically embed CA_1 into $P_{\mathbf{B}}(\mathbf{A})'$. This proves the theorem.

Actually our theorem generalizes the case of monadic algebras as the two-element Boolean algebra has the required formula $\varepsilon(x, y, u, v)$ given by $(x\Delta y) \vee (u\Delta v)' = 1$, where Δ is symmetric difference, and since $(BA)^c$ is a coding expansion of BA-namely let $\varepsilon_0(x, y, u, v)$ be $c(u\Delta v) \leq c(x\Delta y)$.

A *discriminator expansion* K' of a class K is an expansion for which there is a discriminator term for K'. Perhaps the simplest discriminator expansion of a class K is obtained by adding a discriminator function t to each member of K – the resulting class is denoted by K^t. From a discriminator term $t(x, y, z)$ we construct the encoding formula $t(x, y, u) \approx t(x, y, v)$. Thus for $\mathbf{A}$ a nontrivial finite algebra, $(P_{\mathbf{B}}(\mathbf{A}^{t_1}))^{t_2}$ has a hereditarily undecidable theory, where t_1, t_2 denote expansions by discriminator functions. If V_0 is a discriminator variety then $V(V_0^t)$ is called an *iterated discriminator variety*. Clearly we have the following.

COROLLARY. *Nontrivial iterated discriminator varieties have hereditarily undecidable theories.*

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