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BOOLEAN CONSTRUCTIONS

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When I was first studying decidability results in the late 1960's and early 1970's it seemed there must be some fundamental connection between classes of algebras having a decidable first-order theory and classes of algebras having a fairly transparent structure theory. We have made alot of progress in recent years, and at present it seems that the only good candidates for constructions to describe a good structure theory are certain Boolean constructions called Boolean products.

Since Boolean products were so useful in obtaining positive decidability results it came as somewhat of a surprise that a modification of the more specialized Boolean power construction would lead to sweeping undecidability results. The tale of these developments, and many others, will be sketched in the following survey.

CONTENTS

§1. AN EXAMPLE CONCERNING BOOLEAN CONSTRUCTIONS

As mentioned in the introduction I became interested in structure theorems about ten years ago because of work on decidability. A basic example is that of a variety generated by a finite Abelian group G -- certainly this has a very good structure theory since every member of the variety is isomorphic to a direct sum of copies of cyclic subgroups of G. And the variety $V(Z_2)$ generated by the ring Z_2 of integers modulo 2, where the language is $\{+,\cdot,-,0,1\}$, is considered to be well-behaved, being the variety of Boolean rings. Now let us look at the variety $V(Z_3)$ generated by the ring Z_3. At first one might be inclined to think that this variety is quite different from Boolean rings -- we shall see that there is a very tight bond between them.

By a result of McCoy and Montgomery [1937] we know that every member of $V(Z_3)$ is isomorphic to a subdirect power of Z_3, so let $R \leq (Z_3)^I$ be a subdirect power of Z_3. For $X \subseteq I$ let us define χ_X by

$$\chi_X = \begin{cases} 1 & \text{if } i \in X \\ 0 & \text{if } i \notin X. \end{cases}$$

Now the set $B = \{X \subseteq I: \chi_X \in R\}$ is a field of subsets of I as

$$\chi_{X\cup Y} = \chi_X + \chi_Y - \chi_X \cdot \chi_Y$$

$$\chi_{X\cap Y} = \chi_X \cdot \chi_Y$$

$$\chi_{-X} = 1 - \chi_X.$$

Next, for $f \in R$ let

$$f_0 = 1 - f^2$$

$$f_1 = 1 - (f-1)^2$$

$$f_2 = 1 - (f+1)^2.$$

Then one can check that $\chi_{f^{-1}(i)} = f_i \in R$ for $i = 0,1,2$. Thus we see that there is a one-one mapping ϕ from R to B^3 defined by $\phi(f) = \langle f^{-1}(0), f^{-1}(1), f^{-1}(2)\rangle$. If now $f \in R$ and $\phi(f) = \langle X,Y,Z\rangle$ then clearly

$$X \cap Y = X \cap Z = Y \cap Z = \emptyset \tag{1}$$

$$X \cup Y \cup Z = I. \tag{2}$$

Conversely, if $\langle X,Y,Z\rangle$ is any triple from B^3 satisfying (1) and (2) then $f = \chi_Y + 2\chi_Z$ is an element of R, and $\phi(f) = \langle X,Y,Z\rangle$. Thus letting R^* be the set of triples from B^3 which satisfy (1) and (2) we see that ϕ is a bijection from R to R^*. Next we wish to endow R^* with the unique ring structure which makes ϕ an isomorphism. We can express this in the language of sets by observing that, for $f,g \in R$,

$$(f+g)^{-1}(0) = [f^{-1}(0) \cap g^{-1}(0)] \cup [f^{-1}(1) \cap g^{-1}(2)] \cup [f^{-1}(2) \cap g^{-1}(1)]$$
$$(f+g)^{-1}(1) = [f^{-1}(0) \cap g^{-1}(1)] \cup [f^{-1}(1) \cap g^{-1}(0)] \cup [f^{-1}(2) \cap g^{-1}(2)]$$
$$(f+g)^{-1}(2) = [f^{-1}(0) \cap g^{-1}(2)] \cup [f^{-1}(2) \cap g^{-1}(0)] \cup [f^{-1}(1) \cap g^{-1}(1)].$$

Thus given $\langle X,Y,Z\rangle$, $\langle U,V,W\rangle \in R^*$ we must define

(3) $\langle X,Y,Z\rangle + \langle U,V,W\rangle = \langle (X\cap U)\cup(Y\cap W)\cup(Z\cap V),\ (X\cap V)\cup(Y\cap U)\cup(Z\cap W),\ (X\cap W)\cup(Z\cap U)\cup(Y\cap V)\rangle.$

Likewise we can derive

(4) $\langle X,Y,Z\rangle \cdot \langle U,V,W\rangle = \langle X\cup U,\ (Y\cap V)\cup(Z\cap W),\ (Y\cap W)\cup(Z\cap V)\rangle$

(5) $-\langle X,Y,Z\rangle = \langle X,Z,Y\rangle.$

The zero element of R^* is $\langle I,\emptyset,\emptyset\rangle$, and the identity is $\langle\emptyset,I,\emptyset\rangle$.

Since the definitions (1)-(5) are phrased entirely in terms of union and intersection it is clear that, given B, we can readily reconstruct R. Furthermore, given any Boolean algebra $\langle B,\vee,\wedge,',0,1\rangle$ we can use (1)-(5), replacing $\cup$ by $\vee$ and $\cap$ by $\wedge$, to obtain a member of $V(Z_3)$. Thus, in the language of logicians, we have interpreted the theory of $V(Z_3)$ into the theory of Boolean algebras.

