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Free algebras as subdirect products

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If $\mathcal{K}$ is a class of algebras (of a given type) let $F_{\mathcal{K}}(X)$ be the free algebra which is freely generated by X in the variety generated by $\mathcal{K}$. A well-known result of Birkhoff says that $F_{\mathcal{K}}(X)$ can be realized as a subalgebra of a product of algebras from $\mathcal{K}$, i.e. $F_{\mathcal{K}}(X) \in \mathbf{ISP}(\mathcal{K})$. For X sufficiently large we can sharpen this as follows.

THEOREM. *Given a class $\mathcal{K}$ of algebras there is a cardinal m such that if $|X| \geq m$ then $F_{\mathcal{K}}(X)$ can be realized as a subdirect product of algebras from $\mathcal{K}$, i.e. $F_{\mathcal{K}}(X) \in \mathbf{IP}_s(\mathcal{K})$.*

Proof. First let $\mathcal{K}^{\star}$ be a set of algebras from $\mathcal{K}$ which generates the same variety as $\mathcal{K}$, and let m be an infinite upper bound on the size of members of $\mathcal{K}^{\star}$. For X such that $|X| \geq m$ we claim that $F_{\mathcal{K}}(X)$ is in $\mathbf{IP}_s(\mathcal{K}^{\star})$. Indeed if $a, b \in F_{\mathcal{K}}(X)$ and $a \neq b$ then there is a homomorphism $\phi : F_{\mathcal{K}}(X) \to A$ for some $A \in \mathcal{K}^{\star}$ such that $\phi(a) \neq \phi(b)$. As there is a finite $X_1 \subseteq X$ such that a, b both belong to the subalgebra generated by X_1, and as $|X| \geq m$, we can easily modify ϕ to make it onto, completing the proof.

COROLLARY (Kogalovskiĭ (see [1], p. 153)). *For all $\mathcal{K}$, $\mathbf{HSP}(\mathcal{K}) = \mathbf{HP}_s(\mathcal{K})$.*

Proof: Clearly $\mathbf{HSP}(\mathcal{K}) = \mathbf{H}(\{F_{\mathcal{K}}(X) : |X| \geq m\})$, for m as in the Theorem. Thus $\mathbf{HSP}(\mathcal{K}) \subseteq \mathbf{HP}_s(\mathcal{K})$, so $\mathbf{HSP}(\mathcal{K}) = \mathbf{HP}_s(\mathcal{K})$.

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The Theorem was inspired by some recent work of J. Lawrence on regular rings [2].

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