Embedding the Dual of Π_∞ in the Lattice of Equational Classes of Semigroups

S. BURRIS and E. NELSON

Reprint from
algebra universalis
Vol. 1, fasc. 2, 1971 BIRKHÄUSER VERLAG BASEL pages 248–253

Embedding the Dual of Π_∞ in the Lattice of Equational Classes of Semigroups

S. BURRIS[1]) and E. NELSON[2])

0. Introduction

This paper is an addition to the many investigations which have been made into the structure of lattices of equational theories. (For an excellent survey of recent work in equational logic we refer the reader to A. Tarski [12].) As an introduction to this paper we would like to mention a few results which have influenced the curiosity and work of the authors. In [7] E. Jacobs and R. Schwabauer gave a description of the lattice of equational classes of mono-unary algebras; however, much of the recent progress has been made in the study of equational classes of semigroups. J. A. Gerhard has presented in [5] a detailed (and deep) analysis of equational classes of idempotent semigroups. P. Perkins [10] proved that every equational theory of commutative semigroups is finitely based (hence the lattice of equational classes of commutative semigroups is countable). In [4], the authors showed that this lattice does not satisfy any special lattice identities. The next logical step would be to study the uniformly periodic semigroups (i.e. those satisfying an identity of the form $x^m = x^{m+n}$); our main result shows that the structure of the lattice of equational classes of semigroups satisfying $x^2 = x^3$ is considerably more complex than in the idempotent case – namely, the equation $x = x^2$ leads to a countable distributive lattice of width three, whereas the lattice for $x^2 = x^3$ contains a subinterval isomorphic to the dual of the partition lattice of a denumerable set. (Similar results were obtained in considerably simpler cases by S. Burris [3].) It should be pointed out that K. Baker [1] and R. McKenzie [9] have published a related result – namey they have embedded the lattice of subsets of a denumerable set onto a subinterval of the lattice of equational theories of lattices.

1. Preliminaries

Let $\mathfrak{F}(X)$ be the *free semigroup* on the countable set

$$X = \{x_i \mid i \geqslant 0\},$$

and let $\mathfrak{F}(X)^*$ be $\mathfrak{F}(X)$ with a unit, λ, adjoined. A *semigroup equation* (henceforth referred to simply as an *equation*) is a pair of elements of $\mathfrak{F}(X)$. Let E be the set of all

[1]) Research supported by a Summer Research Stipend from the University of Waterloo.
[2]) Research supported by NRC grant A-2985.

Presented by R. McKenzie. Received February 20, 1971. Accepted for publication in final form May 12, 1971.

endomorphisms of $\mathfrak{F}(X)$, $E^\perp$ the set of those mappings in E which, when restricted to X, are one-to-one and onto X. An equation (p, q) is obtained from an equation (r, s) by *substitution* if there exists $\sigma\in E$ such that $p=\sigma(r)$, $q=\sigma(s)$. (p, q) is obtained from (r, s) by *multiplication* if there exist $t_1, t_2\in\mathfrak{F}(X)^*$ such that $p=t_1rt_2$, $q=t_1st_2$. A set of equations is *closed* if it is a fully invariant congruence relation on $\mathfrak{F}(X)$, that is, if it is reflexive, symmetric, and transitive, and closed under substitution and multiplication. For a set Σ of equations we will write $\Gamma\Sigma$ for the closure of Σ.

We will give an embedding of Π_∞ (the lattice of partitions of the natural numbers) into the lattice of closed sets of equations containing the equation (x_0^2, x_0^3). The following theorem, which may be found in [8], will be needed.

THEOREM 1. *If Σ is a symmetric set of equations, then $(p, q)\in\Gamma\Sigma$ iff there exist $r_1,\ldots, r_k, u_1,\ldots, u_{k-1}, v_1,\ldots, v_{k-1}$ in $\mathfrak{F}(X)$, $s_1,\ldots, s_{k-1}, t_1,\ldots, t_{k-1}$ in $\mathfrak{F}(X)^*$, and $\sigma_1,\ldots, \sigma_{k-1}$ in E such that*

$$r_1=p,\ r_k=q,\ r_i=s_i\sigma_i(u_i)\,t_i,\ r_{i+1}=s_i\sigma_i(v_i)\,t_i \text{ and } (u_i, v_i)\in\Sigma,$$

for $1\leqslant i\leqslant k-1$; or $p=q$.