These same ideas show that the theory of $V(Z_p)$, for p a prime, can be interpreted into the theory of Boolean algebras, by using p-tuples instead of 3-tuples. Thus the reader can readily see that the study of a number of interesting varieties reduces to the study of Boolean algebras. We will return to this construction in §4.

One might try similar ideas on $V(Z_4)$. The only nontrivial subdirectly irreducibles in this variety are Z_2 and Z_4. But there is a major obstacle here.

§2. THE DEVELOPMENT OF BOOLEAN ALGEBRA

When Boole first presented his mathematical analysis of the laws of human thought in 1847 he did not have the notion of an algebraic structure $\langle B,\vee,\wedge,',0,1\rangle$, but rather he was concerned with the syntactic side of Boolean algebras. This seems to have been the vantage point until Huntington's 1904 paper on postulates for the algebra of logic.

G. BOOLE 1847, 1854

Boole introduced a mathematical analysis into logic which was analogous to existing work in algebra. He was primarily concerned with equations and deductions from equations -- however in modern terminology he was not working with equations in the language of Boolean algebras since his $+$ was a partial operation which could only be applied to disjoint elements.

W.S. JEVONS 1864

C.S. PEIRCE 1867

They suggest that Boole's exclusive + be replaced by the now common inclusive +.

C.S. PEIRCE 1880

Peirce presented a mathematical development of the algebra of logic based on the copula, denoted by $-\!\!<$. On pages 32-33 of this paper he gives what appears to be a flawless development of the equational axioms for lattices based on what we now call a partial order with g.l.b.s. and l.u.b.s. (He does not isolate the concept that we call lattices in any way -- this is just a fragment of his development of an algebra of logic.) There are two difficulties with his treatment. First, although the copula is treated like a partial order in this development, in previous pages and subsequent pages it acts like a binary operation, for example on page 34 one sees the expression $(a+b -\!\!< a) -\!\!< (b -\!\!< a)$. The second problem is that he claims the distributive laws follow from his definitions.

It is interesting to note the significance attached to each identity discovered. Peirce is careful about allocating credits, and his list includes:

$x = x + x;\ x \times x = x$	(Jevons 1864)
$a + b = b + a;\ a \times b = b \times a$	(Boole, Jevons)
$(a+b)+c = a+(b+c);\ a\times(b\times c) = (a\times b)\times c$	(Boole, Jevons)
$(a+b)\times c = (a\times c) + (b\times c)$	(Boole, Jevons)
$(a\times b)+c = (a+c)\times(b+c)$	(Peirce 1867)
$a+(a\times b) = a;\ a\times(a+b) = a$	(Grassmann, Schröder)

E. SCHRÖDER 1890 - 1895

Schröder was strongly influenced by Peirce's axiomatic approach to the algebra of logic and wrote three volumes (approximately 2,000 pages) on the subject. He followed the aforementioned development of Peirce, but used subsumption, denoted by $\Subset$, as his fundamental notion. Also he realized the distributive law could not be derived as Peirce had claimed, and indeed gives two counterexamples in the appendices to volume one. The difficulty he encountered in constructing a counterexample seems to indicate that he never achieved a thoroughly abstract view of the algebra of logic -- the counterexamples he constructed were intimately connected with deductive systems. The first counterexample was obtained by looking at deductively closed subsets of a collection of 990 quasigroup equations. (We note that Schröder had invented, equationally defined, and studied abstract quasigroups previously.) His second counterexample uses, in modern terminology, the subalgebras of the free Boolean algebra with three free generators.

R. DEDEKIND 1897

Dedekind realized that Schröder's development of the laws of logic overlapped considerably with his unpublished investigations into the laws which govern the combinations of modules (in algebraic number theory) using g.c.d. and l.c.m. Then, as a common abstraction, he defines in full generality what we now call lattices (he called them dualgroups) using the following set of axioms (replacing his operation symbols $+,-$ by $\vee,\wedge$):

(1')	$\alpha \vee \beta = \beta \vee \alpha$
(1")	$\alpha \wedge \beta = \beta \wedge \alpha$
(2')	$(\alpha\vee\beta)\vee\gamma = \alpha\vee(\beta\vee\gamma)$
(2")	$(\alpha\wedge\beta)\wedge\gamma = \alpha\wedge(\beta\wedge\gamma)$
(3')	$\alpha\vee(\alpha\wedge\beta) = \alpha$
(3")	$\alpha\wedge(\alpha\vee\beta) = \alpha.$

Shortly after listing these axioms he proceeds to find the smallest examples of lattices which fail to satisfy (1) the modular law, and (2) the distributive law, namely the lattices we now call N_5 and M_5.

A.N. WHITEHEAD 1898

Whitehead's book Universal Algebra was an attempt to unify the study of the important algebraic systems -- all had two binary operations called addition and multiplication, and these operations satisfied several basic laws such as $a + b = b + a$, etc. Such algebras were divided into two basic types, those of numerical genus and those of non-numerical genus, the latter satisfying $a + a = a$. His only example of an algebra of non-numerical genus was the algebra of symbolic logic. Linear associative algebras give examples of algebras of the numerical genus. He does not treat scalar multiplication by complex numbers as fundamental operations requiring further axioms, but rather as a natural extension of writing $2a$ for $a + a$. By current standards his scope was extremely narrow, not even including groups.

