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Boolean powers

Stanley Burris

The source of our investigations is Boolean algebras – indeed we were initially interested in proving that the first-order theory of the class $\mathscr{K}$ of congruence lattices of Boolean algebras is decidable (the question is still open, but see Burris and Sankappanvar [5] for a fuller discussion of the problem). We noted that $\mathscr{K}$ could also be thought of as the class of congruence lattices of the variety $\mathscr{V}$ generated by the ring $\mathfrak{Z}_p$ of integers modulo a given prime p. A little more reflection and it was clear that every algebra in $\mathscr{V}$ is a Boolean power of $\mathfrak{Z}_p$, and furthermore there is a natural translation of first-order statements about $\mathscr{V}$ into statements about Boolean algebras. In the course of analyzing these findings we were led to some new theorems about Boolean powers, and, thanks to Higgs, Djoković and Macintyre, an extensive bibliography has been compiled. Along with new results we try to organize the subject of Boolean powers so that the reader will also find a useful survey. The contents fall rather clearly into two areas: algebraic and logical aspects of Boolean powers. Let us first give a sketch of the history and results.

In 1942[1]) Rosenbloom [41] axiomatized the variety of n-valued Post algebras and noted many similarities with Boolean algebras, for example every finite algebra in such a variety is a direct power of the n-valued Post algebra of order n, $\mathfrak{P}_n$. He goes on to say ([41], p. 187): "The structure of Post algebras of infinite order is much more complicated and it seems as though a far from trivial extension of Stone's methods will be necessary to determine their structure as completely as he has determined the structure of infinite Boolean algebras."

In 1953 Foster [13] showed that there is a striking connection between n-valued Post algebras and Boolean algebras, namely, up to isomorphism the n-valued Post algebras are precisely the Boolean powers (or Boolean extensions) $\mathfrak{P}_n[\mathfrak{B}]$. Actually Tarski [46] had invented an equivalent construction in 1949 for the case of Boolean semigroups (see [48], p. 219). In brief Foster's two main results were: (1) Boolean powers of certain finite algebras are isomorphic to normal subdirect powers, and vice-versa, and (2) every algebra in a variety generated by a primal algebra is isomorphic to a Boolean power of the given primal. (The $\mathfrak{P}_n$ mentioned above are primal.) In 1967

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[1]) Dates refer to the year of publication.

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Gould and Grätzer [15] extended Foster's result (1) to cover all finite algebras. (It is interesting to note that it is still not known for which finite algebras $\mathfrak{A}$ it is true that every algebra in the variety generated by $\mathfrak{A}$ can be represented as a Boolean power of $\mathfrak{A}$.)

The definition of a Boolean power $\mathfrak{A}[\mathfrak{B}]$ suffers from one defect, namely if $\mathfrak{A}$ has an infinite universe then $\mathfrak{B}$ is assumed to be a complete Boolean algebra [2]), so Foster also introduced the concept of a bounded Boolean power $\mathfrak{A}[\mathfrak{B}]^*$. In 1972 Quackenbush [38] proved that if $\mathfrak{B}_0$ and $\mathfrak{B}_1$ are two Boolean algebras then $\mathfrak{B}_0[\mathfrak{B}_1]^*$ is isomorphic to $\mathfrak{B}_0 * \mathfrak{B}_1$, the free product of $\mathfrak{B}_0$ and $\mathfrak{B}_1$ relative to the class of Boolean algebras. This leads to our basic result: $(\mathfrak{A}[\mathfrak{B}_0]^*)[\mathfrak{B}_1]^* \cong \mathfrak{A}[\mathfrak{B}_0 * \mathfrak{B}_1]^*$. Hence iterated bounded Boolean powers are bounded Boolean powers.

In 1953 Kinoshita [26] gave examples of denumerable Boolean algebras $\mathfrak{B}_0$, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ such that $\mathfrak{B}_0 \cong \mathfrak{B}_0 \times \mathfrak{B}_1 \times \mathfrak{B}_2$ but $\mathfrak{B}_0 \not\cong \mathfrak{B}_0 \times \mathfrak{B}_1$, i.e. there are two denumerable Boolean algebras such that each is a direct factor of the other, but they are not isomorphic. Hanf [17] modified this construction in 1957 to describe two denumerable Boolean algebras $\mathfrak{B}_0$ and $\mathfrak{B}_1$ such that $\mathfrak{B}_0 \cong \mathfrak{B}_0 \times \mathfrak{B}_1 \times \mathfrak{B}_1$ but $\mathfrak{B}_0 \not\cong \mathfrak{B}_0 \times \mathfrak{B}_1$. Tarski pointed out that this implies the existence of two denumerable Boolean algebras $\mathfrak{B}$, $\hat{\mathfrak{B}}$ such that $\mathfrak{B} \times \mathfrak{B} \cong \hat{\mathfrak{B}} \times \hat{\mathfrak{B}}$, but $\mathfrak{B} \not\cong \hat{\mathfrak{B}}$. Jónsson [21] and Tarski [48] realized that if $\mathfrak{A}$ is an algebra such that $\mathfrak{B}_0 \not\cong \mathfrak{B}_1$ implies $\mathfrak{A}[\mathfrak{B}_0]^* \not\cong \mathfrak{A}[\mathfrak{B}_1]^*$ then one could derive similar direct product results for the class of bounded Boolean powers of $\mathfrak{A}$. In the text we refer to such an $\mathfrak{A}$ as *B*-separating, and our basic result on this topic is to find a condition on simple algebras $\mathfrak{A}$ which ensure that they are *B*-separating (Theorem 3.5). An immediate application is to show a non-trivial congruence distributive variety $\mathscr{V}$ has a *B*-separating algebra, and we use this to show that such a $\mathscr{V}$ has 2^λ isomorphism types of power λ for λ at least the power of the language of $\mathscr{V}$. (Not only does this generalize known results for Boolean algebras, it also answers a special case of Vaught's question [50] concerning the number of countable models of a countable first-order theory.) Another easy application of our sufficient conditions for *B*-separating algebras is to give a complete description of all algebras $\mathfrak{A}$ such that every congruence on $\mathfrak{A}[\mathfrak{B}]^*$ is determined by a filter on $\mathfrak{B}$ (Corollary 3.6). This was done by Neumann and Yamamuro [33] in 1964 when $\mathfrak{A}$ is a group.

The logical aspects of Boolean powers rest on two pillars, first the positive decidability results for the theory of Boolean algebras achieved by Skolem [43] 1919, Tarski [47] 1949, and Rabin [39] 1969, and secondly the technique for translating first-order assertions about a power $\mathfrak{A}^I$ into assertions about the Boolean algebra of subsets of I due to Mostowski [32] 1952 and Feferman-Vaught [11] 1959. A variation of the Feferman-Vaught translation suitable for Boolean powers was introduced by Węglorz [54] in 1968, and refined by Wojciechowska [57] in 1969. We will use her formulation to show that bounded Boolean powers $\mathfrak{A}[\mathfrak{B}]^*$ preserve elementary

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[2]) Recently Banaschewski and Nelson have proposed a remedy for this via a topological approach.

equivalence and elementary substructure in both arguments $\mathfrak{A}$ and $\mathfrak{B}$. With a slight modification we obtain partial results of this nature for Boolean powers – Theorem 4.3 gives a survey of what we know, including the fascinating connections with reduced powers and Horn sentences made by Waszkiewicz and Węglorz [51] in 1968. With these tools we prove that relatively free products of Boolean algebras preserve elementary equivalence and elementary substructure. Later we extend this to varieties generated by any primal algebra. Also we show any two bounded Boolean powers (or reduced powers) of an algebra have isomorphic reduced powers.

In 1967 Ershov [10] discovered that when studying the first-order properties of a variety generated by a primal algebra the Feferman-Vaught translation could be replaced by a remarkably simple translation. Using this we are able to show that if $\mathfrak{A}$ is a finite algebra, then $\mathfrak{B}$ is complete implies $\mathfrak{A}[\mathfrak{B}]$ is equationally compact, and if $\mathfrak{B}$ is κ-saturated ($\kappa \geqslant \omega$) then $\mathfrak{A}[\mathfrak{B}]$ is κ-saturated. A surprising property of finite B-separating algebras is that not only do they separate isomorphism types, they also separate elementary types!

With a slight extension of Ershov's translation we are able to improve on results of Comer [7] and show that the theory of countable m-rings with quantification over ideals is decidable, extending the aforementioned theorem of Rabin on Boolean algebras.[3] The paper concludes with a last look at varieties generated by primal algebras, and several interesting problems are posed.

§1. Preliminaries

An *algebra* $\mathfrak{A}$ is a pair $\langle A, \mathscr{F}\rangle$ where $\mathscr{F}$ is a family of *finitary* fundamental operations f_γ, indexed by ordinals γ less than α, for some α, where the *rank* of each function f_γ is n_γ, i.e. $f_\gamma : A^{n_\gamma} \to A$. A is the *universe* of $\mathfrak{A}$ (which we will often designate by $|\mathfrak{A}|$). The *type* of $\mathfrak{A}$ is the sequence $\langle n_0, n_1, \dots, n_\gamma, \dots\rangle_{\gamma<\alpha}$. For convenience we will occasionally use the notation $f_\gamma(\vec{a})$ for $f_\gamma(a_0, \dots, a_{n_\gamma-1})$. An algebra $\mathfrak{A}$ is *trivial* if Card $|\mathfrak{A}|=1$. A subset B of A closed under the operations f_γ is a *subuniverse*, and the pair $\langle B, \mathscr{F}\rangle$ denotes the corresponding *subalgebra* of $\mathfrak{A}$. Con $\mathfrak{A}$ is the *lattice of congruences* of $\mathfrak{A}$ (we also use Con $\mathfrak{A}$ for the set of congruences). $\mathfrak{A}$ is finite if $|\mathfrak{A}|$ is finite.