2. Asymmetric Sequences and a Special Class of Polynomials

In [13] A. Tue constructs a countable sequence consisting of the numbers 1, 2, and 3 which is asymmetric, that is, it does not contain any subsequence $a_i\ldots a_{i+k}$ of the form AA.[3] It follows that there are countably many different elements of $\mathfrak{F}(X)$ containing only the variables x_0, x_1, x_2, that start with x_2, end with x_0, and are square-free (that is, they do not contain any subterm of the form p^2). Thus we may choose, for each $n\in N$ (where N is the set of natural numbers), $p_n\in\mathfrak{F}(X)$ such that:

(1) p_n contains the variables x_0, x_1, x_2, and only these;

(2) $x_2p_nx_0$ is square-free;

(3) $m\neq n$ implies the length of p_m is not equal to the length of p_n.

For each $n\in N$ let

$$P_n=\{x_1x_2^{n_2}p_nx_0^{n_0}x_1 \mid n_0, n_2\geqslant 2\}.$$

For an arbitrary set of equations Σ, let

$$\Sigma^+=\Sigma\cup\{(x_0^2, x_0^3), (x_0^3, x_0^2)\},$$

and let

$$\Sigma^\perp=\{(\sigma(p), \sigma(q)) \mid (p, q)\in\Sigma,\ \sigma\in E^\perp\}.$$

For $S\subseteq N\times N$, let

$$\Sigma_S=\{(p, q) \mid \text{there exist}\,(m, n)\in S \text{ with } p\in P_m,\ q\in P_n\}.$$

[3]) The authors are indebted to J. A. Brzozowski for this reference.

LEMMA 1. *If* $(p, q)\in\Gamma\{(x_0^2, x_0^3)\}$ *and* $p\in P_n$, *then* $q\in P_n$.

Proof. This is an easy consequence of the observation that the only subterms of $x_1x_2^{n_2}p_nx_0^{n_0}x_1$ which are of the form r^k, $k\geqslant 2$, are contained in the two subterms $x_2^{n_2}$ and $x_0^{n_0}$.

LEMMA 2. *If* $s, t\in\mathfrak{F}(X)^*$, $\sigma\in E$, $p\in P_m$ *and* $s\sigma(p)t\in P_n$ *then one of the following holds:*

(1) $s=t=\lambda$, $m=n$, *and* $\sigma(x_i)=x_i$ *for* $i=0, 1, 2$;

(2) $\sigma(x_i)=x_2^{k_i}$ *for some* $k_i\geqslant 1$, $i=0,1,2$, $s=x_1x_2^k$ *for some* $k\geqslant 0$, *and* $x_1x_2^2t\in P_n$;

(3) $\sigma(x_i)=x_0^{k_i}$ *for some* $k_i\geqslant 1$, $i=0,1,2$, $t=x_0^kx_1$ *for some* $k\geqslant 0$, *and* $sx_0^2x_1\in P_n$.

Proof. If $p\in P_m$ then $p=x_1x_2^{m_2}p_mx_0^{m_0}x_1$ for some $m_0, m_2\geqslant 2$. Thus there exist $n_0, n_2\geqslant 2$ such that

$$x_1x_2^{n_2}p_nx_0^{n_0}x_1 = s\sigma(x_1)\,\sigma(x_2)^{m_2}\,\sigma(p_m)\,\sigma(x_0)^{m_0}\,\sigma(x_1)\,t\,.$$

The result follows from the fact that $x_2p_mx_0$ and $x_2p_nx_0$ are square-free, and of different lengths if $m\neq n$.

LEMMA 3. *If* $S\subseteq N\times N$ *is reflexive, symmetric, and transitive, and if* $(p, q)\in\Gamma\Sigma_S^+$ *and* $p\in P_n$ *then* $(p, q)\in\Sigma_S$.

Proof. In view of Theorem 1 it suffices to show that the following statement holds for all $k\geqslant 1$:

(T_k): for all $r_1,\ldots, r_k, u_1,\ldots, u_{k-1}, v_1,\ldots, v_{k-1}$ in $\mathfrak{F}(X)$, $s_1,\ldots, s_{k-1}, t_1,\ldots, t_{k-1}$ in $\mathfrak{F}(X)^*$, and $\sigma_1,\ldots,\sigma_{k-1}$ in E, if

$$r_i = s_i\sigma_i(u_i)\,t_i,\quad r_{i+1} = s_i\sigma_i(v_i)\,t_i \quad\text{and}\quad (u_i, v_i)\in\Sigma_S^+$$

for $1\leqslant i\leqslant k-1$, and if $r_1\in P_n$, then $(r_1, r_k)\in\Sigma_S$.