E.V. HUNTINGTON 1904

Huntington gives three sets of postulates for the algebra of logic, one based on $\oplus, \odot$, one on $\oslash$, and one on $\oplus$. He does not use the word 'algebra' to describe a set with operations. However he does introduce systems $\langle K,\oplus,\odot\rangle$, $\langle K,\oslash\rangle$, and $\langle K,\oplus\rangle$ to prove his postulates are independent. In the appendix he suggests that one use the name logical field (analogous to Galois field) for systems $\langle K,\oplus,\odot,\oslash\rangle$ which satisfy the laws of the algebra of logic. Then he shows (1) every finite logical field has 2^m elements, and (2) for each m there is exactly one logical field with 2^m elements.

E.L. POST 1921

The completeness of the propositional calculus is proved, and n-valued logics are introduced.

O. FRINK 1928

He shows that every Boolean algebra can be considered as a ring using the operations symmetric difference and meet.

M.H. STONE 1935, 1936

Stone rediscovers the result of Frink above, and proves that conversely every Boolean ring can be considered as a Boolean algebra.

G. BIRKHOFF 1933

He proves that every distributive lattice can be represented as a ring of sets.

M.H. STONE 1934, 1936

Every Boolean algebra can be represented as a field of sets.

M.H. STONE 1934, 1937

Stone develops the duality between Boolean algebras and Boolean spaces.

M.H. STONE 1938

"I believe it would be accurate to say that of the many books, memoirs, notes, and reviews (more than one hundred seventy-five in number [6]) which deal with Boolean algebras the great majority draw their inspiration directly or indirectly from the work of Boole. The orientation of these studies toward symbolic logic is apparent in their preoccupation with algorithms, identities, and equations, or with the logical interrelations of the formal properties of the various Boolean operations. Recently there has emerged a different tendency, namely, to view Boolean algebras structurally, as organic systems, rather than algorithmically. Although this tendency might naturally have been expected to take its origin either in the rich experience of algebraists or in the needs of mathematicians concerned with the calculus of classes, it sprang, in fact, from quite different sources as a recognizable, if somewhat remote, consequence of the work of Hilbert. The most intensive exploitation of this new tendency is due to Tarski and myself [28]-[39]. Tarski's theory of deductive systems, which is but one illustration of the way in which logic has been enriched by the sort of metamathematical inquiry first seriously attempted by Hilbert, deals with systems of propositions which are complete with respect to logical inference; from a mathematical point of view, it is therefore a theory of the relations between special subalgebras of a Boolean algebra. My own investigations are a systematic attempt to discuss the structure of Boolean algebras by the methods which have thrown so much light on far deeper algebraic problems. The need for investigations of this character was suggested to me by the theory of operator-rings in Hilbert space: there, as in other rings and linear algebras, the 'spectral' representation as a 'direct sum' of irreducible subrings reposes in essence upon the construction of an abstract Boolean algebra; and this construction, trivial for rings with strong chain conditions, is not trivial in the case of operator-rings."

§3. THE DEVELOPMENT OF STRUCTURE THEOREMS FOR VARIETIES

N.H. McCOY and D. MONTGOMERY 1937

They point out that Stone's representation of Boolean rings by rings of sets is clearly equivalent to the statement that every Boolean ring is isomorphic to a subring of a direct sum of rings F_2. They go on to prove that every p-ring (a commutative ring satisfying $a^p = a$ and $pa = 0$) is isomorphic to a subring of a direct sum of rings F_p. One of their basic observations is that given (abstract) algebras A and B, $A \in ISP(B)$ iff there are homomorphisms from A to B which separate points.

I. GELFAND 1941

Gelfand showed that certain Banach algebras are isomorphic to the algebra of continuous functions $C(X,R)$ or $C(X,\mathbb{C})$, where X is a compact Hausdorff space and R is the reals, $\mathbb{C}$ the complex numbers.

P.C. ROSENBLOOM 1942

Rosenbloom introduces the equational class P_n of n-valued Post algebras, for $n = 1,2,...,$ and proves that the finite members of P_n are finite powers of the smallest nontrivial member P_n. He does not know if $P_n = ISP(P_n)$.

G. BIRKHOFF 1944

Birkhoff introduces the concepts of subdirect product (he calls it subdirect union) and subdirectly irreducible, and proves that every algebra is a subdirect product of subdirectly irreducible algebras.

L.I. WADE 1945

He proves that P_n is the only nontrivial subdirectly irreducible n-valued Post algebra.

§4. THE INTRODUCTION OF BOOLEAN POWERS

R.F. ARENS and I. KAPLANSKY 1948

They blend the ideas of Stone and Gelfand, and thus introduce the first Boolean constructions. "Stone [23, Theorem 1] has shown that a Boolean ring with unit is the set of all open and closed sets in a compact zero-dimensional space. In slightly different terminology: a Boolean ring with unit is the set of all continuous functions from a compact zero-dimensional space to the field GF(2) of two elements." Then they proceed to look at representations of the form $C(X,R)$ where X is a Boolean space and R is a simple ring or algebra, and generalizations of such representations. Thus they introduced, for rings, what is now the most popular definition of a bounded Boolean power, namely the algebra of continuous functions $C(X,R)$ where X is a Boolean space and R is given the discrete topology. (This gives a subdirect power of R.)

