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The first order theory of Boolean algebras with a distinguished group of automorphisms

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The theory of Boolean algebras with a sequence of distinguished ideals is known to be decidable (Rabin [4], 1969), and the theory of Boolean algebras with a distinguished subalgebra is hereditarily undecidable (Rubin [5], 1976). In this paper we look at, for a group $\langle G, \cdot, e\rangle$, the class $\mathcal{BA}(G)$ of algebras $\langle B, \vee, \wedge, ', 0, 1, (g)_{g\in G}\rangle$ which satisfy, for $g, h \in G$,

(1) $\langle B, \vee, \wedge, ', 0, 1\rangle$ is a Boolean algebra

(2) $g(x \vee y) \approx g(x) \vee g(y)$
$g(x') \approx g(x)'$
$g(h(x)) \approx (g \cdot h)(x)$
$e(x) \approx x.$

We will let B_G denote such an algebra, and B the reduct to the Boolean operations. Thus B_G is a *Boolean algebra B with a group G of automorphisms acting on it.* The only result on the theory of $\mathcal{BA}(G)$ that we know of is due to Wolf ([7], 1975): *If G is a finite solvable group then $\mathcal{BA}(G)$ has a decidable theory.* His proof is based on Arens and Kaplansky's theorem that a finite solvable group of homeomorphisms acting on the Boolean space of a countable Boolean algebra has a fundamental domain. We take another approach, introducing monadic algebras into the study, and arrive at a more general result.

THEOREM 1. *Let G be a finite group. Then $\mathcal{BA}(G)$ is a finitely generated discriminator variety and consequently has a decidable first order theory.*

Proof. Let us define a unary term $c(x)$ by

$$c(x) = \bigvee_{g\in G} g(x).$$

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As the identity function belongs to G it is clear that

$$x \leq c(x), \tag{C1}$$

and since automorphisms are order preserving we have

$$x \leq y \rightarrow c(x) \leq c(y). \tag{C2}$$

If h is any member of G then

$$\begin{aligned} h(c(x)) &= h\Big(\bigvee_{g \in G} g(x)\Big) \\ &= \bigvee_{g \in G} hg(x) \\ &= \bigvee_{g \in G} g(x) \\ &= c(x); \end{aligned}$$

hence

$$c(c(x)) = c(x) \tag{C3}$$

holds. Next note that

$$\begin{aligned} c(x \vee y) &= \bigvee_{g \in G} g(x \vee y) \\ &= \bigvee_{g \in G} (g(x) \vee g(y)) \\ &= \Big(\bigvee_{g \in G} g(x)\Big) \vee \Big(\bigvee_{g \in G} g(y)\Big), \end{aligned}$$

which establishes

$$c(x \vee y) = c(x) \vee c(y). \tag{C4}$$

Finally

$$\begin{aligned} c(c(x)') &= c\Big(\bigwedge_{g\in G} g(x)'\Big) \\ &= c\Big(\bigwedge_{g\in G} g(x')\Big) \\ &= \bigvee_{h\in G} h\Big(\bigwedge_{g\in G} g(x')\Big) \\ &= \bigvee_{h\in G} \Big(\bigwedge_{g\in G} hg(x')\Big) \\ &= \bigvee_{h\in G} \Big(\bigwedge_{g\in G} g(x')\Big) \\ &= \bigwedge_{g\in G} g(x') \end{aligned}$$

shows that we also have

$$c(c(x)') = c(x)'. \tag{C5}$$

For any algebra B_G in $\mathscr{BA}(G)$ let B_c be the corresponding algebra $(B, \vee, \wedge, ', 0, 1, c)$. Properties (C1)–(C5) say that B_c is a monadic algebra, and the closed elements of B_c are precisely the fixed points of G acting on B.

