BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 4, Number 3, May 1981

DECIDABLE VARIETIES WITH MODULAR CONGRUENCE LATTICES

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ABSTRACT. For a large collection of varieties we show that if the first-order theory of such a variety is decidable then the variety decomposes into the product of two well-known highly specialized varieties. For many varieties the decidability question then reduces to a decidability question about modules.

A *variety* is a class of (abstract) algebras (belonging to some language) closed under the formation of direct products, subalgebras and homomorphic images. A variety V is *locally finite* if every finitely-generated member of V is finite. A class of algebras K *generates a variety* V if V is the smallest variety containing K—then we say that V *is generated by* K, written $V = V(K)$. A variety is *finitely generated* if it is generated by finitely many finite algebras, or equivalently by a single finite algebra. The kernel of a homomorphism is called a *congruence,* and the congruences of any algebra form a lattice. A variety is *congruence modular,* or we prefer to say just *modular,* if the lattice of congruences of every algebra in the variety satisfies the modular law. (Most of the well-studied varieties are modular; for example varieties of groups, rings, modules and lattices are modular. However,the variety of semigroups is not modular.)

A variety V is *a product of two subvarieties* V_1, V_2 if $V_1 \cup V_2$ generates V and there is a term $b(x, y)$ such that $V_1 \models b(x, y) = x$, $V_2 \models b(x, y) = y$. If this is so we write $V = V_1 \otimes V_2$, and then for every algebra A in V there are (up to isomorphism) unique algebras $A_1 \in V_1$, $A_2 \in V_2$ such that $A \cong A_1 \times A_2$. A class K of first-order structures has a *decidable theory* if there is an effective procedure to determine precisely which first-order sentences are true of every member of K.

A variety V is a *discriminator variety* if it is generated by a set K for which there exists a ternary term $t(x, y, z)$ such that K satisfies $t(x, y, z) = x$ if $x \neq y$; $= z$ if $x = y$. In everyday mathematics such varieties appear only as highly specialized varieties of rings, or varieties associated with algebraic logics.

Received by the editors December 29, 1980.

1980 *Mathematics Subject Classification.* Primary 03B25, 08B10, 08A05.

Key words and phrases. Decidable theory, modular variety, discriminator variety, abelian variety, product of varieties, unitary left R-modules.

[1]Research supported by NSERC Grant No. A7256.

[2]Research supported by NSF Grant No. MCS 77-22913.

The center Z_A of an algebra A is the binary relation defined by

$$(a, b) \in Z_A \text{ iff } \forall t \forall \widetilde{u} \forall \widetilde{v} [t(\widetilde{u}, a) = t(\widetilde{v}, a) \longleftrightarrow t(\widetilde{u}, b) = t(\widetilde{v}, b)],$$

where t denotes an $m + 1$-ary term and $\widetilde{u}, \widetilde{v}$ are m-ary sequences of variables, for any $m < \omega$. Z_A is actually a congruence on A. An algebra is *abelian* if $V(A)$ is modular and $Z_A = A \times A$. Given any modular variety V the abelian algebras in V form a subvariety V_{ab}.

Let $\mathcal{BP}^1$ denote the class of structures $(B, B_0, \vee, \wedge, ', 0, 1)$ where $(B, \vee, \wedge, ', 0, 1)$ is an atomic Boolean algebra and $(B_0, \vee, \wedge, ', 0, 1)$ is a subalgebra containing all the atoms of B.

THEOREM 1. *The theory of* $\mathcal{BP}^1$ *is undecidable.*

PROOF. The class of finite graphs can be interpreted in $\mathcal{BP}^1$ along the lines of a construction introduced by M. Rubin [6]. □

THEOREM 2. *If V is a locally finite modular variety such that every reduct of V to a finite language has a decidable theory, then V has a subvariety V_{ds} which is a discriminator variety and $V = V_{ds} \otimes V_{ab}$.*

PROOF. We focus our attention on three special kinds of finite algebras.

(Type I) A is subdirectly irreducible and there is a subalgebra B of A and a congruence θ of B such that $Z_A \cap (B \times B) < \theta < B \times B$.

(Type II) A is subdirectly irreducible nonabelian with $Z_A = 0$, and there is an abelian subalgebra B of A with $|B| > 1$.

(Type III) A is directly indecomposable but not simple and $V(A)$ is congruence distributive (i.e. every algebra in $V(A)$ has a lattice of congruences which satisfies the distributive law).

If V contains an algebra A of Type I or III then, using a modification of the Boolean power construction, we can interpret $\mathcal{BP}^1$ into the class of subdirect powers of A. If V contains an algebra A of Type II then, extending a technique pioneered by Zamjatin [8] for the study of groups, finite bipartite graphs can be interpreted into the class of subdirect powers of A.

It follows that no finite member of V can be of Type I, II or III. Then using A. Pixley's characterization [5] of finitely-generated discriminator varieties plus the *modular commutator* introduced by J. Hagemann, C. Herrmann and J. D. H. Smith [3], [4], [7] and two results of R. Freese and R. McKenzie [2], [2a], one can prove that V has the desired decomposition. □

THEOREM 3. *The question of 'which modular varieties $V(A)$, generated by a finite algebra A belonging to a finite language, have a decidable theory' effectively reduces to the question of 'which finite rings R are such that the class of unitary left R-modules has a decidable theory'.*

PROOF. Given a finite algebra A of finite type one can effectively determine if $V(A)$ is modular, and also if $V(A)$ is of the form $V_{ds} \otimes V_{ab}$ where V_{ds} is a discriminator variety and V_{ab} is abelian. H. Werner (see [1]), extending results of S. Comer, proved that every finitely-generated discriminator variety belonging to a finite language has a decidable theory. If $V(A) = V_{ds} \otimes V_{ab}$ then one can effectively construct a finite ring R, given A, such that a decision procedure for the theory of V_{ab} yields a decision procedure for the theory of unitary left R-modules, and vice-versa. $\square$

REFERENCES

1. S. Burris and H. Werner, *Sheaf constructions and their elementary properties,* Trans. Amer. Math. Soc. **248** (1979), 269–309.

2. R. Freese and R. McKenzie, *The commutator, an overview* (preprint).

2a. R. Freese and R. McKenzie, *Residually small varieties with modular congruence lattices,* Trans. Amer. Math. Soc. 264 (1981), 419–430.

3. J. Hagemann and C. Herrmann, *A concrete ideal multiplication for algebraic systems and its relation to congruence distributivity,* Arch. Math. (Basel) 32 (1979), 234–245.

4. C. Herrmann, *Affine algebras in congruence modular varieties,* Acta Sci. Math. (Szeged) **41** (1979), 119–125.

5. A. F. Pixley, *Completeness in arithmetical algebras,* Algebra Universalis 2 (1972), 179–196.

6. M. Rubin, *The theory of Boolean algebras with a distinguished subalgebra is undecidable,* Ann. Sci. Univ. Clermont No. 60 Math. No. 13 (1976), 129–134.

7. J. D. H. Smith, *Mal'cev varieties,* Lecture Notes in Math., vol. 554, Springer-Verlag, Berlin and New York, 1976.

8. A. P. Zamjatin, *A nonabelian variety of groups has an undecidable elementary theory,* Algebra and Logic 17 (1978), 13–17.

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