A.L. FOSTER 1953a

Foster was apparently working completely independently of Arens and Kaplansky when he presented his version of Boolean powers. Given any algebra A and Boolean algebra B he defined the universe of the Boolean power $A[B]$ to be

$$\{f \in B^A: f(a_1) \wedge f(a_2) = 0 \quad \text{if} \quad a_1 \neq a_2, \bigvee_{a \in A} f(a) = 1\}.$$

If A is infinite then B is required to be a complete Boolean algebra. The fundamental operations are defined on $A[B]$ by

$$F(f_1,\ldots,f_n)(a) = \bigvee_{F(a_1,\ldots,a_n) = a} f_1(a_1) \wedge \ldots \wedge f_n(a_n).$$

(In the case that A is Z_3 note that this is exactly the construction we developed in §1.)

Foster also introduced a notion of normal subdirect power, and proved that for special finite algebras (so-called f-algebras), normal subdirect powers were essentially the same as Boolean powers.

A.L. FOSTER 1961

Foster introduces the bounded Boolean power construction $A[B]^*$ by adding the requirement "$|\{a \in A: f(a) \neq 0\}| < \omega$" to the definition of $A[B]$. Thus for A finite, $A[B] = A[B]^*$.

B. JÓNSSON 1962

In his review of Foster's 1961 paper, Jónsson points out that $A[B]^*$ is, in a natural manner, isomorphic to $C(X,A)$, where X is the Boolean space of B.

M. GOULD and G. GRÄTZER 1967

They give a new, more general definition of normal subdirect power (based on the normal transform) and prove that this construction is equivalent to the bounded Boolean power.

C.J. ASH 1975

S. BURRIS 1975

If we let $P_B(K)$ denote the class of all bounded Boolean powers of algebras in K then one has $P_B \cdot P_B \le I \cdot P_B$. This is based on the observation that $(A[B_1]^*)[B_2]^* \cong A[B_1 * B_2]^*$, $B_1 * B_2$ being the free product of B_1 and B_2.

§5. BOUNDED BOOLEAN POWER REPRESENTATION THEOREMS FOR VARIETIES

In the following $\mathcal{BA}$ denotes the variety of Boolean algebras.

M.H. STONE 1934, 1936

(rephrased) Let $\underset{\sim}{2}$ be a two-element Boolean algebra. Then $\mathcal{BA} = IP_B(\underset{\sim}{2})$.

R.F. ARENS and I. KAPLANSKY 1948

For F a finite field, $\{$algebras over F$\} = IP_B(F)$.

A.L. FOSTER 1953a/b

For A a primal algebra, $V(A) = IP_B(A)$.

R. QUACKENBUSH 1980

For finite algebras A, $V(A) = IP_B(A)$ iff A is quasiprimal with no proper subalgebras or A is simple modular-Abelian with a trivial subalgebra.

S. BURRIS and R. McKENZIE 1981

A variety V can be expressed as $IP_B(K)$ for some finite set K of finite algebras iff $V = V(A)$ where (a) A is quasiprimal with no proper subalgebras, or (b) A is a finite simple modular-Abelian algebra.

§6. DIRECT PRODUCT PHENOMENA AND THE NUMBER OF MODELS

Bounded Boolean powers have been extremely useful to show that the various direct product phenomena observed by Hanf in $\mathcal{BA}$ transfer to numerous other varieties, and to show that many varieties have the maximum possible number of isomorphism types of algebras in all suitably high powers. The key concept is that of a B-separating algebra -- A is such an algebra if for any $B_1, B_2 \in \mathcal{BA}$, $A[B_1]^* \cong A[B_2]^* \Rightarrow B_1 \cong B_2$.

W. HANF 1957

Hanf proved the following results:

(1) There exist denumerable Boolean algebras B_1 and B_2, given a positive integer n, such that for any positive integers m,k we have $B_1{}^m \cong B_2{}^m \times B_2{}^k$ iff $n|k$.

(2) There exist denumerable Boolean algebras B_1 and B_2, given n, such that $B_1{}^k \cong B_2{}^k$ iff $n|k$. (This was pointed out by Tarski.)

(3) There exists, for any $n \geq 2$, a Boolean algebra B such that $B \cong B \times \underset{\sim}{2}^n$ but $B \not\cong B \times \underset{\sim}{2}^k$ for $k = 1,\ldots,n-1$.

(4) There exists, for any $n \geq 3$, a Boolean algebra B such that $B \cong B^n$ but $B \not\cong B^k$ for $k = 2,3,\ldots,n-1$.

(5) There exist Boolean algebras B_1 and B_2 such that $B_1 \times B_2 \cong B_1 \times B_2 \times \underset{\sim}{2}$ but neither $B_1 \cong B_1 \times \underset{\sim}{2}$ nor $B_2 \cong B_2 \times \underset{\sim}{2}$.

A. TARSKI 1957

Letting A be the semigroup $\langle\omega,+\rangle$ Tarski shows that $A[B_1 \times B_2]^* \cong A[B_1]^* \times A[B_2]^*$, and also that A is B-separating. From this he concludes that the Hanf phenomena above (replacing $\underset{\sim}{2}$ by A) apply to commutative semigroups.

B. JÓNSSON 1957

Jónsson shows that indecomposable centerless countable algebras (defined within a special class of algebras) are B-separating, and hence again we have the Hanf phenomena.

G. BERGMAN 1972

He shows that if M is any module and B_1, B_2 are Boolean algebras of the same cardinality then $M[B_1]^* \cong M[B_2]^*$.