The proof is by induction on k. The case $k=1$ is trivial since S is reflexive, so assume (T_{k-1}) holds for some $k\geqslant 2$, and that we have the r_i, u_i, v_i, s_i, t_i, and σ_i as in the hypotheses of (T_k).

If $(u_1, v_1)\in\{(x_0^2, x_0^3), (x_0^3, x_0^2)\}$, then, by Lemma 1, $(r_1, r_2)\in\Gamma\{(x_0^2, x_0^3)\}$ and $r_1\in P_n$ implies $r_2\in P_n$. By the induction hypothesis, $(r_2, r_k)\in\Sigma_S$, hence $(r_1, r_k)\in\Sigma_S$.

Otherwise, there exist h, m with $u_1\in P_h$, $v_1\in P_m$ and $(h, m)\in S$. But $s_1\sigma_1(u_1)\,t_1 = r_1\in P_n$ and $u_1\in P_h$, thus we can apply Lemma 2. If $s_1=t_1=\lambda$, $n=h$, and $\sigma_1(x_i)=x_i$ for $i=0, 1, 2$, then $r_1=u_1$, $r_2=v_1$, thus $(r_1, r_2)\in\Sigma_S$. By the induction hypothesis, $(r_2, r_k)\in\Sigma_S$. Since S is transitive it follows that $(r_1, r_k)\in\Sigma_S$. If $\sigma_1(x_i)=x_2^{j_i}$ for $i=0, 1, 2$, $s_1=x_1x_2^j$ for some $j\geqslant 0$, and $x_1x_2^2t_1\in P_n$, then $r_2=s_1\sigma_1(v_1)\,t_1\in P_n$. Applying the induction hypothesis, $(r_2, r_k)\in\Sigma_S$, and thus $(r_1, r_k)\in\Sigma_S$. Similarly, if $\sigma_1(x_i)=x_0^{j_i}$ for $i=0, 1, 2$, $t_1=x_0^jx_1$ for some $j\geqslant 0$ and $s_1x_0^2x_1\in P_n$, then $(r_1, r_k)\in\Sigma_S$. This completes the proof.

COROLLARY. *If* $S\subseteq N\times N$ *is reflexive, transitive, and symmetric, and if*

$(p, q)\in\Gamma\Sigma_S^+$ and $p=\sigma(p')$ for $p'\in P_n$, $\sigma\in E^\perp$, then there exists q' such that $q=\sigma(q')$ and $(p', q')\in\Sigma_S$.

Let

$$R = N \times N - \{(n, n) \mid n\in N\}$$

and let

$$\mathbf{T} = \Gamma\Sigma_{N\times N}^+ - \Sigma_R^\perp .$$

LEMMA 4. **T** *is closed.*

Proof. If **T** is not closed, then there exist $\sigma\in E^\perp$, $p\in P_m$, $q\in P_n$, $m\neq n$, with $(\sigma(p), \sigma(q))\in\Gamma\mathbf{T}$. But then $(p, q)\in\Gamma\mathbf{T}$. **T** is clearly reflexive and symmetric, and, by Lemma 3, is also transitive. Thus it is enough to show that if $(p, q)\in\mathbf{T}$ and if $s\sigma(p)\,t\in P_m$ then $s\sigma(q)\,t\in P_m$.

Let s, $t\in\mathfrak{F}(X)^*$, $\sigma\in E$ be arbitrary, but fixed for the rest of this proof. Again by Theorem 1, it is enough to prove that the following statement holds for all $k\geqslant 1$:

(S_k): For all $r_1,\ldots, r_k, u_1,\ldots, u_{k-1}, v_1,\ldots, v_{k-1}$ in $\mathfrak{F}(X)$, $s_1,\ldots, s_{k-1}, t_1,\ldots, t_{k-1}$ in $\mathfrak{F}(X)^*$, $\sigma_1,\ldots, \sigma_{k-1}$ in E, if

$$r_i = s_i\sigma_i(u_i)\,t_i,\; r_{i+1} = s_i\sigma_i(v_i)\,t_i \quad\text{and}\quad (u_i, v_i)\in\Sigma_{N\times N}^+$$

for $1\leqslant i\leqslant k-1$ and if $(r_1, r_k)\in\mathbf{T}$ and if $s\sigma(r_1)\,t\in P_m$, then $s\sigma(r_k)\,t\in P_m$.

The proof is by induction on k. The case $k=1$ is trivial, so assume (S_{k-1}) holds for some $k\geqslant 2$. If we are given the r_i, s_i, t_i, u_i, v_i, and σ_i as in the hypotheses of (S_k), then we consider two cases.