A *Boolean algebra* $\mathfrak{B}$ is an algebra $\langle B, \vee, \wedge, ', 0, 1\rangle$ with the usual properties. Let $\mathfrak{A}=\langle A, \mathscr{F}\rangle$ be an arbitrary algebra and $\mathfrak{B}$ a Boolean algebra. The *Boolean power* $\mathfrak{A}[\mathfrak{B}]$ has as its universe the set of all $\xi \in B^A$ such that

(i) if $a_0, a_1 \in A$, $a_0 \neq a_1$, then $\xi(a_0) \wedge \xi(a_1)=0$,

and

(ii) $\bigvee_{a\in A} \xi(a)=1$,

[3]) Recently Werner and the author have shown that the theory of the countable models of any variety generated by a weakly independent family of quasi-primal algebras with quantification over congruences is decidable.

and the fundamental operations f_γ are defined by

$$f_\gamma(\xi_0, \ldots, \xi_{n_\gamma-1})(a) = \bigvee_{f_\gamma(a_0, \ldots, a_{n_\gamma-1})=a} [\xi_0(a_0) \wedge \ldots \wedge \xi_{n_\gamma-1}(a_{n_\gamma-1})].$$

If A is infinite we always require that $\mathfrak{B}$ be a complete Boolean algebra. A straightforward consequence of these definitions is that if $p(x_0, \ldots, x_n)$ is a polynomial on $\mathfrak{A}$, then on $\mathfrak{A}[\mathfrak{B}]$

$$p(\xi_0, \ldots, \xi_n)(a) = \bigvee_{p(a_0, \ldots, a_n)=a} [\xi_0(a_0) \wedge \ldots \wedge \xi_n(a_n)].$$

The *bounded Boolean power* $\mathfrak{A}[\mathfrak{B}]^*$ has as its universe the set of $\xi \in B^A$ satisfying (i) and (ii) above and also

(iii) $\{a \in A : \xi(a) \neq 0\}$ is finite.

The fundamental operations f_γ are defined as in Boolean powers.

§2. Some basic properties

Let $\mathbf{2}$ denote a two-element Boolean algebra, and throughout the paper the letter $\mathfrak{B}$ will denote a Boolean algebra.

PROPOSITION 2.1. *For any algebra $\mathfrak{A}$ we have*

(i) $\mathfrak{A}[\mathfrak{B}] = \mathfrak{A}[\mathfrak{B}]^*$ *if $\mathfrak{A}$ is finite or $\mathfrak{B}$ is finite,*

(ii) $\mathfrak{A}[\mathbf{2}] = \mathfrak{A}[\mathbf{2}]^* \cong \mathfrak{A}$,

(iii) $\mathfrak{A}[\prod_{i\in I} \mathfrak{B}_i] \cong \prod_{i \in I} \mathfrak{A}[\mathfrak{B}_i]$, *(where the $\mathfrak{B}_i$ are complete if $|\mathfrak{A}|$ is infinite),*

(iv) $\mathfrak{A}[\mathfrak{B}_0 \times \cdots \times \mathfrak{B}_n]^* \cong \mathfrak{A}[\mathfrak{B}_0]^* \times \cdots \times \mathfrak{A}[\mathfrak{B}_n]^*$,

and

(v) $\mathbf{2}[\mathfrak{B}] = \mathbf{2}[\mathfrak{B}]^* \cong \mathfrak{B}$.

Proof. Part (i) is obvious. For (ii) define $\xi_a \in |\mathfrak{A}[\mathbf{2}]|$ by $\xi_a(a) = 1$ and $\xi_a(x) = 0$ if $x \neq a$. Then the mapping $a \to \xi_a$ is the desired isomorphism. For (iii) it is enough to show that the map

$$\lambda : |\mathfrak{A}[\prod_{i \in I} \mathfrak{B}_i]| \to |\prod_{i \in I} \mathfrak{A}[\mathfrak{B}_i]|$$

defined by

$$\lambda(\xi)(i)(a) = \xi(a)(i) \quad \text{for} \quad i \in I,\ a \in A \quad \text{and} \quad \xi \in |\mathfrak{A}[\prod_{i \in I} \mathfrak{B}_i]|$$

is an isomorphism. The proof of (iv) is similar. For (v) the map λ from $|\mathbf{2}[\mathfrak{B}]|$ to $|\mathfrak{B}|$ defined by $\lambda(\xi) = \xi(1)$ is the desired isomorphism.

COROLLARY 2.2. *For any algebra* $\mathfrak{A}$,

$$\mathfrak{A}[\mathbf{2}^I] \cong \mathfrak{A}^I \quad \textit{for any } I.$$

PROPOSITION 2.3. *If* $\mathfrak{B}_0$ *is a subalgebra of* $\mathfrak{B}_1$ *then*

(i) $\mathfrak{A}[\mathfrak{B}_0]$ *is a subalgebra of* $\mathfrak{A}[\mathfrak{B}_1]$ *(provided* $\mathfrak{B}_0$ *is a complete subalgebra of* $\mathfrak{B}_1$ *if* $|\mathfrak{A}|$ *is infinite),*
and

(ii) $\mathfrak{A}[\mathfrak{B}_0]^*$ *is a subalgebra of* $\mathfrak{A}[\mathfrak{B}_1]^*$.

If $\mathfrak{B}$ is a Boolean algebra, F a filter on $\mathfrak{B}$ and $\mathfrak{A}$ an algebra define the relations θ_F and θ_F^* by

$$\theta_F = \{\langle \xi, \eta\rangle \in |\mathfrak{A}[\mathfrak{B}]|^2 : \bigvee_{a \in A} \xi(a) \wedge \eta(a) \in F\},$$

and θ_F^* is θ_F restricted to $\mathfrak{A}[\mathfrak{B}]^*$.

PROPOSITION 2.4. (a) θ_F *is a congruence on* $\mathfrak{A}[\mathfrak{B}]$, *and the mapping* $F \to \theta_F$ *embeds the lattice of filters of* $\mathfrak{B}$ *into* $\operatorname{Con} \mathfrak{A}[\mathfrak{B}]$ *as a complete sublattice if* $\mathfrak{A}$ *is non-trivial;*

(b) θ_F^* *is a congruence on* $\mathfrak{A}[\mathfrak{B}]^*$, *and the mapping* $F \to \theta_F^*$ *embeds the lattice of filters of* $\mathfrak{B}$ *into* $\operatorname{Con} \mathfrak{A}[\mathfrak{B}]^*$ *as a complete sublattice if* $\mathfrak{A}$ *is non-trivial.*

Both θ_F and θ_F^* will be called *filter congruences.* We write $\mathfrak{A}[\mathfrak{B}]/F$ for $\mathfrak{A}[\mathfrak{B}]/\theta_F$ and $\mathfrak{A}[\mathfrak{B}]^*/F$ for $\mathfrak{A}[\mathfrak{B}]^*/\theta_F^*$.

PROPOSITION 2.5. *If* F *is a filter on* $\mathfrak{B}$ *and* $\mathfrak{A}$ *is any algebra then*

$$\mathfrak{A}[\mathfrak{B}]^*/F \cong \mathfrak{A}[\mathfrak{B}/F]^*.$$

Proof. Let $\xi \in \mathfrak{A}[\mathfrak{B}]^*$ and let $[\xi]_F$ denote the equivalence class of ξ with respect to $\theta_F^*(\mathfrak{A})$, and if $b \in \mathfrak{B}$ let $[b]_F$ denote the equivalence class of b with respect to the congruence on $\mathfrak{B}$ determined by F. Then it is routine to show that the map $\lambda : |\mathfrak{A}[\mathfrak{B}]^*/F| \to |\mathfrak{A}[\mathfrak{B}/F]^*|$ defined by $\lambda([\xi]_F)(a) = [\xi(a)]_F$ is the desired isomorphism.

PROPOSITION 2.6.[4]) *If* $\mathfrak{A}$ *is finite then for any* I *and filter* F *on the subsets of* I *we have*

$$\mathfrak{A}[(\prod_{i \in I} \mathfrak{B}_i)/F] \cong (\prod_{i \in I} \mathfrak{A}[\mathfrak{B}_i])/F.$$

Proof. This follows from Proposition 2.1 (iii) and Proposition 2.5.

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[4]) This proposition was stated by Bacsich [1] for the case F is an ultrafilter.

Now we turn our attention to *iterated bounded Boolean powers*, and show that for a given algebra $\mathfrak{A}$ the class of $\mathfrak{A}[\mathfrak{B}]^*$ is, up to isomorphism, closed under bounded Boolean powers. First we consider the *full* algebra $\mathfrak{P}=\langle P, \mathscr{F}\rangle$, meaning that every finitary function is a fundamental operation. By a *bounded subdirect power* of $\mathfrak{P}$ we mean a subalgebra $\mathfrak{P}_0$ of some $\mathfrak{P}^I$ which is a subdirect power and such that each function in the subdirect power has a finite range. By embedding a given Boolean algebra $\mathfrak{B}$ in a suitable $\mathbf{2}^I$ we can easily show, for any $\mathfrak{A}$, that $\mathfrak{A}[\mathfrak{B}]^*$ is isomorphic to a bounded subdirect power of $\mathfrak{A}$. Also, if $\mathfrak{P}_0$ is a bounded subdirect power of a full $\mathfrak{P}$ then $\{\mu^{-1}(p): p\in P,\ \mu\in|\mathfrak{P}_0|\}$ is a field of sets and this gives the desired Boolean algebra to show $\mathfrak{P}_0$ is isomorphic to a bounded Boolean power of $\mathfrak{P}$. Since bounded subdirect powers of bounded subdirect powers are isomorphic to bounded subdirect powers it follows that $(\mathfrak{P}[\mathfrak{B}_0]^*)[\mathfrak{B}_1]^*$ is isomorphic to some $\mathfrak{P}[\mathfrak{B}_2]^*$. If $\mathfrak{P}$ is non-trivial then by taking a two-element relativized reduct isomorphic to $\mathbf{2}$ we have $(\mathbf{2}[\mathfrak{B}_0]^*)[\mathfrak{B}_1]^*\cong\mathbf{2}[\mathfrak{B}_2]^*$, hence $\mathfrak{B}_0[\mathfrak{B}_1]^*\cong\mathfrak{B}_2$, so $(\mathfrak{P}[\mathfrak{B}_0]^*)[\mathfrak{B}_1]^*\cong \mathfrak{P}[\mathfrak{B}_0[\mathfrak{B}_1]^*]^*$. Now we can easily prove the following.