S. BURRIS 1975

(1) In this paper it is noted that if A is a B-separating algebra then $IP_B(A)$ has 2^κ isomorphism types of algebras for each $\kappa \geq |A|$.

(2) If S is an algebra such that for every positive n, $|\mathrm{Con}\, S^n| = 2^n$, then S is B-separating. In particular this shows that the nontrivial simple algebras in congruence-distributive varieties are B-separating.

J. KETONEN 1978

Ketonen vastly increases the possibilities for curious direct product phenomena in Boolean algebras by showing that any countable commutative semigroup can be embedded into the semigroup of isomorphism types of countable Boolean algebras under direct product.

J. LAWRENCE 1981

Lawrence shows the following results for groups:

(1) Every finite subdirectly irreducible group is B-separating.

(2) $G \times G$ is not B-separating for any group G.

K. HICKIN and J.M. PLOTKIN 1981

They continue the study of groups and prove:

(1) If G is a nonabelian group which (i) is not a central product of two nonabelian groups, or (ii) is finitely subdirectly irreducible, then G is B-separating.

(2) If G is a finitely generated nonabelian group then G has 2^{ω} countable Boolean powers.

J.T. BALDWIN and R.N. McKENZIE [a]

Using Boolean powers to help count the number of models in universal Horn classes they prove:

(1) Every nonabelian subdirectly irreducible algebra in a modular variety is B-separating.

(2) Every countable nonabelian algebra A has 2^{λ} distinct bounded Boolean powers of power λ for every uncountable λ, each of which is elementarily equivalent to $A[F_{BA}(\omega)]^*$.

§7. BOUNDED BOOLEAN POWERS AND INJECTIVES

B.A. DAVEY and H. WERNER 1979

A series of papers, starting in 1972, which show that in certain varieties the injectives, or weak injectives, are of the form (*) $A_1[B_1]^* \times \ldots \times A_n[B_n]^*$, where the A_i's are certain subdirectly irreducible algebras and the B_i's are complete Boolean algebras, are brought under the following theorem.

Let V be a variety, let K be a finite set of finite algebras from V and suppose $V_{SI} \subseteq IS(K)$, where V_{SI} is the class of subdirectly irreducibles in V. If there is a simplicity formula for K and K has factorizable congruences then the following are equivalent:

(i) A is a [weak] injective in V

(ii) A is of the form (*) above where each $A_i \in H(K) \cap V_{SI}$, A_i is a [weak] injective in V, B_i is a complete Boolean algebra, and the A_i are pairwise nonisomorphic.

P.H. KRAUSS [a]

Krauss shows that in filtral varieties with a finite number of nonisomorphic simple members in each finite cardinal the [weak] injectives are characterized as direct products of bounded Boolean powers of [weak] V_{SI}-injectives using complete Boolean algebras.

§8. RIGID ALGEBRAS

An algebra A is rigid if it has exactly one automorphism.

B. JÓNSSON 1951

Answering Problem 74 of Birkhoff's Lattice Theory Jónsson proves that there is an infinite rigid Boolean algebra.

S. BURRIS 1978 [b]

Suppose that S is a finitely generated algebra which is rigid and $|\mathrm{Con}\ S^n| = 2^n$ for all positive n. Then if B is a rigid Boolean algebra it follows that $S[B]^*$ is rigid.

J.D. MONK and W. RASSBACH 1979

They prove there exist 2^κ rigid Boolean algebras in every uncountable cardinal κ.

§9. MATRIX RINGS

In the following R denotes a Boolean ring, B the corresponding Boolean algebra, and S an arbitrary ring.

J.G. ROSENSTEIN 1972

He proves that $GL_2(R) \cong GL_2(F_2)[B]^*$.

H. GONSHOR 1975

Gonshor generalizes and simplifies Rosenstein's work in his proof of $GL_n(R) \cong GL_n(F_2)[B]^*$ for n any positive integer.

S. BURRIS and H. WERNER 1980

Specializations of the results in this paper show that

(1) $M_n(S[B]^*) \cong M_n(S)[B]^*$

(2) $GL_n(S[B]^*) \cong GL_n(S)[B]^*$

(3) $SL_n(S[B]^*) \cong SL_n(S)[B]^*$

(4) $PSL_n(S[B]^*) \cong PSL_n(S)[B]^*$.

§10. FIRST ORDER ASPECTS OF BOOLEAN POWERS

A. TARSKI 1949

The elementary types of Boolean algebras are characterized (using the Tarski invariants). Also the theory of Boolean algebras is proved to be decidable.

YU. L. ERSHOV 1964

Ershov shows that for every Boolean algebra B there is a filter F over ω

such that $B \equiv 2^{\omega}/F$.

YU. L. ERSHOV 1967

He shows that if P is a primal algebra then one can semantically embed the variety $V(P)$ into $\mathbb{BA}$. (We did this in §1 for the case $P = Z_3$.)

A. WOJCIECHOWSKA 1969

She gives a Feferman-Vaught analysis of $A[B]^*$. From this it follows that $A[B]^*$ preserves elementary equivalence and elementary embeddings in both arguments -- see also Burris 1975, Ash 1975.

R. MANSFIELD 1971

Mansfield proves that two structures are elementarily equivalent iff they have isomorphic Boolean ultrapowers.

