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ON THE SIMPLICITY AND SUBDIRECT IRREDUCIBILITY OF BOOLEAN ULTRAPOWERS

BY

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In [2], Frayne et al. gave an example of a simple group with ultrapowers which are not simple. In this paper* we will obtain necessary and sufficient conditions for a Boolean ultrapower to be simple, or subdirectly irreducible, provided the language is countable.

Let $\mathfrak{A} = \langle A, \mathscr{F}\rangle$ be an algebra, and $\mathfrak{B} = \langle B, \vee, \wedge, ', 0, 1\rangle$ a Boolean algebra. Assume that $\mathfrak{B}$ is complete if $\mathfrak{A}$ is infinite. The *Boolean power* $\mathfrak{A}[\mathfrak{B}]$ has as its universe (written $|\mathfrak{A}[\mathfrak{B}]|$) the set of all mappings α of A into B such that

(i) if $a, b \in A$, $a \neq b$, then $\alpha(a) \wedge \alpha(b) = 0$;

(ii) $\bigvee_{a\in A} \alpha(a) = 1$.

The fundamental operations are defined by

(iii) $f(\alpha_0, \ldots, \alpha_{n-1})(a) = \bigvee \{\alpha_0(a_0) \wedge \ldots \wedge \alpha_{n-1}(a_{n-1}) : f(a_0, \ldots, a_{n-1}) = a\}$.

Let $\mathscr{U}$ be an ultrafilter on a Boolean algebra $\mathfrak{B}$. Define the relation $\theta_{\mathscr{U}}(\mathfrak{A})$ on $\mathfrak{A}[\mathfrak{B}]$ by

$$\theta_{\mathscr{U}}(\mathfrak{A}) = \Big\{\langle\alpha, \beta\rangle \in |\mathfrak{A}[\mathfrak{B}]|^2 : \bigvee_{a\in A} \alpha(a) \wedge \beta(a) \in \mathscr{U}\Big\}.$$

It can easily be shown that $\theta_{\mathscr{U}}(\mathfrak{A})$ is a congruence on $\mathfrak{A}[\mathfrak{B}]$. We denote the quotient algebra $\mathfrak{A}[\mathfrak{B}]/\theta_{\mathscr{U}}(\mathfrak{A})$ by $\mathfrak{A}[\mathfrak{B}]/\mathscr{U}$, and call it a *Boolean ultrapower* of $\mathfrak{A}$. For $\xi \in |\mathfrak{A}[\mathfrak{B}]|$ let $[\xi]_{\mathscr{U}}$ denote the image in $|\mathfrak{A}[\mathfrak{B}]/\mathscr{U}|$.

Remark 1. If $\mathfrak{B} \cong 2^I$ for some I (where 2 is the two-element Boolean algebra), then $\mathfrak{A}[\mathfrak{B}] \cong \mathfrak{A}[2^I] \cong \mathfrak{A}^I$. Therefore, $\mathfrak{A}[\mathfrak{B}]/\mathscr{U} \cong \mathfrak{A}^I/\mathscr{U}$, and the Boolean ultrapower in this case is just the familiar ultrapower.

An algebra $\mathfrak{A}$ is *simple* if $|A| > 1$ and the only congruence relations on $\mathfrak{A}$ are Δ_A and ∇_A, where $\Delta_A = \{\langle a, a\rangle : a \in A\}$, and $\nabla_A = A \times A$. An algebra $\mathfrak{A}$ is said to be (a, b)-*irreducible* if $a \neq b$ and every non-trivial

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congruence on $\mathfrak{A}$ identifies a and b. An algebra $\mathfrak{A}$ is said to be *subdirectly irreducible* if there are $a, b \in A$ such that $\mathfrak{A}$ is (a, b)-irreducible. A *simplicity sentence* is a first-order sentence all models of which are simple. Similarly we define a *subdirect irreducibility sentence.*

An ultrafilter $\mathscr{U}$ on a Boolean algebra is said to be *ω-complete* if, whenever $\{x_n: n < \omega\} \subseteq \mathscr{U}$,

$$\bigwedge_{n<\omega} x_n \in \mathscr{U}.$$

$\mathscr{U}$ is said to be *ω-incomplete* if it is not ω-complete.