THEOREM 2.7. *Let $\mathfrak{A}$ be any algebra. Then*

$$(\mathfrak{A}[\mathfrak{B}_0]^*)[\mathfrak{B}_1]^*\cong\mathfrak{A}[\mathfrak{B}_0[\mathfrak{B}_1]^*]^*.$$

Proof. For any $\mathfrak{A}$ we can find a full $\mathfrak{P}$ such that $\mathfrak{A}$ is a reduct of $\mathfrak{P}$.

Quackenbush [38] has shown that $\mathfrak{B}_0[\mathfrak{B}_1]^*$ is isomorphic to the *$\mathscr{BA}$-free product* $\mathfrak{B}_0*\mathfrak{B}_1$, where $\mathscr{BA}$ is *the class of Boolean algebras.*

COROLLARY 2.8. *For any algebra $\mathfrak{A}$,*

$$(\mathfrak{A}[\mathfrak{B}_0]^*)[\mathfrak{B}_1]^*\cong(\mathfrak{A}[\mathfrak{B}_1]^*)[\mathfrak{B}_0]^*\cong\mathfrak{A}[\mathfrak{B}_0*\mathfrak{B}_1]^*.$$

Proof. Combine Theorem 2.7 and Quackenbush's result.

COROLLARY 2.9. *If $\mathfrak{B}_0$, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are Boolean algebras and n is a positive integer then*

(i) $\mathfrak{B}_0*(\mathfrak{B}_1\times\mathfrak{B}_2)\cong(\mathfrak{B}_0*\mathfrak{B}_1)\times(\mathfrak{B}_0*\mathfrak{B}_2)$,

(ii) $\mathfrak{B}_0*\mathfrak{B}_1^n\cong(\mathfrak{B}_0*\mathfrak{B}_1)^n$,

and

(iii) $\mathfrak{B}_0*\mathbf{2}^n\cong\mathfrak{B}_0^n$.

Let $\mathscr{BA}_\kappa/I$ denote the isomorphism types of Boolean algebras with cardinality less than κ, where κ is infinite. Then the algebra $\langle\mathscr{BA}_\kappa/I, \times, *\rangle$, where $\times$ and $*$ are defined in the obvious manner, is a *commutative semi-ring with unity*, i.e. $\times$ and $*$ are both commutative and associative binary operations, $*$ distributes over $\times$, and $\mathbf{2}$ is the unity since $\mathbf{2} * \mathfrak{B} \cong \mathfrak{B}$. Indeed $\langle \mathscr{BA}_\omega / I, \times, * \rangle$ is isomorphic to $\langle \omega, +, \cdot \rangle$, the non-negative integers under addition and multiplication.

§3. *B*-separating algebras

The applications of Boolean powers by Jónsson and Tarski mentioned in the introduction were based on finding algebras $\mathfrak{A}$ such that if $\mathfrak{B}_0 \not\cong \mathfrak{B}_1$ then $\mathfrak{A}[\mathfrak{B}_0]^* \not\cong \mathfrak{A}[\mathfrak{B}_1]^*$. Such an algebra $\mathfrak{A}$ will henceforth be called *B-separating*. Clearly such an $\mathfrak{A}$ must have at least two elements in its universe. However the problem of characterizing *B*-separating algebras is open. We are motivated to examine this question by the thesis that a class $\mathscr{K}$ of algebras closed under bounded Boolean powers and containing a *B*-separating algebra is, in many ways, at least as 'complex' as the class of Boolean algebras. The following theorem contains the two basic applications of *B*-separating algebras.

THEOREM 3.1. *Let $\mathscr{K}$ be a class of algebras closed under bounded Boolean powers and containing a B-separating algebra $\mathfrak{A}$. Then one has*

(a) *there are algebras $\mathfrak{A}_0, \mathfrak{A}_1 \in \mathscr{K}$ such that $\mathfrak{A}_0^2 \cong \mathfrak{A}_1^2$ but $\mathfrak{A}_0 \not\cong \mathfrak{A}_1$, as well as obvious translations of other direct product phenomena in Boolean algebras;*

(b) *$\mathscr{K}$ has at least 2^λ isomorphism types of algebras of power λ, where $\lambda \geqslant \mathrm{Card}\,|\mathfrak{A}|$.*

Proof. For (a) simply note that $\mathfrak{A}[\mathfrak{B}_0 \times \mathfrak{B}_1]^* \cong \mathfrak{A}[\mathfrak{B}_0]^* \times \mathfrak{A}[\mathfrak{B}_1]^*$ and use the direct product results of Boolean algebras. For (b) note that Boolean algebras have 2^λ isomorphism types of power λ, $\lambda \geqslant \omega$ (see [31]).

THEOREM 3.2. (Tarski [48]). *The additive semigroup $\langle \omega, + \rangle$ of non-negative integers is B-separating.*

This result was generalized by Jónsson to a large class of algebras.

THEOREM 3.3. (Jónsson [21]). *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be a countable algebra with two polynomials $+, 0$ such that $\langle A, +, 0 \rangle$ is an indecomposable centerless algebra. (It is assumed that $x + 0 = x = 0 + x$.) Then $\mathfrak{A}$ is B-separating if $|\mathfrak{A}|$ has at least two elements.*

For example the symmetric group on three elements is *B*-separating by this theorem. (And furthermore any group of smaller cardinality is not *B*-separating.) [The definition of an indecomposable centerless algebra $\langle A, +, 0 \rangle$, where $x + 0 = x = 0 + x$, is somewhat technical, but for the flavour of this approach we will say what it means *given that + is associative*. In this case a *direct factor* of $\langle A, +, 0 \rangle$ is a subalgebra $\langle A', +, 0 \rangle$ such that there is another subalgebra $\langle A'', +, 0 \rangle$ with every element $a \in A$ having a unique representation in the form $a' + a''$, $a' \in A'$ and $a'' \in A''$, and furthermore $a' + a'' = a'' + a'$. $\langle A, +, 0 \rangle$ is *indecomposable* if the only direct factors are $\langle A, +, 0 \rangle$

and $\langle \{0\}, +, 0\rangle$. $\langle A, +, 0\rangle$ is *centerless* if $\{0\}$ is the only subuniverse A' of $\langle A, +, 0\rangle$ such that (i) for every $a' \in A'$ there is an $a'' \in A'$ such that $a'+a''=0$, and (ii) $a+a'=a'+a$ for every $a \in A$, $a' \in A'$. Jónsson and Tarski proved that the universes of the direct factors of a centerless algebra form a Boolean algebra (under $\subseteq$).] The basic idea of Jónsson's proof is to show that the universes of subalgebras of $\mathfrak{A}[\mathfrak{B}]^*$ which are direct factors form a Boolean algebra (under $\subseteq$) isomorphic to $\mathfrak{B}$. Our first sufficiency theorem is based on polynomial definability, and our second on a study of congruence lattices. Either of these sets of conditions will lead to B-separating algebras not covered by Jónsson's result above, but at present we are nowhere near necessary and sufficient conditions.

THEOREM 3.4. *Let $\mathfrak{A}$ be an algebra such that we can find an equation $p(x)=q(x)$ and polynomials $p^\vee(x, y)$, $p^\wedge(x, y)$ and $p'(x)$ such that exactly two elements $\hat{a}_0, \hat{a}_1 \in |\mathfrak{A}|$ satisfy $p(x)=q(x)$ and such that $\langle \{\hat{a}_0, \hat{a}_1\}, p^\vee, p^\wedge, p', \hat{a}_0, \hat{a}_1\rangle$ is a Boolean algebra. Then up to isomorphism $\mathfrak{B}$ is first-order definable in $\mathfrak{A}[\mathfrak{B}]^*$, hence $\mathfrak{A}$ is B-separating.*

Proof. Let $\hat{\mathfrak{B}}=\langle \{\xi \in |\mathfrak{A}[\mathfrak{B}]^*| : p(\xi)=q(\xi)\}, p^\vee, p^\wedge, p', \xi_0, \xi_1\rangle$ where $\xi_0(\hat{a}_0)=1$, $\xi_0(x)=0$ otherwise, and $\xi_1(\hat{a}_1)=1$, $\xi_1(x)=0$ otherwise. It is reasonably straight-forward to show that $\hat{\mathfrak{B}}$ is a Boolean algebra, and that the mapping $\lambda: |\mathfrak{B}| \to |\hat{\mathfrak{B}}|$defined by

$$(\lambda(b))(x)=\begin{cases} 0 & \text{if} \quad x \neq \hat{a}_0 \quad \text{and} \quad x \neq \hat{a}_1 \\ b' & \text{if} \quad x=\hat{a}_0 \\ b & \text{if} \quad x=\hat{a}_1 \end{cases}$$

is an isomorphism. Since $\hat{\mathfrak{B}}$ is first-order definable in $\mathfrak{A}[\mathfrak{B}]^*$ the theorem is proved.