J.T. BALDWIN and A.H. LACHLAN 1973

They show that if A is a finite structure and B is the denumerable free Boolean algebra then $A[B]^*$ is ω-categorical.

B. WEGLORZ 1974

He shows that free products of Boolean algebras preserve $\equiv$.

S. BURRIS 1975

This paper contains the following results:

(1) If A is a finite algebra and B is a complete Boolean algebra then $A[B]^*$ is equationally compact.

(2) If A is a finite B-separating algebra then $A[B_1]^* \equiv A[B_2]^*$ implies $B_1 \equiv B_2$.

(3) Every bounded Boolean power of an algebra A is elementarily equivalent to a reduced power of A, and vice-versa.

(4) A first-order sentence is preserved by bounded Boolean powers iff it is equivalent to a disjunction of Horn sentences.

(5) An elementary class $\mathcal{K}$ is closed under bounded Boolean powers iff it is closed under reduced powers iff it is defined by a set of disjunctions of Horn sentences.

S. BURRIS 1978 [a]

We say that $A \equiv_n B$ if A and B satisfy the same sentences with at most n alternations of quantifiers. In this paper it is proved that if A is a finite B-separating algebra then for any positive n there exist Boolean algebras B_1, B_2 such that $A[B_1]^* \equiv_n A[B_2]^*$ but $A[B_1]^* \not\equiv A[B_2]^*$.

B. BANASCHEWSKI and E. NELSON 1980

They show $A[B]$ preserves elementary equivalence and elementary embeddings in both arguments, that $A[B]^* \equiv A[B]$, and the canonical embedding of $A[B]^*$ into $A[B]$ is elementary.

S. GARAVAGLIA and J.M. PLOTKIN [a]

They construct an infinite B-separating structure A and two Boolean algebras B_1 and B_2 such that $A[B_1]^* \equiv A[B_2]^*$ but $B_1 \not\equiv B_2$.

§11. FILTERED BOOLEAN POWERS

R.F. ARENS and I. KAPLANSKY 1948

They realized that bounded Boolean powers were going to be severely limited as a method of proving representation theorems for rings, so they introduced two generalizations of this construction, the first of which we call a filtered Boolean power. This construction proceeds as follows (we describe it for arbitrary algebras, whereas they were only concerned with rings): Let A be an algebra and let A_i, $i \in I$, be the family of its subalgebras indexed by some set I. Then, given a Boolean space X and an indexed family $(X_i)_{i \in I}$ of closed subsets we construct the subalgebra of A^X whose universe is given by $\{f \in C(X,A): f(X_i) \subseteq A_i \text{ for } i \in I\}$. Given a class of algebras K let $P_{FB}(K)$ denote the class of all filtered Boolean powers of members of K. In the following we use $V_{\omega_1}(K)$ to denote the countable members in the variety $V(K)$. Arens and Kaplansky proved the following two results on filtered Boolean powers:

(1) For the variety $V(F_4)$ of rings generated by the 4-element field F_4 one has $V(F_4) \neq IP_{FB}(F_4)$.

(2) For any finite field F, $V_{\omega_1}(F) \subseteq IP_{FB}(F)$.

M.O. RABIN 1969

In this paper Rabin proves, as a corollary to his work on two successor functions, that the theory of countable Boolean algebras with quantification over filters is decidable.

S.D. COMER 1974

Comer realized that the result (2) of Arens and Kaplansky above could be used to interpret the theory of $(x^m = x)$-rings into the theory of countable Boolean algebras with quantification over filters; hence the theory of $(x^m = x)$-rings is decidable.

S.D. COMER 1975

Comer shows that if A is a finite monadic algebra then there is another finite monadic algebra A' such that $V_{\omega_1}(A) \subseteq IP_{FB}(A')$. Then using Rabin's result he concludes that any finitely generated variety of monadic algebras has a decidable theory.

H. WERNER 1978

S. BURRIS and H. WERNER 1979

Werner extends Comer's methods to show that if $V(A)$ is a finitely generated discriminator variety then there is a finite algebra A' such that $V_{\omega_1}(A) \subseteq IP_{FB}(A')$; hence if $V(A)$ is also of finite type then it has a decidable theory.

S. BURRIS and R. McKENZIE 1981

In this monograph there are three fundamental theorems on filtered Boolean powers (= sub-Boolean powers):

(1) For a finite algebra C, $V(C) = IP_{FB}(C)$ iff the following conditions hold:

- (i) $C = A \times D$ where A is modular-Abelian and D generates a discriminator variety.
- (ii) $V(C)$ is congruence permutable.
- (iii) If A and D are both nontrivial then they both have trivial subalgebras.
- (iv) $V(A) = IP_{FB}(A)$ and $V(D) = IP_{FB}(D)$.

(2) If A is a quasiprimal algebra then $V(A) = IP_{FB}(A)$ iff every isomorphism between nontrivial subalgebras of A has a unique extension to an automorphism of A, and the only automorphism of A with a fixed point is the identity map.

(3) If A is a quasiprimal algebra then $V_{\omega_1}(A) \subseteq IP_{FB}(A)$ iff every isomorphism between nontrivial subalgebras of A extends to an automorphism of A.

§12. MODIFYING BOUNDED BOOLEAN POWERS USING HOMEOMORPHISMS

This modification of bounded Boolean powers introduced by Arens and Kaplansky is a more applicable, less tractable construction for representations, as is the Boolean product in the following sections.

