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Remarks on the Fraser–Horn property

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To the memory of András Huhn

In [2] Fraser and Horn studied varieties V with the property that for every $\mathbf{A}, \mathbf{B} \in V$, every congruence θ of $\mathbf{A} \times \mathbf{B}$ is a product congruence $\theta_1 \times \theta_2$. (We will follow the terminology of [1].) This leads us to the following.

DEFINITION. A variety V of algebras has the *Fraser–Horn Property* (FHP) if for every $\mathbf{A}, \mathbf{B} \in V$, all congruences of $\mathbf{A} \times \mathbf{B}$ are product congruences.

Congruence distributive varieties and varieties of rings (with 1) are important examples of classes with the FHP.

The basic result of Fraser and Horn (in the paper cited above) shows that the FHP is a Mal'cev condition for varieties. In this paper we will point out a basic connection between the FHP and principal congruence formulas. Recently H. Riedel [4] has carried out investigations on the role of the FHP in the study of existentially closed algebras.

Our first lemma says that one can check the FHP by looking at principal congruences only.

LEMMA 1. *A variety V has the* FHP *iff for $a_i, b_i, c_i, d_i \in A_i$, $\mathbf{A}_i \in V$, $i = 1, 2$:*

$$\langle a_i, b_i \rangle \in \Theta_{\mathbf{A}_i}(c_i, d_i), \qquad i = 1, 2, \quad \textit{implies} \quad \langle \vec{a}, \vec{b} \rangle \in \Theta_{\mathbf{A}_1 \times \mathbf{A}_2}(\vec{c}, \vec{d}).$$

Proof. ($\Rightarrow$) Since $\Theta_{\mathbf{A}_1 \times \mathbf{A}_2}(\vec{c}, \vec{d})$ decomposes (by the FHP), say $\Theta_{\mathbf{A}_1 \times \mathbf{A}_2}(\vec{c}, \vec{d}) = \theta_1 \times \theta_2$, and $\langle c_i, d_i \rangle \in \theta_i$, it follows that $\Theta_{\mathbf{A}_i}(c_i, d_i) \subseteq \theta_i$. Thus $\langle a_i, b_i \rangle \in \theta_i$, so $\langle \vec{a}, \vec{b} \rangle \in \theta_1 \times \theta_2 = \Theta_{\mathbf{A}_1 \times \mathbf{A}_2}(\vec{c}, \vec{d})$.

($\Leftarrow$) Let θ be a congruence on $\mathbf{A}_1 \times \mathbf{A}_2$, and let $\langle \vec{c}, \vec{d} \rangle \in \theta$. For $\langle a_i, b_i \rangle \in \Theta_{\mathbf{A}_i}(c_i, d_i)$, $i = 1, 2$ we have (by our hypothesis) $\langle \vec{a}, \vec{b} \rangle \in \Theta(\vec{c}, \vec{d}) \subseteq \Theta$. In particular it follows that $\Theta_{\mathbf{A}_1}(c_1, d_1) \times \Delta_{\mathbf{A}_2} \subseteq \theta$. Consequently $\Theta_{\mathbf{A}_1}(\theta|_{\mathbf{A}_1}) \times \Delta_{\mathbf{A}_2} \subseteq \theta$, where $\theta|_{\mathbf{A}_1} = \{\langle c_1, d_1 \rangle : \langle \vec{c}, \vec{d} \rangle \in \theta$ for some $c_2, d_2\}$; and likewise $\Delta_{\mathbf{A}_1} \times \Theta_{\mathbf{A}_2}(\theta|_{\mathbf{A}_2}) \subseteq \theta$, so $\Theta_{\mathbf{A}_1}(\theta|_{\mathbf{A}_1}) \times \Theta_{\mathbf{A}_2}(\theta|_{\mathbf{A}_2}) \subseteq \theta$. But then $\theta = \Theta_{\mathbf{A}_1}(\theta|_{\mathbf{A}_1}) \times \Theta_{\mathbf{A}_2}(\theta|_{\mathbf{A}_2})$.

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This leads to a new characterization of the varieties with equationally definable principal congruences introduced by Fried, Grätzer and Quackenbush [3]. But first some definitions, where V is a variety.

DEFINITION. V has definable principal congruences (DPC) if for some first-order $\phi(x, y, u, v)$ we have, for all $a, b, c, d \in A$, $\mathbf{A} \in V$, $\langle a, b\rangle \in \Theta_{\mathbf{A}}(c, d)$ iff $\mathbf{A} \models \phi(a, b, c, d)$. Such a ϕ is said to *define principal congruences* in V. V has *equationally definable principal congruences* (EDPC) if some positive primitive $\phi(x, y, u, v)$ (i.e. of the form "$\exists$& eqn's") defines principal congruences in V. V has a *uniform congruence scheme* (UCS) if some principal congruence formula $\pi(x, y, u, v)$ defines principal congruences in V.

THEOREM 4. *A variety V has equationally definable principal congruences iff it has the Fraser–Horn Property and definable principal congruences* (*i.e.*, EDPC = FHP + DPC).