An algebra $\mathfrak{A}$ is *simple* if it is non-trivial and has only two congruences.

THEOREM 3.5. *Let $\mathfrak{S}=\langle S, \mathscr{F}\rangle$ be a simple algebra such that*

(i) $\mathrm{Con}(\mathfrak{S}^n)$ *is modular for* $n<\omega$,

and

(ii) $\mathrm{Con}(\mathfrak{S}^2) \cong (\mathrm{Con}\,\mathfrak{S})^2$.

Then $\mathfrak{S}$ is B-separating. Indeed $\theta \in \mathrm{Con}\,\mathfrak{S}[\mathfrak{B}]^*$ implies $\theta=\theta_F^*$ for some filter F of $\mathfrak{B}$.

Proof. First we will show that the principal congruences $\theta(\xi, \eta)$ of $\mathfrak{S}[\mathfrak{B}]^*$ are filter congruences. Let F be the principal filter on $\mathfrak{B}$ generated by $b=\bigvee_{s \in S} \xi(s) \wedge \eta(s)$. Clearly $\theta(\xi, \eta) \leqslant \theta_F^*$. If $\mathfrak{B}_0$ is any finite subalgebra of $\mathfrak{B}$ let $\nabla_0=|\mathfrak{B}_0|^2$. From a previous result of ours [3] every congruence of $\mathfrak{S}[\mathfrak{B}_0]$ is a filter congruence. Clearly

$$\theta_F^* \cap \nabla_0=\{\langle \hat{\xi}, \hat{\eta}\rangle \in \nabla_0 : \bigvee_{s \in S} \hat{\xi}(s) \wedge \hat{\eta}(s) \geqslant b\} \quad \text{if} \quad b \in B_0 .$$

Let us suppose for the rest of the proof that $b \in B_0$ and $\xi(s) \wedge \eta(t) \in B_0$ for $s, t \in S$. Then $\theta(\xi, \eta) \cap \nabla_0=\{\langle \hat{\xi}, \hat{\eta}\rangle \in \nabla_0 : \bigvee_{s \in S} \hat{\xi}(s) \wedge \hat{\eta}(s) \geqslant b_0\}$ for some $b_0 \in B_0$. Since $\langle \xi, \eta\rangle$

$\in\theta(\xi,\eta)\cap\nabla_0$ and $\theta(\xi,\eta)\subseteq\theta_F^*$ it follows that $b=b_0$, so $\theta(\xi,\eta)\cap\nabla_0=\theta_F^*\cap\nabla_0$. But then $\theta(\xi,\eta)=\theta_F^*$, so *every* congruence of $\mathfrak{S}[\mathfrak{B}]^*$ is a filter congruence, hence $\operatorname{Con}\mathfrak{S}[\mathfrak{B}]^*\cong\operatorname{Con}\mathfrak{B}$. Since non-isomorphic Boolean algebras have non-isomorphic congruence lattices it follows that $\mathfrak{S}$ is B-separating.

Using Theorem 3.5 we can easily show that the following are examples of B-separating algebras: (a) $\mathfrak{S}=\langle S,+,\cdot,0,1,\ldots\rangle$, a simple algebra with two binary operations $+,\cdot$ and two nullary operations $0,1$ such that $x\cdot 1=x+0=0+x=x$ and $x\cdot 0=0$ hold in $\mathfrak{S}$ (see Fraser and Horn [14]); (b) any simple algebra $\mathfrak{S}$ in a congruence distributive variety (Magari [30] proved that in a non-trivial variety (i.e. a variety with a non-trivial algebra) one can always find a simple algebra $\mathfrak{S}$ with $\operatorname{Card}|\mathfrak{S}|$ no greater than the power of the language); (c) the seven-element Steiner quasigroup (see [56]).

COROLLARY 3.6. *Let* $\mathfrak{A}$ *be a non-trivial algebra. Then* $\operatorname{Con}\mathfrak{A}[\mathfrak{B}]^*=\{\theta_F^*: F$ *is a filter on* $\mathfrak{B}\}$ *for all Boolean algebras* $\mathfrak{B}$ *iff*

(i) $\mathfrak{A}$ *is simple,*

(ii) $\operatorname{Con}(\mathfrak{A}^n)$ *is modular,* $n<\omega$,

and

(iii) $\operatorname{Con}(\mathfrak{A}^2)\cong(\operatorname{Con}\mathfrak{A})^2$.

Proof. The conditions (i)–(iii) are certainly sufficient by Theorem 3.5. Clearly condition (i) is necessary, and if $\mathfrak{B}=\mathbf{2}^n$ then from Proposition 2.5 we would have $\operatorname{Con}(\mathfrak{A}^n)\cong\operatorname{Con}\mathfrak{A}[\mathfrak{B}]\cong\operatorname{Con}\mathfrak{B}\cong\mathbf{2}^n$, so (ii) and (iii) are also necessary.

As Neumann and Yamamuro [33] pointed out for the case of a non-abelian simple finite group, if $\mathfrak{A}$ satisfies (i)–(iii) and is finite then $\mathfrak{A}^I$ has no denumerable quotients for any I.

§4. Elementary properties

In [57] Wojciechowska showed how to adapt the Feferman-Vaught methods [11] to handle generalized limit powers, and hence as a special case bounded Boolean powers. We will outline this plus an adaptation to cover Boolean powers. In the following $\mathscr{L}(\mathfrak{A})$ is the language of $\mathfrak{A}$ with parameters from $|\mathfrak{A}|$.

Let $\mathfrak{B}$ be a Boolean algebra and F a filter on $\mathfrak{B}$. Let I be an index set and $\lambda\mathfrak{B}$ a copy of $\mathfrak{B}$ embedded by λ into the algebra of subsets of I, with λF the corresponding filter on $\lambda\mathfrak{B}$. For a given algebra $\mathfrak{A}$ let $\mathfrak{A}\langle\lambda\mathfrak{B}\rangle^*$ denote the subalgebra of $\mathfrak{A}^I$ with universe $\{f\in|\mathfrak{A}^I|: f^{-1}(a)\in|\lambda\mathfrak{B}|$ for $a\in A$, and $f^{-1}(a)=\phi$ for all but finitely many $a\}$. Note that $\mathfrak{A}\langle\lambda\mathfrak{B}\rangle^*\cong\mathfrak{A}[\mathfrak{B}]^*$. Let $\lambda\mathfrak{B}_F$ be the structure $\langle|\lambda\mathfrak{B}|,\cup,\cap,{}',\phi,I,\lambda F\rangle$. Let $\theta^*(F)$ be the congruence defined on $\mathfrak{A}\langle\lambda\mathfrak{B}\rangle^*$ by $\langle f,g\rangle\in\theta^*(F)$ iff $\{i\in I: f(i)=g(i)\}\in\lambda F$, and let $[f]^*$ denote the equivalence class on f with respect to this congruence. We will define a translation T^* from $\mathscr{L}(\mathfrak{A}\langle\lambda\mathfrak{B}\rangle^*)$ to sequences $\langle\Phi,\theta_0,\ldots,\theta_n\rangle$, where

$\Phi \in \mathscr{L}(\lambda\mathfrak{B}_F)$ and $\theta_i \in \mathscr{L}(\mathfrak{A}\langle\lambda\mathfrak{B}\rangle^*)$, $0 \leqslant i \leqslant n$, such that the following lemma, our desired analogue of the Fundamental Lemma of Feferman-Vaught, holds. If θ is a sentence in $\mathscr{L}(\mathfrak{A}\langle\lambda\mathfrak{B}\rangle^*)$ with parameters $f_0, \ldots, f_k$, let $\mathrm{Ind}(\theta) = \{i \in I : \mathfrak{A} \models \theta(f_0(i), \ldots, f_k(i))\}$.

LEMMA 4.1. *Let $\sigma(f_0, \ldots, f_k)$ be a sentence in $\mathscr{L}(\mathfrak{A}\langle\lambda\mathfrak{B}\rangle^*)$ with parameters $f_0, \ldots, f_k$. Then*

$$\mathfrak{A}\langle\lambda\mathfrak{B}\rangle^*/\theta^*(F) \models \sigma([f_0]^*, \ldots, [f_k]^*)$$

iff

$$\lambda\mathfrak{B}_F \models \Phi(\mathrm{Ind}(\theta_0), \ldots, \mathrm{Ind}(\theta_n)),$$

where $T^*(\sigma) = \langle \Phi; \theta_0, \ldots, \theta_n \rangle$.

For brevity in defining T^* we will assume that our formulas involve only atomic formulas, the Sheffer stroke $|$, and the existential quantifier $\exists$.