R.F. ARENS and I. KAPLANSKY 1948

They show that if R is a ring of characteristic p such that each element satisfies $x^{p^n} = x$ then there is a Boolean space X with a homeomorphism α whose n^{th} power is the identity such that, letting F be the finite field of order p^n, we have R is isomorphic to the subring of $C(X,F)$ whose universe is

$$\{f \in C(X,F) : f(\alpha x) = (f(x))^p \text{ for } x \in X\}.$$

K. KEIMEL and H. WERNER 1974

They generalize the above to obtain representations for finitely generated discriminator varieties.

A. WOLF 1975

If G is a group, let $\mathcal{BA}(G)$ be the variety of Boolean algebras expanded by the group of automorphisms G. Wolf shows that if G is a finite solvable group then $\mathcal{BA}(G)$ has a decidable theory, and uses this to show that varieties generated by certain quasiprimals have a decidable theory.

S. BURRIS [b]

For G a finite group $\mathcal{BA}(G)$ has a decidable theory.

§13. THE DEFINITION OF BOOLEAN PRODUCTS

We will use the notation $A \underset{s.d.}{\leq} \prod_{x \in X} A_x$ to mean that A is a subdirect product of the indexed family of algebras $(A_x)_{x \in X}$. For $\Phi(x_1,\ldots,x_n)$ a first-order formula and $f_1,\ldots,f_n \in A$, let $[\![\Phi(f_1,\ldots,f_n)]\!] = \{x \in X\colon A_x \models \Phi(f_1(x),\ldots,f_n(x))\}$.

An algebra A is a Boolean product of members of $\mathcal{K}$ if there is a family $(A_x)_{x \in X}$ of algebras $A_x \in \mathcal{K}$ indexed by a Boolean space X such that

(i) $A \underset{s.d.}{\leq} \prod_{x \in X} A_x$

(ii) (atomic extension property) for $f,g \in A$, $[\![f = g]\!]$ is a clopen subset of X.

(iii) (Patchwork property) for $f,g \in A$ and N a clopen subset of X we have $f\restriction_N \cup g\restriction_{X-N} \in A$.

The Boolean product construction was introduced by Burris and Werner [1979] as a reformulation of the Boolean sheaf construction, popularized by Dauns and Hofmann [1966]. Given a class $\mathcal{K}$ of algebras we let $\Gamma^a(\mathcal{K})$ denote the class of Boolean products of members of $\mathcal{K}$. On can check that $P_B \leq P_{FB} \leq \Gamma^a$.

§14. BOOLEAN PRODUCT REPRESENTATION THEOREMS

If $\mathcal{V}$ is a variety let $\mathcal{V}_{SI}$ be the subdirectly irreducible members of $\mathcal{V}$, let $\mathcal{V}_S$ be the simple algebras in $\mathcal{V}$, and let $\mathcal{V}_{DI}$ be the directly indecomposable members of $\mathcal{V}$.

J. DAUNS and K.H. HOFMAN 1966

They prove that for R a biregular ring one has $R \in I\Gamma^a(\mathcal{K})$ where $\mathcal{K}$ is the class of simple rings with 1 (in the language $\{+,\cdot,-,0\}$).

R.S. PIERCE 1967

In this study of Boolean product representations of rings (in the language $\{+,\cdot,-,0,1\}$) he states and proves the following:

(1) "Roughly speaking we would like to obtain a representation of rings by means of indecomposable rings."

(2) For R a commutative ring, $R \in I\Gamma^{a}(CR_{DI})$, CR being the variety of commutative rings.

(3) For R a commutative regular ring, $R \in I\Gamma^{a}(F)$, F being the class of fields.

(4) For any ring R a Boolean product representation is constructed using the central idempotents of R. (This construction is now called the Pierce sheaf of R.)

S.D. COMER 1971

He discusses a general result to the effect that if the factor congruences of an algebra form a Boolean algebra then one obtains a Boolean product representation of the algebra in a natural way.

S.D. COMER 1972

Letting CA_n be the variety of cylindric algebras of dimension n he shows that $CA_n = I\Gamma^{a}((CA_n)_S)$.

K. KEIMEL and H. WERNER 1974

They show that for V a finitely generated discriminator variety we have $V = I\Gamma^{a}(V_S)$.

S. BULMAN-FLEMING and H. WERNER 1977

They improve on the previous result by showing that for any discriminator variety V, $V = I\Gamma^{a}(V_S)$.

W.D. BURGESS and W. STEPHENSON 1978

They show that iterations of the Pierce sheaf construction (applied to rings) need not terminate in finitely many steps with directly indecomposable stalks.

W.D. BURGESS and W. STEPHENSON 1979

Let R be the class of rings in the language $\{+,\cdot,-,0,1\}$. Then $R \in I\Gamma^{a}(R_{DI})$ iff every idempotent of R is central.

H. WERNER 1978

S. BURRIS and H. WERNER 1979

These papers contain Werner's generalization of Comer's key result on monadic algebras, namely Werner shows that for K a finite set of finite algebras there exists a finite algebra A' such that $\{A \in \Gamma^a(K) : |A| \leq \omega\} \subseteq IP_{FB}(A')$. This is a key step in proving the decidability of finitely generated discriminator varieties of finite type.