Let $+$ be the symmetric difference operation on B, that is

$$x + y = (x \wedge y') \vee (x' \wedge y).$$

The congruences of B_G are readily seen to correspond to the ideals I of B which are closed under the action of G. For if θ is a congruence of B_G then certainly $0/\theta$ is an ideal of B, and if $(a, 0) \in \theta$ then $(g(a), g(0)) = (g(a), 0) \in \theta$ for $g \in G$, so $0/\theta$ is closed under g. Conversely given an ideal I of B closed under the action of G the corresponding congruence θ of B defined by $(a, b) \in \theta$ iff $a + b \in I$ is also a congruence for B_G as $(a, b) \in \theta$ implies $a + b \in I$, so $g(a) + g(b) = g(a + b) \in I$, hence $(g(a), g(b)) \in \theta$ for $g \in G$.

The congruences of B_c are known (see [3]) to correspond to the ideals of B which are closed under the operation c. The conditions $h(x) \leq c(x) = \bigvee_{g\in G} g(x)$, for $h \in G$, suffice to show that an ideal I of B is closed under G iff it is closed under c, so B_G and B_c have the same congruences.

A monadic algebra is subdirectly irreducible iff the closure operator c satisfies (see [3]) the implication

$$x \neq 0 \rightarrow c(x) = 1.$$

Now observe that $\mathcal{BA}(G)$ is locally finite as G is finite; for if $X \subseteq B_G \in \mathcal{BA}(G)$ then the subalgebra of B_G generated by X is the subalgebra of B generated by $X \cup \{g(x) : g \in G, x \in X\}$. Consequently if $B_G \in \mathcal{BA}(G)$ is such that $|B_G| > 2^{|G|}$ let A be a finite subalgebra of B_G with $|A| > 2^{|G|}$. If a is an atom of A then $g(a)$ is also an atom. But then $c(a) < 1$ as $|\{g(a) : g \in G\}|$ is less than the number of atoms of A, so B_G is not subdirectly irreducible.

Since the subdirectly irreducible members of $\mathcal{BA}(G)$ have cardinality less than or equal to $2^{|G|}$ it follows that $\mathcal{BA}(G)$ is finitely generated, i.e. generated by finitely many finite algebras. Now, as in the case of monadic algebras (see [6]), the ternary term

$$t(x, y, z) = [x \wedge c(x+y)] \vee [z \wedge (c(x+y))']$$

defines the ternary discriminator on each of the subdirectly irreducible members of $\mathcal{BA}(G)$, that is for a, b, c in B_G we have $t(a, b, c) = a$ if $a \neq b, = c$ if $a = b$. Thus $\mathcal{BA}(G)$ is indeed a discriminator variety, and hence has a decidable first order theory (see [2]). □

Remark. If one replaces the finite group G by a finite monoid M then $c(x)$, defined as before, still satisfies (C1)–(C4), but one can readily find a four-element member $B_{\bar{M}}$ of $\mathcal{BA}(\bar{M})$, where $\bar{M}$ is a two-element monoid, such that (C5) fails in $B_{\bar{M}}$; furthermore $B_{\bar{M}}$ is subdirectly irreducible but not simple, so $\mathcal{BA}(\bar{M})$ has an hereditarily undecidable theory (by Theorem 9.1 of [1]). However for any finite monoid M the subvariety of $\mathcal{BA}(M)$ defined by (C5) is a finitely generated discriminator variety; hence it has a decidable theory. It would be interesting to know for which finite monoids the theory of $\mathcal{BA}(M)$ is decidable.

Now we will apply our first theorem to construct what we think is the first example of a decidable congruence distributive variety which is not a discriminator variety. From [1] we know that such a variety cannot be finitely generated.

THEOREM 2. *The variety $\mathcal{BA}(\sum_\omega Z_2)$ is arithmetical (i.e. congruence distributive and congruence permutable), has a decidable first order theory, but is not semi-simple (i.e. not every subdirectly irreducible member is simple).*