(i) (Atomic formulas) If p and q are polynomials, $\vec{x}$ a sequence of variables $x_{i_0}, \ldots, x_{i_k}$, and $\vec{f}$ a sequence of elements of $\mathfrak{A}\langle\lambda\mathfrak{B}\rangle^*$, then

$$T^*(p(\vec{x}, \vec{f}) = q(\vec{x}, \vec{f})) = \langle \lambda F(X_0); p(\vec{x}, \vec{f}) = q(\vec{x}, \vec{f}) \rangle.$$

(ii) (Sheffer stroke) If $T^*(\alpha) = \langle \Phi(X_0, \ldots, X_m); \theta_0, \ldots, \theta_m \rangle$ and $T^*(\alpha') = \langle \Phi'(X_0, \ldots, X_{m'}); \theta'_0, \ldots, \theta'_{m'} \rangle$ then

$$T^*(\alpha \mid \alpha') = \langle \Phi(X_0, \ldots, X_m) \mid \Phi'(X_{m+1}, \ldots, X_{m+m'+1}); \theta_0, \ldots, \theta_m, \theta'_0, \ldots, \theta'_{m'} \rangle.$$

(iii) ($\exists$ Quantifier) If $T^*(\alpha) = \langle \Phi(X_0, \ldots, X_m); \theta_0, \ldots, \theta_m \rangle$ let $m' = 2^{m+1} - 1$, let $A_0, \ldots, A_{m'}$ be the subsets of $\{0, \ldots, m\}$, and let $S_l = \{i : 0 \leqslant i \leqslant m', l \in A_i\}$, $0 \leqslant l \leqslant m$. Then define

$$\Phi'(X_0, \ldots, X'_m) = \Phi\Big(\bigcup_{i \in S_0} X_i, \ldots, \bigcup_{i \in S_m} X_i\Big),$$

and for $0 \leqslant i \leqslant m'$, define

$$\theta'_i = \Big(\mathop{\&}_{j \in A_i} \theta_j\Big) \,\&\, \Big(\mathop{\&}_{j \notin A_i} \neg\, \theta_j\Big).$$

Let $\mathrm{Part}(Y_0, \ldots, Y_{m'})$ be the predicate

$$\Big(\mathop{\&}_{0 \leqslant i < j \leqslant m'} (Y_i \cap Y_j = \emptyset)\Big) \,\&\, \Big(\bigcup_{0 \leqslant i \leqslant m'} Y_i = I\Big).$$

Then

$$T^*(\exists x_k \alpha) = \langle \exists Y_0 \ldots \exists Y_{m'} \,[\mathrm{Part}(Y_0, \ldots, Y_{m'}) \,\&\, \Big(\mathop{\&}_{0 \leqslant i \leqslant m'} Y_i \subseteq X_i\Big) \,\&$$
$$\& \, \Phi'(Y_0, \ldots, Y_{m'}) \;\; \exists x_k \theta'_0, \ldots, \exists x_k \theta'_{m'} \rangle.$$

This completes our definition of T^*.

The proof of Lemma 4.1 is parallel to the inductive Feferman-Vaught proof, where the crucial step is to show, in part (iii), that from $\bigcup_{0 \leqslant i \leqslant m'} \mathrm{Ind}(\exists x_k \theta'_i) = I$ we can construct an element $g \in |\mathfrak{A}\langle\lambda\mathfrak{B}\rangle^*|$ such that if $j \in \mathrm{Ind}(\exists x_k \theta'_i)$ then $\mathfrak{A} \models \theta'_i(\ldots, g(j), \ldots)$.

This follows easily from the observation that there is a finite number of subsets $C_0, \ldots, C_t$ in $|\lambda\mathfrak{B}|$ which partition I, and which have the property that all parameters of the various θ_i' are constant on each C_j, thus ensuring that we can select g.

Next we modify the previous work to study $\mathfrak{A}[\mathfrak{B}]$, if $\mathfrak{B}$ is complete, using the same notation except that F will now designate a filter on the subsets of I. Let $\mathfrak{A}\langle\lambda\mathfrak{B}\rangle$ denote the subalgebra of $\mathfrak{A}^I$ consisting of all $f \in |\mathfrak{A}^I|$ such that there is a partition $\{b_j\}_{j\in J}$ of unity in $\mathfrak{B}$ with the property that for any $j \in J$ there is an $a \in |\mathfrak{A}|$ with $\lambda b_j \subseteq f^{-1}(a)$. Note that $\mathfrak{A}\langle\lambda\mathfrak{B}\rangle^*$ is a subalgebra of $\mathfrak{A}\langle\lambda\mathfrak{B}\rangle$. Let $\mathfrak{P}_F(I)$ denote the modified algebra of sets $\langle P(I), \cup, \cap, ', \emptyset, I, F, \mathrm{Part}_{\mathfrak{B}}^{(n)}\rangle_{n\in\omega-1}$ ($\mathrm{Part}_{\mathfrak{B}}^{(n)}$ is defined below). Let $\theta(F)$ be the congruence defined on $\mathfrak{A}\langle\lambda\mathfrak{B}\rangle$ by $\langle f, g\rangle \in \theta(F)$ iff $\{i : f(i) = g(i)\} \in F$. For $f \in |\mathfrak{A}\langle\lambda\mathfrak{B}\rangle|$ let $[f]$ denote the equivalence class with respect to this congruence. We will define a translation T from $\mathscr{L}(\mathfrak{A}\langle\lambda\mathfrak{B}\rangle)$ to sequences $\langle\Phi; \theta_0, \ldots, \theta_n\rangle$, where $\Phi \in \mathscr{L}(\mathfrak{P}_F(I))$ and $\theta_i \in \mathscr{L}(\mathfrak{A}\langle\lambda\mathfrak{B}\rangle)$, $0 \leqslant i \leqslant n$, with the following property.

LEMMA 4.2. *Let $\sigma(f_0, \ldots, f_k)$ be a sentence in $\mathscr{L}(\mathfrak{A}\langle\lambda\mathfrak{B}\rangle)$ with parameters $f_0, \ldots, f_k$. Then*

$$\mathfrak{A}\langle\lambda\mathfrak{B}\rangle/\theta(F) \models \sigma([f_0], \ldots, [f_k])$$

iff

$$\mathfrak{P}_F(I) \models \Phi(\mathrm{Ind}(\theta_0), \ldots, \mathrm{Ind}(\theta_n)),$$

where $T(\sigma) = \langle\Phi; \theta_0, \ldots, \theta_n\rangle$.

T is defined as follows:

(i) (Atomic formulas) If p and q are polynomials, $\vec{x}$ a sequence of variables $x_{i_0}, \ldots, x_{i_k}$, and $\vec{f}$ a sequence of elements from $\mathfrak{A}\langle\lambda\mathfrak{B}\rangle$ then

$$T(p(\vec{x}, \vec{f}) = q(\vec{x}, \vec{f})) = \langle F(X_0); p(\vec{x}, \vec{f}) = q(\vec{x}, \vec{f})\rangle.$$

(ii) (Sheffer stroke) Just repeat the part (ii) for T^*.

(iii) ($\exists$ Quantifier) Repeat part (iii) of T^* with one modification, namely replace $\mathrm{Part}(Y_0, \ldots, Y_{m'})$ by $\mathrm{Part}_{\mathfrak{B}}^{(m')}(Y_0, \ldots, Y_{m'})$, the latter defined to mean 'Part$(Y_0, \ldots, Y_{m'})$ and there is a partition $\{b_j\}_{j\in J}$ of unity in $\mathfrak{B}$ such that for each $j \in J$, $\lambda b_j \subseteq Y_i$ for some i'.

If $\mathscr{K}$ is a class of algebras of a given similarity type then $\mathrm{Th}(\mathscr{K})$ will designate *the first-order theory of* $\mathscr{K}$, i.e. the set of first-order sentences in the language of $\mathscr{K}$ which are true of every algebra in $\mathscr{K}$. If $\mathscr{K}$ is a singleton $\{\mathfrak{A}\}$ then we write simply $\mathrm{Th}(\mathfrak{A})$. Two algebras $\mathfrak{A}_0$ and $\mathfrak{A}_1$ are *elementarily equivalent* if $\mathrm{Th}(\mathfrak{A}_0) = \mathrm{Th}(\mathfrak{A}_1)$, and we abbreviate this by $\mathfrak{A}_0 \equiv \mathfrak{A}_1$. $\mathfrak{A}_1$ is an *elementary extension* of $\mathfrak{A}_0$, written $\mathfrak{A}_0 \prec \mathfrak{A}_1$, if $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}_1$ and every first-order sentence in the language of $\mathfrak{A}_0$, with parameters from $|\mathfrak{A}_0|$, is true in $\mathfrak{A}_0$ iff it is true in $\mathfrak{A}_1$.

In the following theorem parts (i) and (ii), for $\mathfrak{A}_0 = \mathfrak{A}_1$, were stated by Wojciechowska [57], and parts (v)–(ix) are essentially in Waszkiewicz and Węglorz's [51] work on generalized limit powers.