S. BURRIS and R. McKENZIE 1981

The limitations of using Boolean constructions for representation theorems is most clearly set forth in the following result from this monograph: Let V be a finitely generated variety. Then there is a finite set K of finite algebras such that $V = I\Gamma^a(K)$ iff $V = V_{ab} \otimes V_{discr}$ and the ring $R(V_{ab})$ associated with V_{ab} is of finite representation type. (V_{ab} is an Abelian subvariety of V, V_{discr} is a discriminator subvariety of V.)

S. BURRIS [a]

D.M. CLARK and P.H. KRAUSS [a]

Suppose V is a congruence-distributive variety. Then $V = I\Gamma^a(V_{SI})$ iff V is a discriminator variety.

§15. FIRST-ORDER ASPECTS OF BOOLEAN PRODUCTS

A. MACINTYRE 1973

Macintyre uses Boolean products to study the model companions of classes of rings, giving conditions which ensure that K has a model-complete theory implies certain Boolean products of K have a model-complete theory.

S.D. COMER 1974

Let Γ^e be the class operator obtained by replacing condition (ii) of the definition of Γ^a by: (ii') $[\![\phi(\vec{f})]\!]$ is clopen for any first-order formula. Comer presents a Feferman-Vaught theorem for Γ^e.

S.D. COMER 1976

Comer generalizes Macintyre's conditions to study model companions of varieties of monadic algebras.

S. BURRIS and H. WERNER 1979

This paper is devoted to the elementary properties of Boolean products and includes the following results:

(1) $ISP_R = IS\Gamma_0^e S^{(\kappa)} P_U$.

(2) Any finitely generated universal Horn class has a model companion.

(3) If V is a discriminator variety and V_S has a model companion then V has a model companion.

(4) If K is a model-complete elementary class with a discriminator formula then $ISP_R(K)$ has a model companion, and the algebraically closed members form an elementary class. Both are described using Boolean products.

(5) A sentence is preserved by Γ^e iff it is equivalent to a Horn sentence. (This answers Prob. 1 of Mansfield [1977].)

P.H. KRAUSS [a]

In this paper Krauss uses Boolean products to characterize the algebraically closed and the existentially closed members of filtral varieties, and to describe injectives in these varieties.

S. BURRIS and R. McKENZIE 1981

A discriminator formula for the class of simple aglebras in a variety with equationally definable principal congruences is given.

S. BURRIS [c]

Using the Boolean product construction the following are proved:

(1) If T^* is the model companion of a finitely generated universal Horn class $ISP(K)$ then the following are equivalent: (a) T^* is ω - categorical, (b) T^* is complete, (c) $ISP(K)$ has the joint embedding property, (d) $ISP(K) = ISP(A)$ for some finite A. (We assume the language is finite.)

(2) For T^* as above, T^* admits a primitive recursive elimination of quantifiers. As the theory T^* is primitive recursive, it follows that T^* is decidable.

§16. DOUBLE BOOLEAN POWERS

I discovered double Boolean powers in late 1978 while constructing directly indecomposable algebras, and shortly thereafter realized their value in proving undecidability results. McKenzie modified this construction yet further in order to prove the wide-ranging undecidability results of our 1981 Memoir. It is quite remarkable that both our best decidability and our best undecidability results depend on Boolean constructions.

M. RUBIN 1976

Rubin answers a longstanding question on monadic algebras by showing their first-order theory is undecidable. As a consequence the theory of Boolean pairs BP, the class of Boolean algebras with a distinguished subalgebra, is undecidable.

Let us use this result to show that the variety generated by the three-element

Heyting algebra $H = \langle\{0,e,1\},\vee,\wedge,\rightarrow,0,1\rangle$ has an undecidable theory. Given a field B of subsets of an index set I and a subfield B_0 of B let us define the subalgebra $H(B,B_0)$ of H^I by letting its universe be

$$\{f \in H^I\colon f^{-1}(0) \in B_0,\ f^{-1}(1) \in B\}.$$

For $X \subseteq I$ let χ_X be defined by

$$\chi_X = \begin{cases} 1 & \text{if } i \in X \\ e & \text{if } i \notin X\ , \end{cases}$$

and let e^* be the constant function in $H(B,B_0)$ with value e. Then

$$C \overset{\text{def}}{=} \{\chi_X\colon X \in B\} = \{f \vee e^*\colon f \in H(B,B_0)\}$$

$$C_0 \overset{\text{def}}{=} \{\chi_X\colon X \in B_0\} = \{(e^* \rightarrow f) \vee e^*\colon f \in H(B,B_0)\}.$$

Now we clearly have an isomorphism between the structures $\langle B,B_0,\subseteq\rangle$ and $\langle C,C_0,\leq\rangle$; hence we can semantically embed $\mathcal{BP}$ into $V(H)$.

S. BURRIS and R. McKENZIE 1981

Double Boolean powers are used along with a modification of a technique of Zamjatin and results from the study of the modular comutator to show that if a locally finite modular variety has a decidable theory then it decomposes into the product of an Abelian variety and a discriminator variety. Then it is shown that for finitely generated varieties of finite type which are modular the decidability question reduces to the decidability question for all unitary left R-modules, for R a finite ring with 1.

S. BURRIS and J. LAWRENCE [a]

Double Boolean powers are used to give brief proofs of Ershov's theorem on decidable varieties of groups and Zamjatin's results on decidable varieties of rings with 1.

S. BURRIS [b]

Using double Boolean powers this paper shows that if G is not a locally finite group then $\mathcal{BA}(G)$ has an undecidable theory.

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