Remark 2. Principal ultrafilters on Boolean algebras are always ω-complete. Therefore, $\mathfrak{B}$ must be infinite in order that ω-incomplete ultrafilters may exist.

An algebra $\mathfrak{A}$ is *α-saturated* if every set of formulae $\{\sigma_i(x_0): i \in I\}$ in the language of $\mathfrak{A}$, with fewer than α parameters from $|\mathfrak{A}|$, which is finitely satisfiable in $\mathfrak{A}$ is also satisfiable in $\mathfrak{A}$.

From now on we assume that the language of $\mathfrak{A}$ is countable.

LEMMA 1. *An ω-saturated algebra $\mathfrak{A}$ satisfies a simplicity (subdirect irreducibility) sentence iff $\mathfrak{A}$ is simple (subdirectly irreducible).*

Proof. For the non-trivial direction, assume that $\mathfrak{A}$ does not satisfy a simplicity sentence. Taylor has shown in [4] that, for any $a, b, c, d \in A$, $(c, d) \in \theta(a, b)$ iff there exists an existential positive formula $\varphi(x, y, u, v)$ which satisfies certain conditions, and $\mathfrak{A} \models \varphi(a, b, c, d)$. Let $\{\varphi_i(x, y, u, v)\}_{i<\omega}$ be all such formulae in our language. From [4] one can conclude the following:

(1) $\mathfrak{A}$ is not simple iff, for some $a, b, c, d \in A$,

$$\mathfrak{A} \models \neg\varphi_i(a, b, c, d) \& a \neq b \quad \text{for all } i < \omega;$$

(2) for every choice of $\varphi_{i_0}(x, y, u, v), \ldots, \varphi_{i_n}(x, y, u, v)$,

$$\forall xyuv(x \neq y \to \varphi_{i_0} \vee \ldots \vee \varphi_{i_n})$$

is a simplicity sentence.

Therefore, suppose that $\mathfrak{A}$ does not satisfy a simplicity sentence. Then, for any $i_0, \ldots, i_n < \omega$,

(3) $\mathfrak{A} \models \neg\forall xyuv(x \neq y \to \varphi_{i_0} \vee \ldots \vee \varphi_{i_n})$.

Let Γ be the set of all formulae of the form

$$(x \neq y) \& (\neg\varphi_i(x, y, u, v)).$$

From (3) we see that Γ is finitely satisfiable in $\mathfrak{A}$. Since $\mathfrak{A}$ is ω-saturated, and the members of Γ contain no parameters from $|\mathfrak{A}|$, Γ is satisfiable in $\mathfrak{A}$. (1) now implies that $\mathfrak{A}$ is not simple. A similar proof holds for the subdirect irreducibility case.

Now we need two results in [3].

LEMMA 2. *If $\mathscr{U}$ is an ω-incomplete ultrafilter on a Boolean algebra $\mathfrak{B}$, then $\mathfrak{A}[\mathfrak{B}]/\mathscr{U}$ is ω_1-saturated.*

LEMMA 3. *Let $\xi_0, \ldots, \xi_n \in |\mathfrak{A}[\mathfrak{B}]|$ and suppose that $\sigma([\xi_0]_{\mathscr{U}}, \ldots, [\xi_n]_{\mathscr{U}})$ is a sentence. Then*

$$\mathfrak{A}[\mathfrak{B}]/\mathscr{U} \models \sigma([\xi_0]_{\mathscr{U}}, \ldots, [\xi_n]_{\mathscr{U}})$$

iff

$$\bigvee \{\xi_0(a_0) \wedge \ldots \wedge \xi_n(a_n) \colon \mathfrak{A} \models \sigma(a_0, \ldots, a_n)\} \in \mathscr{U}.$$

THEOREM 1. *Let $\mathfrak{B}$ be a Boolean algebra, and $\mathscr{U}$ an ultrafilter on $\mathfrak{B}$. Then $\mathfrak{A}[\mathfrak{B}]/\mathscr{U}$ is simple (subdirectly irreducible) iff either $\mathscr{U}$ is ω-complete and $\mathfrak{A}$ is simple (subdirectly irreducible) or $\mathfrak{A}$ satisfies a simplicity (subdirect irreducibility) sentence.*