THEOREM 4.3. *For any algebras* $\mathfrak{A}$, $\mathfrak{A}_0$, *and* $\mathfrak{A}_1$ *we have*

(i) $\mathfrak{A}_0 \equiv \mathfrak{A}_1$, $\mathfrak{B}_0 \equiv \mathfrak{B}_1$ *implies* $\mathfrak{A}_0[\mathfrak{B}_0]^* \equiv \mathfrak{A}_1[\mathfrak{B}_1]^*$,

(ii) $\mathfrak{A}_0 \prec \mathfrak{A}_1$, $\mathfrak{B}_0 \prec \mathfrak{B}_1$ *implies* $\mathfrak{A}_0[\mathfrak{B}_0]^* \prec \mathfrak{A}_1[\mathfrak{B}_1]^*$,

(iii)[5]) *if* $\mathfrak{B}$ *is complete then* $\mathfrak{A}_0 \equiv \mathfrak{A}_1$ ($\mathfrak{A}_0 \prec \mathfrak{A}_1$) *implies* $\mathfrak{A}_0[\mathfrak{B}] \equiv \mathfrak{A}_1[\mathfrak{B}]$ ($\mathfrak{A}_0[\mathfrak{B}] \prec \mathfrak{A}_1[\mathfrak{B}]$),

(iv) *if* $\mathscr{K}$ *is a class of Boolean algebras with* $\mathrm{Th}(\mathscr{K})$ *decidable, and if* $\mathrm{Th}(\mathfrak{A})$ *is decidable, then* $\mathrm{Th}(\{\mathfrak{A}[\mathfrak{B}]^* : \mathfrak{B} \in \mathscr{K}\})$ *is decidable,*

(v) $\mathfrak{A}[2^I/F]^*$ *is isomorphic to an elementary substructure of* $\mathfrak{A}^I/F$, *for any* I *and* F,

(vi) *every bounded Boolean power of* $\mathfrak{A}$ *is elementarily equivalent to a reduced power of* $\mathfrak{A}$, *and vice-versa,*

(vii) *a first-order sentence is preserved under bounded Boolean powers iff it is equivalent to a disjunction of Horn sentences,*

(viii) *an elementary class* $\mathscr{K}$ *is closed under bounded Boolean powers iff it is closed under reduced powers iff it is definable by a set of sentences, each of which is a disjunction of Horn sentences,*

(ix) *if* $\mathrm{Th}(\mathfrak{A})$ *and* $\mathrm{Th}(\mathfrak{B})$ *are* ω*-categorical then* $\mathrm{Th}(\mathfrak{A}[\mathfrak{B}]^*)$ *is* ω*-categorical,*

and

(x) *if* $\mathfrak{A}$ *is finite and B-separating then* $\mathfrak{A}[\mathfrak{B}_0] \equiv \mathfrak{A}[\mathfrak{B}_1]$ *implies* $\mathfrak{B}_0 \equiv \mathfrak{B}_1$.

Proof. (i), (ii) and (iv) are immediate from Lemma 4.1. In Lemma 4.2, if we set

$$F = \{K \subseteq I: \text{ for some partition } \{b_j\}_{j\in J} \text{ of unity in } \mathfrak{B}, \bigcup_{j\in J} \lambda b_j \subseteq K\},$$

then $\mathfrak{A}\langle\mathfrak{B}\rangle/\theta(F) \cong \mathfrak{A}[\mathfrak{B}]$, hence (iii) follows. If we let $\mathfrak{B} = \lambda\mathfrak{B} = \langle P(I), \cup, \cap, ', \emptyset, I\rangle$ then $\mathrm{Part}_{\mathfrak{B}}^{(m')}(Y_0, \ldots, Y_{m'})$ in part (iii) of the definition of T is the same as $\mathrm{Part}(Y_0, \ldots, Y_{m'})$, hence (v) holds. For (vi) we use Ershov's result that every Boolean algebra is elementarily equivalent to a reduced power of **2**, and then (vii) and (viii) follow from well-known preservation theorems for reduced powers (see [6]). (ix) can be easily proved using the methods of Baldwin and Lachlan [2], p. 105. For (x) note that by Shelah [42] there is an index set I and an ultrafilter U on $\mathbf{2}^I$ such that $(\mathfrak{A}[\mathfrak{B}_0])^I/U \cong (\mathfrak{A}[\mathfrak{B}_1])^I/U$. From Proposition 2.6 we have $\mathfrak{A}[\mathfrak{B}_0^I/U] \cong \mathfrak{A}[\mathfrak{B}_1^I/U]$, hence $\mathfrak{B}_0^I/U \cong \mathfrak{B}_1^I/U$, so $\mathfrak{B}_0 \equiv \mathfrak{B}_1$.

Recently Olin [35] has shown that elementary equivalence need not be preserved by free products of semigroups, and Jónsson and Olin [23] have shown the same for any non-trivial variety of lattices. We know that $\mathfrak{B}_0 * \mathfrak{B}_1 \cong \mathfrak{B}_0[\mathfrak{B}_1]^*$, hence Theorem 4.3 guarantees the following[6]).

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[5]) Banaschewski and Nelson have recently proved that $\mathfrak{A}[\mathfrak{B}]$ preserves elementary equivalence and elementary substructure in both arguments $\mathfrak{A}$ and $\mathfrak{B}$; also they have shown $\mathfrak{A}[\mathfrak{B}]^* \prec \mathfrak{A}[\mathfrak{B}]$.

[6] Olin has informed the author that he has proved 4.4(i) using the Tarski invariants for Boolean algebras, and Sabbagh has 4.4 as a special case of results on Λ-associative algebras. Also Sabbagh pointed out that an operation preserving elementary substructure will preserve elementary equivalence.

COROLLARY 4.4. (i) (Węglorz [55] 1974) *If* $\mathfrak{B}_0 \equiv \hat{\mathfrak{B}}_0$ *and* $\mathfrak{B}_1 \equiv \hat{\mathfrak{B}}_1$ *then* $\mathfrak{B}_0 * \mathfrak{B}_1 \equiv \hat{\mathfrak{B}}_0 * \hat{\mathfrak{B}}_1$, *and* (ii) *if* $\mathfrak{B}_0 \prec \hat{\mathfrak{B}}_0$ *and* $\mathfrak{B}_1 \prec \hat{\mathfrak{B}}_1$ *then* $\mathfrak{B}_0 * \mathfrak{B}_1 \prec \hat{\mathfrak{B}}_0 * \hat{\mathfrak{B}}_1$.

Next we show that any two bounded Boolean powers of a given $\mathfrak{A}$ have isomorphic reduced powers.

COROLLARY 4.5. *For any* $\mathfrak{A}$, $\mathfrak{B}_0$ *and* $\mathfrak{B}_1$ *(the Boolean algebras being nontrivial) there is an index set I and filter F such that*

$$(\mathfrak{A}[\mathfrak{B}_0]^*)^I/F \cong (\mathfrak{A}[\mathfrak{B}_1]^*)^I/F.$$

Proof. First let D denote the filter of cofinite subsets of ω. Then $\mathfrak{B}_0^\omega/D \equiv \mathfrak{B}_1^\omega/D$ since they are both atomless. Thus, by Shelah [42], there is an index set J and ultrafilter U such that $(\mathfrak{B}_0^\omega/D)^J/U \cong (\mathfrak{B}_1^\omega/D)^J/U$, hence for suitable K and E, $\mathfrak{B}_0^K/E \cong \mathfrak{B}_1^K/E$. Now, using Theorems 2.7 and 4.3, we have $\mathfrak{B}_i^K/E \equiv \mathfrak{B}_i[\mathbf{2}^K/E]^*$, $i=0, 1$, hence

$$\mathfrak{A}[\mathfrak{B}_i^K/E]^* \equiv \mathfrak{A}[\mathfrak{B}_i[\mathbf{2}^K/E]^*]^* \cong (\mathfrak{A}[\mathfrak{B}_i]^*)\,[\mathbf{2}^K/E]^* \equiv (\mathfrak{A}[\mathfrak{B}_i]^*)^K/E, \quad i=0,1,$$

and this gives $(\mathfrak{A}[\mathfrak{B}_0]^*)^K/E \equiv (\mathfrak{A}[\mathfrak{B}_1]^*)^K/E$. Again, using Shelah [42] we have, for some I and F, $(\mathfrak{A}[\mathfrak{B}_0]^*)^I/F \cong (\mathfrak{A}[\mathfrak{B}_1]^*)^I/F$.

Using similar techniques we can show any two reduced powers of $\mathfrak{A}$ have isomorphic reduced powers.

COROLLARY 4.6. *For any* $\mathfrak{A}$, I_0, F_0, I_1, *and* F_1, *there is an I and F such that*

$$(\mathfrak{A}^{I_0}/F_0)^I/F \cong (\mathfrak{A}^{I_1}/F_1)^I/F.$$

Ershov made the basic observation that if $\mathfrak{A}$ is a finite algebra then there is an unusually transparent technique for translating assertions about $\mathfrak{A}[\mathfrak{B}]$ into assertions about $\mathfrak{B}$, namely he gives an effective translation t from $\mathscr{L}(\mathfrak{A}[\mathfrak{B}])$ into $\mathscr{L}(\mathfrak{B})$ such that the following is true.

THEOREM 4.7. (Ershov [10]) *Let* σ *be a sentence in* $\mathscr{L}(\mathfrak{A}[\mathfrak{B}])$, *where* $\mathfrak{A}$ *is a finite algebra. Then* $\mathfrak{A}[\mathfrak{B}] \models \sigma$ *iff* $\mathfrak{B} \models t(\sigma)$.

To describe t suppose that the universe of $\mathfrak{A}$ is $\{0, 1, \ldots, n-1\}$. Let $\mathrm{Part}(x_0, \ldots, x_{n-1})$ be the predicate in $\mathscr{L}(\mathfrak{B})$ defined by

$$\mathrm{Part}(x_0, \ldots, x_{n-1}) \leftrightarrow (x_0 \vee \ldots \vee x_{n-1} = 1) \ \& \ \mathop{\&}_{i<j} (x_i \wedge x_j = 0).$$

The following procedure suffices to define t recursively (if σ is first put into a suitable form):

(i) $$t(f_\gamma(x, y, \ldots) = z) = \mathop{\&}_{i \leqslant n-1} \Big[\bigvee_{f_\gamma(i_0, \ldots, i_{n-1}) = i} (x_{i_0} \wedge y_{i_1} \wedge \ldots) = z_i \Big],$$

where the $x, y, \ldots, z$ are either variables or members of $|\mathfrak{A}[\mathfrak{B}]|$. If x is a variable, then $x_0, \ldots, x_{n-1}$ are n new variables, and if $x \in |\mathfrak{A}[\mathfrak{B}]|$, then $x_i = x(i)$, etc.;

(ii) $t(\sigma \mid \tau) = t(\sigma) \mid t(\tau)$, where $\mid$ is the Sheffer stroke;

and

(iii) $t(\exists x\sigma) = \exists x_0 \ldots \exists x_{n-1}\,(\mathrm{Part}(x_0, \ldots, x_{n-1})\ \&\ t(\sigma))$.

