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An example concerning definable principal congruences

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In [1] Baldwin and Berman ask if a variety generated by a finite algebra $\mathfrak{A}$ has definable principal congruences, i.e. is there a first-order formula $\Phi(x, y, u, v)$ such that for any $\mathfrak{B} \in \mathcal{V}(\mathfrak{A})$ and a, b, c, d in $\mathfrak{B}$ we have $\langle c, d\rangle \in \theta(a, b)$ iff $\mathfrak{B} \models \Phi(a, b, c, d)$. *In the following we describe a four-element algebra $\mathfrak{A}$ such that $\mathcal{V}(\mathfrak{A})$ has distributive congruence lattices and does not have definable principal congruences.*

Let $\mathfrak{A} = \langle A, +, t\rangle$ where $A = \{0, 1, 2, 3\}$ and the operations are given by

$$x + y = \begin{cases} 3 & \text{if} \quad \langle x, y\rangle = \langle 2, 1\rangle \\ x & \text{if} \quad \langle x, y\rangle \neq \langle 2, 1\rangle \end{cases}$$

and

$$t(x, x, y) = t(x, y, x) = t(y, x, x) = x,$$
$$t(x, y, z) = 0 \quad \text{otherwise.}$$

Let $\mathfrak{A}_0$ be the subalgebra $\langle\{0, 2, 3\}, +, t\rangle$ of $\mathfrak{A}$. Note that $\langle 0, 3\rangle \notin \theta_{\mathfrak{A}_0}(0, 2)$, whereas $\langle 0, 3\rangle \in \theta_{\mathfrak{A}}(0, 2)$.

Next, for $n \geq 1$ $(n = \{0, 1, \ldots, n-1\})$, define $\mathfrak{A}_n = \langle A_n, +, t\rangle$ to be the subdirect product of $\mathfrak{A}$ with universe

$$A_n = \{0, 2, 3\}^n \cup \{h_i : 0 \leq i \leq n-1\},$$

where

$$h_i(j) = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases}.$$

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Let **0**, **2**, and **3** be those constant functions in A_n whose values are 0, 2, and 3 respectively. Since $(\cdots(\mathbf{0}+h_0)+\cdots+h_{n-1})=\mathbf{0}$ and $(\cdots(\mathbf{2}+h_0)+\cdots+h_{n-1})=\mathbf{3}$ it follows that $\langle\mathbf{0},\mathbf{3}\rangle\in\theta_{\mathfrak{A}_n}(\mathbf{0},\mathbf{2})$. From our previous observation on $\mathfrak{A}_0$ and the fact that only the h_i's can assume the value 1 we see that to derive $\langle\mathbf{0},\mathbf{3}\rangle\in\theta_{\mathfrak{A}_n}(\mathbf{0},\mathbf{2})$ using the usual Mal'cev sequences of algebraic functions we are forced to incorporate all the h_i's as parameters in the algebraic functions. Now letting $\mathcal{U}$ be a non-principal ultrafilter on the natural numbers N the last observation implies $\langle\mathbf{0},\mathbf{3}\rangle\notin\theta_{\mathfrak{A}_\infty}(\mathbf{0},\mathbf{2})$ where $\mathfrak{A}_\infty=\prod_{n\in N}\mathfrak{A}_n/\mathcal{U}$; hence principal congruences are not definable in $\mathcal{V}(\mathfrak{A})$.

Remark. Recently McKenzie [2] has proved (February, 1976) that every non-distributive variety of lattices does not have definable principal congruences. However the following is still open.

PROBLEM. Is there an example of a variety $\mathcal{V}(\mathfrak{A})$ where $\mathfrak{A}$ is a finite algebra and $\mathcal{V}(\mathfrak{A})$ has permutable congruences but $\mathcal{V}(\mathfrak{A})$ does not have definable principal congruences?

REFERENCES

[1] JOHN T. BALDWIN and JOEL BERMAN, *The number of subdirectly irreducible algebras in a variety.* Alg. Univ. 5 (1975), 379–389.

[2] RALPH MCKENZIE, *Para primal varieties: a study of finite axiomatizability and definable principal congruences in locally finite varieties.* Preprint.

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