Proof. ($\Rightarrow$) First EDPC $\Rightarrow$ FHP, for if $\phi(x, y, u, v)$ is a positive primitive formula defining principle congruences in V, then for a_i, b_i, c_i, $d_i \in A_i$, $\mathbf{A}_i \in V$, $i = 1, 2$, $\mathbf{A}_i \models \phi(a_i, b_i, c_i, d_i)$ implies $\mathbf{A}_1 \times \mathbf{A}_2 \models \phi(\vec{a}, \vec{b}, \vec{c}, \vec{d})$. So Lemma 1 applies. Clearly EDPC $\Rightarrow$ DPC.

($\Leftarrow$) Since DPC holds we can choose PCF's $\pi_1, \ldots, \pi_n$ such that $\pi_1 \vee \cdots \vee \pi_n$ defines principal congruences. As we have Dir PCF's, let π be a PCF such that $\pi_1 \vee \cdots \vee \pi_n \rightarrow \pi$. Then π defines principal congruences, so we have EDPC.

COROLLARY 5. (Fried/Grätzer/Quackenbush [3]). *A variety V has equationally definable principal congruences iff it has a uniform congruence scheme.*

Proof. ($\Leftarrow$) (Clear).

($\Rightarrow$) Use the ($\Leftarrow$) part of the previous proof.

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Recall that *principal congruence formulas* (PCF's) $\pi(x, y, u, v)$ are special first-order formulas of the form $\exists\&$ atomic which formalize Mal'cev's description of how to generate the elements in a principal congruence. If π is a principal congruence formula and $\mathbf{A} \models \pi(a, b, c, d)$ then π is said *to witness* the fact that $\langle a, b\rangle \in \Theta_{\mathbf{A}}(c, d)$.

The next lemma describes what it means to witness in algebraic terms, i.e., in terms of the existence of a homomorphism.

LEMMA 2. *For $\pi(x, y, u, v)$ a principal congruence formula, say $\exists z_1 \cdots \exists z_m [\&_{1\le i\le n}\, p_i(x, y, u, v, \vec{z}) \approx q_i(x, y, u, v, \vec{z})]$, there is an algebra $\mathbf{A}_\pi \in V$ and elements $a_\pi, b_\pi, c_\pi, d_\pi \in A_\pi$ such that for any $a, b, c, d \in A$, $\mathbf{A} \in V$, $\mathbf{A} \models \pi(a, b, c, d)$ iff there is a homomorphism*

$$\alpha : \langle A_\pi, a_\pi, b_\pi, c_\pi, d_\pi\rangle \to \langle A, a, b, c, d\rangle.$$

Proof. Let $\mathbf{A}_\pi$ be the algebra in V with the presentation

$$\langle a_\pi, b_\pi, c_\pi, d_\pi, g_1, \ldots, g_m : p_i(a_\pi, b_\pi, c_\pi, d_\pi, \vec{g}) \approx q_i(a_\pi, b_\pi, c_\pi, d_\pi, \vec{g})\rangle \qquad 1 \le i \le n.$$

DEFINITION. We say $\langle A_\pi, a_\pi, b_\pi, c_\pi, d_\pi\rangle$ is *universal for* π.

DEFINITION. A variety V has directed principal congruence formulas (Dir PCF) if for any principal congruence formulas π_1, π_2 there is a principal congruence formula π such that $V \models \pi_1 \vee \pi_2 \to \pi$.

Now we can prove our main result.

THEOREM 3. *A variety V has the Fraser Horn Property iff it has directed principal congruence formulas.*

Proof. ($\Rightarrow$) Let π_1, π_2 be PCF's. Let $\langle A_1, a_1, b_1, c_1, d_1\rangle$, $\langle A_2, a_2, b_2, c_2, d_2\rangle$ be universal for π_1, resp. π_2. Then by the FHP (see Lemma 1) $\langle \vec{a}, \vec{b}\rangle \in \Theta_{\mathbf{A}_1\times\mathbf{A}_2}(\vec{c}, \vec{d})$. Let π be a PCF which witnesses $\langle \vec{a}, \vec{b}\rangle \in \Theta_{\mathbf{A}_1\times\mathbf{A}_2}(\vec{c}, \vec{d})$. Then $\mathbf{A}_i \models \pi(a_i, b_i, c_i, d_i)$ for $i = 1, 2$, so by Lemma 2, $\pi_1 \vee \pi_2 \to \pi$.

($\Leftarrow$) For $a_i, b_i, c_i, d_i \in A_i$ with $\langle a_i, b_i\rangle \in \Theta_{\mathbf{A}_i}(c_i, d_i)$, $i = 1, 2$, choose PCF's π_i with $\mathbf{A}_i \models \pi_i(a_i, b_i, c_i, d_i)$. Choose a PCF π such that $\pi_1 \vee \pi_2 \to \pi$. Then π witnesses $\langle \vec{a}, \vec{b}\rangle \in \Theta_{\mathbf{A}_1\times\mathbf{A}_2}(\vec{c}, \vec{d})$. By Lemma 1, V has the FHP.