Proof. Certainly this variety is arithmetical as Boolean algebras are a reduct of it. Now we claim that for any $n<\omega$, the reduct of $\mathcal{BA}(\sum_\omega Z_2)$ to the language $L_n=\{\vee, \wedge, ', 0, 1, \{g\}_{g\in\sum_n Z_2}\}$, where we make the obvious identification of $\sum_n Z_2$ with the subgroup of $\sum_\omega Z_2$ consisting of those elements which are zero on all $i\geq n$, is just $\mathcal{BA}(\sum_n Z_2)$. The containment from left to right is clear. For the converse choose a member of $\mathcal{BA}(\sum_n Z_2)$, and extend it to a member of $\mathcal{BA}(\sum_\omega Z_2)$ by letting all the additional automorphisms g, with $g(i)=0$ for $i<n$, be the identity map. As each $\mathcal{BA}(\sum_n Z_2)$, $n<\omega$, has a decidable first order theory by Theorem 1 it follows that the theory of $\mathcal{BA}(\sum_\omega Z_2)$ is also decidable (as every reduct to a finite language is decidable).

To finish the proof let $G=\sum_\omega Z_2$, and let B be the power set Boolean algebra $P(G)$. Define the action of G on B to be

$$g(X)=\{g\cdot h: h\in X\}$$

for $g\in G$. Then the smallest non-zero ideal of B closed under G is the ideal of all finite subsets of G. Thus B_G is subdirectly irreducible but not simple, and clearly B_G is in $\mathcal{BA}(G)$. □

The proof of Theorem 2 obviously goes through for any countable recursive abelian group of finite exponent. Thus the next result stands in sharp relief.

THEOREM 3. *If G is not a locally finite group then the first order theory of $\mathcal{BA}(G)$ is hereditarily undecidable.*

Proof. Let B be the power set algebra $P(G)$, and let G act on $P(G)$ as in the proof of the previous theorem, giving B_G. Choose finitely many elements $g_1,\ldots,g_n\in G$ such that $g_1,\ldots,g_n$ generate an infinite subgroup H of G. Let $a=\{g_1\}$, and let $b=H$. Then $0<a<b$, a is an atom of B, and b is an infinite subset of G.

Let θ be the congruence of B_G determined by the ideal of finite subsets of G (this ideal is indeed closed under the action of G). Given an arbitrary field F of subsets of a set I and a subfield F_0 of F define the member $B_G[F, F_0, \theta]^*$ of $\mathcal{BA}(G)$ by

$$B_G[F, F_0, \theta]^*=\{f\in {}^I(B_G): f^{-1}(e)\in F, f^{-1}(e/\theta)\in F_0, \text{ for } e\in B, \text{ and } |f(I)|<\omega\}.$$

(This is, indeed, a subdirect power of B_G.) For $e\in B_G$ let $\bar{e}$ denote the constant function in ${}^I(B_G)$ with value e. Since a is an atom of B_G the interval $\hat{F}=[0, \bar{a}]$ in $B_G[F, F_0, \theta]^*$ is order-isomorphic to F under the map $f\mapsto f^{-1}(a)$; and as the only elements in the interval $[0, b]$ which are fixed by all of $g_1,\ldots,g_n$ are the elements

0 and b, it follows that F_0 is order-isomorphic to $\tilde{F}_0 = \{x \in [0, \bar{b}] : g_i(x) = x, 1 \leq i \leq n\}$ under the map $f \mapsto f^{-1}(b) = f^{-1}(b/\theta)$, and hence to $\hat{F}_0 = \{\bar{a} \wedge x : x \in [0, \bar{b}], g_i(x) = x, 1 \leq i \leq n\}$. From this one can check that $\langle F, F_0, \subseteq \rangle \cong \langle \hat{F}, \hat{F}_0, \leq \rangle$, Thus one can easily write down a first order interpretation of the Boolean pair $(F, F_0, \subseteq)$ into $G_B[F, F_0, \theta]^*$, and as the theory of such pairs is hereditarily undecidable (see [1]), so is the theory of $\mathcal{BA}(G)$. □

An interesting problem is to determine for precisely which groups G the theory of $\mathcal{BA}(G)$ is decidable. Of course the groups must be recursive and locally finite; if finitely presented it must have a solvable word problem. So far we do not even know if the theory of $\mathcal{BA}(Z_{p^\infty})$ is decidable.

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