An algebra $\mathfrak{A}$ is *κ-saturated* [27] if every set of formulas $\{\sigma_i(x_0): i \in I\}$, in the language of $\mathfrak{A}$, with fewer than κ parameters from $|\mathfrak{A}|$, which is finitely satisfiable in $\mathfrak{A}$ is also satisfiable in $\mathfrak{A}$.

COROLLARY 4.8. *If $\mathfrak{A}$ is a finite algebra and $\mathfrak{B}$ is a κ-saturated Boolean algebra, $\kappa \geqslant \omega$, then $\mathfrak{A}[\mathfrak{B}]$ is* κ-saturated.

Proof. Let $\{\sigma_i(x_0): i \in I\}$ have fewer than κ parameters from $|\mathfrak{A}[\mathfrak{B}]|$ and suppose it is finitely satisfiable in $\mathfrak{A}[\mathfrak{B}]$. Then $\{t(\sigma_i): i \in I\}$ has (at most) $n = \mathrm{Card}\,|\mathfrak{A}|$ free variables appearing and fewer than κ parameters from $\mathfrak{B}$, hence it is satisfiable in $\mathfrak{B}$, and from a solution $b_0, \ldots, b_{n-1}$ in $|\mathfrak{B}|$ we can construct a solution $\xi \in |\mathfrak{A}[\mathfrak{B}]|$ of $\{\sigma_i(x_0): i \in I\}$.

Thus if $\mathfrak{A}$ is finite and $\mathfrak{B}$ is saturated it follows that $\mathfrak{A}[\mathfrak{B}]$ is saturated. Pacholski has shown (see [52]) that $\mathfrak{A}$ and $\mathfrak{B}$ are both countably saturated need not imply that $\mathfrak{A}[\mathfrak{B}]^*$ is countably saturated.

An algebra $\mathfrak{A}$ is *equationally compact* [53] if any set of atomic formulas in the language of $\mathfrak{A}$, with parameters from $|\mathfrak{A}|$, which is finitely satisfiable in $\mathfrak{A}$ is also satisfiable in $\mathfrak{A}$.

COROLLARY 4.9. *If $\mathfrak{A}$ is a finite algebra and $\mathfrak{B}$ is a complete Boolean algebra then $\mathfrak{A}[\mathfrak{B}]$ is equationally compact.*

Proof. If $p = q$ is an atomic formula in the language of $\mathfrak{A}$ with parameters from $|\mathfrak{A}[\mathfrak{B}]|$ then $t(p = q)$ is equivalent to a conjunction of atomic formulas in the language of Boolean algebras with parameters from $|\mathfrak{B}|$. With this we only need to note that $\mathfrak{B}$ is equationally compact iff it is complete [53].

Remark. One might hope to extend Corollary 4.9 by replacing '$\mathfrak{A}$ is finite' by '$\mathfrak{A}$ is equationally compact'; however Mansfield [29] points out that if $\mathfrak{B}_0$ and $\mathfrak{B}_1$ are two complete Boolean algebras then $\mathfrak{B}_0[\mathfrak{B}_1]$ need not be complete, hence not equationally compact. Also $\mathbf{2}^\omega[\mathbf{2}^\omega]^*$ is not a complete Boolean algebra, so we cannot extend Corollary 4.9 using bounded Boolean powers.

§5. A theorem of M.O. Rabin generalized to *m*-rings

In 1967 Rabin [38] proved that the theory of countable Boolean rings with quantification over ideals is decidable. Comer [8] combined this with a sheaf-theoretic generalization of the Feferman-Vaught results and known representation theorems for m-rings (see below) to show that the first-order theory of m-rings (where m is fixed) is decidable. In this section we will substitute the direct techniques of Ershov (see § 4) for the Feferman-Vaught methods to show that, *for a given m, the theory of countable m-rings with quantification over ideals is decidable.*

An *m-ring* $(m \geqslant 2)$ $\mathfrak{R} = \langle R, +, \cdot \rangle$ is a (commutative) ring with unity satisfying $x^m = x$.

Let $\mathscr{BA}_{\omega_1}$ and $\mathscr{R}^{(m)}_{\omega_1}$ denote the class of countable Boolean algebras, respectively m-rings. Let $\mathscr{L}^{\mathscr{I}}_{\mathscr{BA}}$ and $\mathscr{L}^{\mathscr{I}}_{\mathscr{R}}$ denote the first-order language of Boolean algebras, respectively rings, augmented by quantification over ideals. (We use capital letters for ideals.)

THEOREM 5.1. (Rabin) *$\{\sigma \in \mathscr{L}^{\mathscr{I}}_{\mathscr{BA}} : \mathscr{BA}_{\omega_1} \models \sigma\}$ is decidable.*

Remarks. It is interesting to note that Theorem 5.1 is equivalent to the statement: $\mathrm{Th}\{\mathrm{Con}(\mathfrak{B}) : \mathfrak{B} \in \mathscr{BA}_{\omega_1}\}$ is decidable. Comer recently pointed out to the author that $\mathfrak{B}_0 \equiv \mathfrak{B}_1$ does not lead to $\mathrm{Con}(\mathfrak{B}_0) \equiv \mathrm{Con}(\mathfrak{B}_1)$.

The remainder of this section is devoted to proving the following.

THEOREM 5.2. *$\{\sigma \in \mathscr{L}^{\mathscr{I}}_{\mathscr{R}} : \mathscr{R}^{(m)}_{\omega_1} \models \sigma\}$ is decidable.*

First we need the two ring representation results used by Comer.

LEMMA 5.3. *If $\mathfrak{R}$ is an m-ring then $\mathfrak{R}$ is isomorphic to a product of $p_i^{e_i}$-rings $\mathfrak{R}_i$, where $(p_i^{e_i} - 1) \mid (m-1)$.*

LEMMA 5.4. (Pierce [37]) *Let p be a prime number, e a positive integer, and suppose $\mathfrak{R}$ is a countable p^e-ring. Let $\mathfrak{F}_1, \ldots, \mathfrak{F}_e$ be the subfields of $\mathfrak{G}\mathfrak{F}(p^e)$, where $\mathfrak{F}_1$ is the prime subfield. Then there exists a countable Boolean algebra $\mathfrak{B}$ and ideals $I_1, \ldots, I_e$, with $I_1 = |\mathfrak{B}|$, such that $\mathfrak{R}$ is isomorphic to the subalgebra $\overline{\mathfrak{R}}$ of $\mathfrak{G}\mathfrak{F}(p^e)\,[\mathfrak{B}]$ where*

$$|\overline{\mathfrak{R}}| = \{\xi \in |\mathfrak{G}\mathfrak{F}(p^e)\,[\mathfrak{B}]| : \xi(a) \in \bigvee \{I_j : a \in |\mathfrak{F}_j|\}\}. \tag{5.5}$$

Furthermore these ideals satisfy

$$F_j \subseteq F_k \textit{ implies } I_j \supseteq I_k. \tag{5.6}$$

First let it be noted that if I is an ideal of $\mathfrak{B}$ then

$$\{\xi \in \overline{\mathfrak{R}} : \bigvee_{a \neq 0} \xi(a) \in I\} \tag{5.7}$$

is an ideal of $\overline{\mathfrak{R}}$, and furthermore every ideal of $\overline{\mathfrak{R}}$ is of this form. Also if $I_1, \ldots, I_e$

are ideals satisfying (5.6) and $I_1=|\mathfrak{B}|$, then the set of elements determined by the right-hand side of (5.5) is a subuniverse of $\mathfrak{G}\mathfrak{F}(p^e)\,[\mathfrak{B}]$. Let $0=a_0,\dots,a_{p^e-1}$ be the elements of $|\mathfrak{G}\mathfrak{F}(p^e)|$. It is straight-forward to write out a predicate $\mathbf{D}(x,y,\dots;X_1,\dots,X_e)$ in $\mathscr{L}^{\mathscr{I}}_{\mathscr{B}\mathscr{A}}$ such that ξ is in the right side of (5.5) iff $\mathfrak{B}\models\mathbf{D}(\xi(a_0),\dots,\xi(a_{p^e-1});I_1,\dots,I_e)$, so let us proceed to define a translation τ from $\mathscr{L}^{\mathscr{I}}_{\mathscr{R}}$ to $\mathscr{L}^{\mathscr{I}}_{\mathscr{B}\mathscr{A}}$ as follows, where the variables $X_1,\dots,X_e$ appear only as introduced in the fourth line below:

$$\tau(f_\gamma(x,y,\dots)=z)=t(f_\gamma(x,y,\dots)=z),\qquad \text{(see §4)}$$
$$\tau(x\in X)=x_1\vee\dots\vee x_{p^e-1}\in X,$$
$$\tau(\sigma_0\mid\sigma_1)=\tau(\sigma_0)\mid\tau(\sigma_1),$$
$$\tau(\exists x\sigma)=\exists x_0\dots\exists x_{p^e-1}(\mathbf{D}(x_0,\dots,x_{p^e-1};X_1,\dots,X_e)\ \&\ \tau(\sigma)),$$

and

$$\tau(\exists X\sigma)=\exists X\tau(\sigma).$$

With this translation it is routine to verify the following, where $[X_i/I_i]$ means that X_i is to be replaced by I_i, $1\leqslant i\leqslant e$.