Proof. We will consider the case of simplicity — the subdirect irreducibility case has a similar treatment. If $\mathfrak{A}$ satisfies a simplicity sentence, then, since $\mathfrak{A}$ can elementarily be embedded in $\mathfrak{A}[\mathfrak{B}]/\mathscr{U}$ (see [3]), $\mathfrak{A}[\mathfrak{B}]/\mathscr{U}$ is simple. So assume that $\mathfrak{A}$ is simple and $\mathscr{U}$ is ω-complete. Let the formulae φ mentioned in the proof of Lemma 1 be enumerated as follows: $\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots$ $(n < \omega)$. Let

$$S_i = \{\langle a_0, a_1, c_0, c_1\rangle \colon \mathfrak{A} \models \varphi_i(a_0, a_1, c_0, c_1)\}.$$

Let ξ, η, α, β be arbitrary elements of $|\mathfrak{A}[\mathfrak{B}]|$ such that

$$\bigvee_{a_0 \neq a_1} \xi(a_0) \wedge \eta(a_1) \in \mathscr{U}$$

(i.e. $[\xi]_{\mathscr{U}} \neq [\eta]_{\mathscr{U}}$ in $|\mathfrak{A}[\mathfrak{B}]/\mathscr{U}|$). Then, since $\mathfrak{A}$ is simple,

$$\bigvee_{i<\omega} \bigvee \{\xi(a_0) \wedge \eta(a_1) \wedge \alpha(c_0) \wedge \beta(c_1) \colon \langle a_0, a_1, c_0, c_1\rangle \in S_i\} \in \mathscr{U}.$$

Hence (by ω-completeness) for some $i < \omega$ we have

$$\{\xi(a_0) \wedge \eta(a_1) \wedge \alpha(c_0) \wedge \beta(c_1) \colon \langle a_0, a_1, c_0, c_1\rangle \in S_i\} \in \mathscr{U},$$

i.e.

$$\mathfrak{A}[\mathfrak{B}]/\mathscr{U} \models \varphi_i([\xi]_{\mathscr{U}}, [\eta]_{\mathscr{U}}, [\alpha]_{\mathscr{U}}, [\beta]_{\mathscr{U}}),$$

so $\mathfrak{A}[\mathfrak{B}]/\mathscr{U}$ is simple.

Now, assume that $\mathfrak{A}[\mathfrak{B}]/\mathscr{U}$ is simple and $\mathscr{U}$ is ω-incomplete. By Lemma 2, $\mathfrak{A}[\mathfrak{B}]/\mathscr{U}$ is ω_1-saturated, hence ω-saturated. By Lemma 1, together with the fact that $\mathfrak{A}$ is isomorphic to an elementary substructure of $\mathfrak{A}[\mathfrak{B}]/\mathscr{U}$, the proof is complete.

COROLLARY. *For a given algebra $\mathfrak{A}$ and for a given infinite Boolean algebra $\mathfrak{B}$, $\mathfrak{A}[\mathfrak{B}]/\mathscr{U}$ is simple (subdirectly irreducible) for all $\mathscr{U}$ iff $\mathfrak{A}$ satisfies a simplicity sentence (subdirect irreducibility sentence).*

In view of Remark 1 the above applies to a special case of ultrapowers. Indeed, a similar result can be stated for ultraproducts.

THEOREM 2. *Let $\mathscr{U}$ be an ultrafilter on a given infinite set I, and assume that the language of our algebras $\mathfrak{A}_i$ is countable. Then $\prod_{i\in I}\mathfrak{A}_i/\mathscr{U}$ is simple (subdirectly irreducible) iff either $\mathscr{U}$ is ω-complete and*

$$\{i\in I\colon \mathfrak{A}_i \textit{ is simple (subdirectly irreducible)}\}\in\mathscr{U}$$

or, for some simplicity sentence (subdirect irreducibility sentence) σ,

$$\{i\in I\colon \mathfrak{A}_i\models\sigma\}\in\mathscr{U}.$$

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