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Discriminator polynomials and arithmetical varieties

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A finite simple algebra in an arithmetical variety is functionally complete, and thus there is a polynomial which defines the ternary discriminator on it (see Werner [6], [7]). For locally finite semisimple arithmetical varieties, certain terms exist which provide discriminator polynomials (in a uniform manner) for all the finitely generated simple algebras in the variety. This leads to a new proof of a result of Pixley characterizing finitely generated discriminator varieties.

We follow the notation of [2]. If V is a variety, V_S denotes the class of simple algebras in V. An algebra is *hereditarily simple* if every subalgebra is simple. For $\mathbf{A}$ an algebra, a polynomial $t(x, y, z, a_1, \ldots, a_n)$ is a *discriminator polynomial* for $\mathbf{A}$ if $\mathbf{A}$ satisfies

$$[x \approx y \rightarrow t(x, y, z, a_1, \ldots, a_n) \approx z] \,\&\, [x \not\approx y \rightarrow t(x, y, z, a_1, \ldots, a_n) \approx x].$$

THEOREM 1. *Let V be a locally finite semisimple arithmetical variety. Then for each $n < \omega$ there is a term $t_n(x, y, z, u_1, \ldots, u_n)$ such that for $\mathbf{S} \in V_S$ and $\{s_1, \ldots, s_n\}$ a set of generators for $\mathbf{S}$ the polynomial $t_n(x, y, z, s_1, \ldots, s_n)$ is a discriminator polynomial for $\mathbf{S}$.*

Proof. From Foster-Pixley [4] we know that each finite member of V is isomorphic to a direct product of simple algebras. Thus the congruence lattice of each finite member is a Boolean lattice. Consider the free algebra $F_V(\bar{x}, \bar{y}, \bar{z}, \bar{u}_1, \ldots, \bar{u}_n)$. Let θ^* be the complement of $\theta(\bar{x}, \bar{y})$ in the interval $[\Delta, \theta(\bar{x}, \bar{y}, \bar{z})]$. Then $\langle \bar{z}, \bar{x} \rangle \in \theta(\bar{x}, \bar{y}) \vee \theta^*$, so we can choose a term $t_n(x, y, z, u_1, \ldots, u_n)$ such that

$$\bar{z}\theta(\bar{x}, \bar{y})t_n(\bar{x}, \bar{y}, \bar{z}, u_1, \ldots, \bar{u}_n)\theta^*\bar{x}.$$

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Let $\mathbf{S} \in V_S$ be generated by n-elements $s_1, \ldots, s_n$. Then for $a, b, c \in S$ consider the homomorphism $\alpha : \mathbf{F}_V(\bar{x}, \bar{y}, \bar{z}, \bar{u}_1, \ldots, \bar{u}_n) \to \mathbf{S}$ defined by $\alpha(\bar{x}) = a$, $\alpha(\bar{y}) = b$, $\alpha(\bar{z}) = c$, $\alpha(\bar{u}_i) = s_i$, $1 \le i \le n$. If $a = b$ then $\langle \bar{x}, \bar{y} \rangle \in \ker \alpha$, so $\langle \bar{z}, t_n(\bar{x}, \bar{y}, \bar{z}, \bar{u}_1, \ldots, \bar{u}_n) \rangle \in \ker \alpha$. Thus $a = b$ implies $t_n(a, b, c, s_1, \ldots, s_n) = c$. If $a \neq b$ then $\langle \bar{x}, \bar{y} \rangle \notin \ker \alpha$. Since $\ker \alpha$ is a maximal congruence this implies $\theta(\bar{x}, \bar{y}) \vee \ker \alpha = \nabla$; hence $\ker \alpha = \Delta \vee \ker \alpha = [\theta(\bar{x}, \bar{y}) \wedge \theta^*] \vee \ker \alpha = \theta^* \vee \ker \alpha$ (use the distributive law). This leads to $\theta^* \subseteq \ker \alpha$, so $\langle t_n(\bar{x}, \bar{y}, \bar{z}, \bar{u}_1, \ldots, \bar{u}_n), \bar{x} \rangle \in \ker \alpha$. Thus $a \neq b$ implies $t_n(a, b, c, s_1, \ldots, s_n) = a$. Consequently $t_n(x, y, z, s_1, \ldots, s_n)$ is a discriminator polynomial for $\mathbf{S}$. □

COROLLARY 2. (Pixley [5]). *Let V be an arithmetical variety generated by a set K of finitely many finite simple algebras. Then the following are equivalent*:

(A) *V is a discriminator variety.*

(B) *Each member of K is hereditarily simple.*

Proof. (A) ⇒ (B) is just the usual one line proof that quasiprimal algebras are hereditarily simple (see [2]).

Suppose (B) holds. By Jónsson's theorem V is semisimple. Choose $t_3(x, y, z, u_1, u_2, u_3)$ as in Theorem 1. Then $t(x, y, z) = t_3(x, y, z, x, y, z)$ is readily seen to be a discriminator term for K; hence V is a discriminator variety. □

Remark. One can use essentially the same proof to show that if a variety V has the property that *for every $\mathbf{A} \in V$ the principle congruences of $\mathbf{A}$ form a sublattice of* **Con A** *which is relatively complemented distributive and permutable* then V is a discriminator variety. One uses $\mathbf{F}_V(x, y, z, \ldots, u_\alpha \ldots)_{\alpha < \kappa}$ (and the downward Löwenheim-Skolem theorem). Thus, in view of known results about discriminator varieties, the italicized words above characterize discriminator varieties. Similar characterizations have been given by Fried and Kiss [3], and Blok and Pigozzi [1]. We are particularly fond of the above characterization of discriminator varieties because it leads directly to the fundamental Boolean product representation (see Chap. IV of [2]).

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