Case 1. $(u_1, v_1)\in\{(x_0^2, x_0^3), (x_0^3, x_0^2)\}$. Then

$$(s\sigma(s_1\sigma_1(u_1)\,t_1)\,t,\, s\sigma(s_1\sigma_1(v_1)\,t_1)\,t)\in\Gamma\{(x_0^2, x_0^3)\}.$$

Since $s\sigma(s_1\sigma_1(u_1)\,t_1)\,t=s\sigma(r_1)\,t\in P_m$ it follows by Lemma 1 that $s\sigma(r_2)\,t=s\sigma(s_1\sigma_1(v_1)\,t_1)\,t\in P_m$. Moreover, $(r_1, r_2)\in\Gamma\{(x_0^2, x_0^3)\}\subseteq\mathbf{T}$, and thus, since **T** is transitive, $(r_2, r_k)\in\mathbf{T}$. By the induction hypothesis, $s\sigma(r_k)\,t\in P_m$.

Case 2. $(u_1, v_1)\in\Sigma_{N\times N}$. Then $u_1\in P_j$, $v_1\in P_h$ for some j, h. Since $s\sigma(s_1)\,\sigma\sigma_1(u_1)\times\times\sigma(t_1)\,t\in P_m$ we can apply Lemma 2.

If $\sigma\sigma_1(x_i)=x_i$ for $i=0, 1, 2$, $s=s_1=t_1=t=\lambda$ and $m=j$, then σ_1 maps x_0, x_1, and x_2 one-to-one into X, and $r_1=\sigma_1(u_1)$. But then, since $(\sigma_1(u_1), r_k)\in\mathbf{T}$, by the corollary to Lemma 3 it follows that there exists an r such that $\sigma_1(r)=r_k$ and $(u_1, r)\in\Sigma_{N\times N}$. Since $(\sigma_1(u_1), \sigma_1(r))\in\mathbf{T}$ it follows that $r\in P_m$. But then $s\sigma(r_k)\,t=\sigma\sigma_1(r)=r\in P_m$.

If $\sigma\sigma_1(x_i)=x_2^{a_i}$ for $i=0, 1, 2$, and $s\sigma(s_1)=x_1x_2^a$ for some $a\geqslant 0$ and $x_1x_2^2\sigma(t_1)\,t\in P_m$, then $s\sigma(r_2)\,t=s\sigma(s_1)\,\sigma\sigma_1(v_1)\,\sigma(t_1)\,t\in P_m$, and the result follows by the induction hypothesis. Similarly the result follows in the case $\sigma\sigma_1(x_i)=x_0^{a_i}$ for $i=0, 1, 2$, $\sigma(t_1)\,t=x_0^a x_1$ for some $a\geqslant 0$, and $s\sigma(s_1)\,x_0^2x_1\in P_m$. This completes the proof.

LEMMA 5. *If $S \subseteq N \times N$ is an equivalence relation on N, then $\mathbf{T} \cup \Sigma_S^{\perp}$ is closed.*

Proof. If $(p, q) \in \Sigma_S^{\perp}$, $s, t \in \mathfrak{F}(X)^*$, and $\sigma \in E$, then if $s = t = \lambda$ and $\sigma \in E^{\perp}$, then $(s\sigma(p)\, t, s\sigma(q)\, t) \in \Sigma_S^{\perp}$; otherwise we have $(s\sigma(p)\, t, s\sigma(q)\, t) \in \mathbf{T}$. Thus $\mathbf{T} \cup \Sigma_S^{\perp}$ is closed under multiplication and substitution. $\mathbf{T} \cup \Sigma_S^{\perp}$ is clearly reflexive, transitive, and symmetric – and this completes the proof.

3. The Embedding

Let $\mathscr{L}$ be the lattice of closed sets of equations, and $\mathscr{L}_E$ the lattice of equivalence relations on N (where, in both cases, the partial ordering is set-theoretical containment). Let $\wedge$ and $\vee$ denote the meet, respectively join, in the two lattices.