LEMMA 5.8. *Let σ be a sentence in $\mathscr{L}^{\mathscr{I}}_{\mathscr{R}}$ and let $\mathfrak{B}$ be a Boolean algebra with ideals $I_1,\dots,I_e$ satisfying* (5.6) *and $I_1=|\mathfrak{B}|$. Let $\mathfrak{R}$ be the ring determined by the right side of* (5.5). *Then*

$$\mathfrak{R}\models\sigma\quad\textit{iff}\quad\mathfrak{B}\models\tau(\sigma)\,[X_i/I_i].$$

Let $\mathbf{ID}(X_1,\dots,X_e)$ be a predicate in $\mathscr{L}^{\mathscr{I}}_{\mathscr{B}\mathscr{A}}$ expressing '$F_j\subseteq F_k$ implies $X_j\supseteq X_k$, and $X_1=|\mathfrak{B}|$'.

LEMMA 5.9. *Let σ be a sentence in $\mathscr{L}^{\mathscr{I}}_{\mathscr{R}}$. Then*

$$\mathscr{R}^{(p^e)}_{\omega_1}\models\sigma\quad\textit{iff}\quad\mathscr{B}\mathscr{A}_{\omega_1}\models\forall X_1\dots\forall X_e(\mathbf{ID}(X_1,\dots,X_e)\rightarrow\tau(\sigma)).$$

Now to prove Theorem 5.2 it suffices to note that every countable m-ring $\mathfrak{R}$ can be written as a product $\mathfrak{R}_1\times\cdots\times\mathfrak{R}_l$, where $\mathfrak{R}_i$ is a $p_i^{e_i}$-ring, and any ideal I of $\mathfrak{R}_1\times\cdots\times\mathfrak{R}_l$ can be expressed as a product $I_1\times\cdots\times I_l$, where I_j is an ideal of $\mathfrak{R}_j$, $1\leqslant j\leqslant l$, for with these observations it is routine to reduce the question of $\mathscr{R}^{(m)}_{\omega_1}\models\sigma$ to the decision procedures for $\mathscr{R}^{(p_i^{e_i})}_{\omega_1}$, $1\leqslant i\leqslant l$, and then via Lemma 5.9 to $\mathscr{B}\mathscr{A}_{\omega_1}$. Also, exactly as Rabin did we can establish the next corollary.

COROLLARY 5.10. *The theory of m-rings with a sequence of distinguished ideals is decidable.*

§6. Boolean extensions of primal algebras

A *primal algebra* $\mathfrak{A}$ is a non-trivial finite algebra such that for any function $f: |\mathfrak{A}|^n \to |\mathfrak{A}|$ there is a polynomial p such that $p(a_0, \ldots, a_{n-1}) = f(a_0, \ldots, a_{n-1})$, $a_0, \ldots, a_{n-1} \in |\mathfrak{A}|$. As stated in the introduction Foster noted that every algebra in the variety generated by a primal $\mathfrak{A}$ is isomorphic to $\mathfrak{A}[\mathfrak{B}]$ for some Boolean algebra $\mathfrak{B}$. We will choose for our primal algebra with an n-element universe, $n \geqslant 2$, the algebra $\mathfrak{P}_n = \langle P_n, \vee, \wedge, 0, a_1, \ldots, a_{n-2}, 1, \chi_{\neq} \rangle$, where $P_n = \{0, a_1, \ldots, a_{n-2}, 1\}$, the operations $\vee$, $\wedge$ are lattice operations with $0 < a_1 < \cdots < a_{n-2} < 1$, $\chi_{\neq}(x, y) = 1$ iff $x \neq y$, $\chi_{\neq}(x, y) = 0$ if $x = y$. Referring to Theorem 3.4 with $p(x) = x$, $q(x) = \chi_{\neq}(0, x)$, $p^{\vee} = \vee$, $p^{\wedge} = \wedge$, $p'(x) = \chi_{\neq}(1, x)$, *we can use first order formulas to recapture* $\mathfrak{B}$ *from* $\mathfrak{P}_n[\mathfrak{B}]$. But this, along with the fact that $\mathfrak{P}_n$ satisfies the conditions of Theorem 3.5, immediately leads to a strengthening of the results in §4 and §5 for $\mathscr{V}(\mathfrak{P}_n)$, the variety generated by $\mathfrak{P}_n$.

THEOREM 6.1. (i) *We can replace the words 'Boolean algebra' by 'a member of* $\mathscr{V}(\mathfrak{P}_n)$*' in Hanf's direct product results,*

(ii) $\mathscr{V}(\mathfrak{P}_n)$ *has* 2^{λ} *isomorphism types of algebras of power* λ, $\lambda \geqslant \omega$,

(iii) *if* $\mathscr{K}$ *is a class of Boolean algebras then* $\mathrm{Th}(\mathscr{K})$ *is decidable iff* $\mathrm{Th}(\{\mathfrak{P}_n[\mathfrak{B}] : \mathfrak{B} \in \mathscr{K}\})$ *is decidable (one direction of this was stated by* Ershov [10]),

(iv) $\mathfrak{P}_n[\mathfrak{B}_0] \equiv \mathfrak{P}_n[\mathfrak{B}_1]$ *iff* $\mathfrak{B}_0 \equiv \mathfrak{B}_1$, *(similarly for* $\prec$*)*

(v) $\mathfrak{P}_n[\mathfrak{B}]$ *is* κ*-saturated iff* $\mathfrak{B}$ *is* κ*-saturated,* $\kappa \geqslant \omega$,

(vi) $\mathfrak{P}_n[\mathfrak{B}]$ *is equationally compact iff* $\mathfrak{B}$ *is complete,*

(vii) *every congruence of* $\mathfrak{P}_n[\mathfrak{B}]$ *is of the form* θ_F, F *a filter on* $\mathfrak{B}$.

It is straight-forward to verify that free products in $\mathscr{V}(\mathfrak{P}_n)$ are given by

$$\mathfrak{P}_n[\mathfrak{B}_0] * \mathfrak{P}_n[\mathfrak{B}_1] \cong \mathfrak{P}_n[\mathfrak{B}_0 * \mathfrak{B}_1],$$

and thus using Theorem 6.1 (iv) and Corollary 4.4 we have the following.

THEOREM 6.3. *Free products preserve elementary equivalence and elementary substructure in* $\mathscr{V}(\mathfrak{P}_n)$.

§7. Concluding remarks and problems

The reader will no doubt have recognized that many of the results make sense for the more general class of first-order structures, and a detailed study of the impact of Boolean powers on Horn classes of structures would be of interest. Steve Schmidt has informed us that the full three-graph is B-separating.

PROBLEM 1. For which finite algebras $\mathfrak{A}$ is the variety generated by $\mathfrak{A}$ just the

Boolean powers of $\mathfrak{A}$, up to isomorphism? (This is closely related to Taylor's [49] work on the fine spectrum of a variety.)

In 1957 Jónsson [22] published examples of Abelian groups displaying some of the Hanf phenomena.

PROBLEM 2. Is there a B-separating Abelian group?

PROBLEM 3. Find conditions on $\mathfrak{A}[\mathfrak{B}]$ such that $\mathrm{Con}\,\mathfrak{A}[\mathfrak{B}]=\{\theta_F\colon F$ a filter on $\mathfrak{B}\}$. (See Burris and Jeffers [4] for partial results.)

PROBLEM 4. If $\mathfrak{A}$ is a finite B-separating algebra and $\mathfrak{A}[\mathfrak{B}]$ is equationally compact, is $\mathfrak{B}$ complete? If $\mathfrak{A}[\mathfrak{B}]$ is κ-saturated ($\kappa \geqslant \omega$), is $\mathfrak{B}$ κ-saturated?

PROBLEM 5. If $\mathfrak{A}$ is B-separating and $\mathfrak{A}[\mathfrak{B}_0]^* \equiv \mathfrak{A}[\mathfrak{B}_1]^*$, does it follow that $\mathfrak{B}_0 \equiv \mathfrak{B}_1$?

Olin [34] has an effective procedure for computing the Tarski invariants of $\mathfrak{B}_0 * \mathfrak{B}_1$ given those of $\mathfrak{B}_0$ and $\mathfrak{B}_1$. (In 1973 Omarov [36] calculated the invariants of $\mathfrak{B}^I/F$ given those of $\mathfrak{B}$ and $\mathbf{2}^I/F$. Olin informed the author that Omarov's calculations agreed with his own, which is not surprising in view of $\mathfrak{B} * \mathbf{2}^I/F \cong \mathfrak{B}[\mathbf{2}^I/F]^* \equiv \mathfrak{B}^I/F$, as we have shown.) From this it follows that the predicate 'σ is preserved under free products of Boolean algebras' is arithmetical. From our previous considerations, if σ is preserved under reduced powers of Boolean algebras then it preserved under free products. However the sentence 'there is a unique atom' is preserved under free products of Boolean algebras but not under direct powers.

PROBLEM 6. Characterize those sentences preserved under free products of Boolean algebras.

Higgs kindly communicated the construction used for $\mathfrak{A}[\mathfrak{B}]$ in Theorem 4.3(iii), pointing out that the basic ideas are partly contained in [40]. Not only does this explicitly give $\mathfrak{A}[\mathfrak{B}] \in \mathbf{HSP}(\mathfrak{A})$, but upon closer inspection, $\mathfrak{A}[\mathfrak{B}] \in \mathbf{SP}_R(\mathfrak{A})$, where $\mathbf{H}$, $\mathbf{S}$, $\mathbf{P}$ and $\mathbf{P}_R$ are the operators homomorphism, subalgebra, product and reduced product (see [16]).

PROBLEM 7. Characterize those sentences preserved under Boolean powers[7]).

PROBLEM 8. Characterize those Horn theories which contain B-separating algebras.

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[7]) It follows from the results of Banaschewski and Nelson – see footnote (5) – that disjuncts of Horn sentences are preserved.

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