Define the mapping θ from $\mathscr{L}_E$ into $\mathscr{L}$ by $\theta(S) = \mathbf{T} \cup \Sigma_S^{\perp}$. If S and S' are equivalence relations on N, then

$$\theta(S \wedge S') = \mathbf{T} \cup \Sigma_{S \wedge S'}^{\perp} = \mathbf{T} \cup (\Sigma_S^{\perp} \cap \Sigma_{S'}^{\perp}) = (\mathbf{T} \cup \Sigma_S^{\perp}) \cap (\mathbf{T} \cup \Sigma_{S'}^{\perp}) = \theta(S) \wedge \theta(S').$$

Also

$$\theta(S) \vee \theta(S') = \Gamma(\mathbf{T} \cup \Sigma_S^{\perp} \cup \Sigma_{S'}^{\perp}) \subseteq \Gamma(\mathbf{T} \cup \Sigma_{S \vee S'}^{\perp}) = \mathbf{T} \cup \Sigma_{S \vee S'}^{\perp} = \theta(S \vee S').$$

On the other hand, $\Sigma_{S \vee S'}$ is contained in the transitive closure of $\Sigma_S \cup \Sigma_{S'}$, hence in $\Gamma(\Sigma_S \cup \Sigma_{S'})$. But then $\mathbf{T} \cup \Sigma_{S \vee S'}^{\perp} \subseteq \Gamma(\mathbf{T} \cup \Sigma_S^{\perp} \cup \Sigma_{S'}^{\perp})$, hence $\theta(S \vee S') \subseteq \theta(S) \vee \theta(S')$. Thus θ is a lattice homomorphism.

If Σ is a closed subset of semigroup equations with $\mathbf{T} \subseteq \Sigma \subseteq \mathbf{T} \cup \Sigma_{N \times N}^{\perp}$, then one can easily verify that $\Sigma = \mathbf{T} \cup \Sigma_S^{\perp}$ for some equivalence relation S on N. Hence the $\mathbf{T} \cup \Sigma_S^{\perp}$ exhaust the subinterval $[\mathbf{T}, \mathbf{T} \cup \Sigma_{N \times N}^{\perp}]$ of $\mathscr{L}$.

Furthermore, if $S \neq S'$, then $\Sigma_S^{\perp} \neq \Sigma_{S'}^{\perp}$, and consequently $\theta(S) \neq \theta(S')$. Thus θ is an embedding of $\mathscr{L}_E$ onto a subinterval of $\mathscr{L}$. Since $(x_0^2, x_0^3) \in \theta(S)$ for $S \in \mathscr{L}_E$, and since $\mathscr{L}_E$ is isomorphic to Π_∞ (the lattice of partitions of N) and $\mathscr{L}$ is dually isomorphic to the lattice of equational classes of semigroups, we have the following theorem.

THEOREM 2. *The lattice of equational classes of semigroups satisfying $x^2 = x^3$ contains a subinterval isomorphic to the dual of Π_∞.*

COROLLARY. *The lattice of equational classes of semigroups satisfying $x^2 = x^3$ does not satisfy any special lattice laws.*

Proof. D. Sachs [11] proved Π_∞ does not satisfy any special lattice laws.

REFERENCES

[1] K. Baker, *Equational classes of modular lattices*, Pacific J. Math. *28* (1968), 9–15.
[2] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Publ., vol. *25*.
[3] S. Burris, *On the structure of the lattice of equational classes* $\mathscr{L}(\tau)$, Alg. Universalis (to appear).

[4] S. Burris, and E. Nelson, *Embedding the dual of Π_m in the lattice of equational classes of commutative semigroups*, Proc. Amer. Math. Soc. *30* (1971) 37–39.
[5] J. A. Gerhard, *The lattice of equational classes of idempotent semigroups*, J. Algebra *15* (1970), 195–224.
[6] G. Grätzer, *Universal Algebra*, (Van Nostrand, Princeton, New Jersey, 1968).
[7] E. Jacobs and R. Schwabauer, *The lattice of equational classes of algebras with one unary operation*, Amer. Math. Monthly *71* (1964), 151–154.
[8] A. Kurosh, *General Algebra*, (Chelsea, New York, 1963).
[9] R. McKenzie, *Equational bases for lattice theories*, Math. Scand. *26* (1970), 24–38.
[10] P. Perkins, *Bases for equational theories of semigroups*, J. of Algebra *11* (1969), 298–314.
[11] D. Sachs, *Identities in finite partition lattices*, Proc. Amer. Math. Soc. *12* (1969), 944–945.
[12] A. Tarski, *Equational logic and equational theories of algebrac*, Proceedings of the Logic Colloquium, Hannover, 1966, pp. 275–288.
[13] A. Thue, *Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen*. Videnskabsselskabets Skrifter, I Mat.-nat. K., Christiania 1912.

University of Waterloo,
Waterloo, Ontario, Canada
and
McMaster University,
Hamilton, Ontario, Canada