Algebra Universalis, **24** (1987) 186–187 0002-5240/87/020186-02$01.50 + 0.20/0

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A note on algebraically and existentially closed structures

STANLEY BURRIS

Algebraically and existentially closed structures can be rather elusive. In this note we look at a couple of simple properties of these structures. For K a class of first-order structures let K^{ac} be the members of K which are algebraically closed with respect to K; and let K^{ec} denote the members of K which are existentially closed with respect to K.

THEOREM 1. *If K is a class of algebras and $A \in K^{ac}$ then A is congruence extensile in K.*

Proof. Let A, $B \in K$ with $A \leq B$, and let θ be a congruence of A. Then let $\hat{\theta}$ be the congruence on B generated by θ. If $\langle a_1, a_2 \rangle \in \hat{\theta} \cap A \times A$ then using Mal'cev's description of the congruence generated by a set of elements it follows that there must be a primitive positive formula $\pi(x, y, x_1, y_1, \ldots, x_n, y_n)$ such that for any algebra C

$$C \models \pi(c_1, c_2, c_{11}, c_{12}, \ldots, c_{n1}, c_{n2}) \quad \text{implies} \quad \langle c_1, c_2 \rangle \in \Theta_C(\{\langle c_{11}, c_{12} \rangle, \ldots, \langle c_{n1}, c_{n2} \rangle\});$$

and we have, for suitable $\langle a_{i1}, a_{i2} \rangle \in \theta$,

$$B \models \pi(a_1, a_2, a_{11}, a_{12}, \ldots, a_{n1}, a_{n2}).$$

Since $A \in K^{ac}$ it follows that

$$A \models \pi(a_1, a_2, a_{11}, a_{12}, \ldots, a_{n1}, a_{n2}),$$

and thus $\langle a_1, a_2 \rangle \in \theta$. This ensures $\theta = \hat{\theta} \cap A \times A$, so A is congruence extensile in K. □

This research has been supported by NSERC Grant no. A7256.

Presented by Bjarni Jónsson. Received April 10, 1986. Accepted for publication in final form November 24, 1986.

The following well-known result follows.

COROLLARY 2. *If each member of K can be embedded in a simple member of K then every member of K^{ac} is simple.*

Now we turn to existentially closed structures. Given a primitive formula

$$\Phi(\vec{u}) = \exists\vec{v}\left[\Phi^{+}(\vec{u}, \vec{v}) \wedge \bigwedge_{i=1}^{n} \neg\alpha_i(\vec{u}, \vec{v})\right],$$

where Φ^{+} is a conjunction of atomic formulas and each α_i is atomic, let

$$\Phi_0(\vec{u}) = \exists\vec{v}\Phi^{+}(\vec{u}, \vec{v})$$

$$\Phi_i(\vec{u}) = \exists\vec{v}[\Phi^{+}(\vec{u}, \vec{v}) \wedge \neg\alpha_i(\vec{u}, \vec{v})], \quad 1 \le i \le n.$$

In Macintyre's sheaf-theoretic analysis of the model companion of expansions of certain rings he made crucial use of the fact that a Boolean power of a structure using an atomless Boolean algebra would satisfy

$$\Phi(\vec{u}) \leftrightarrow \bigwedge_{i=0}^{n} \Phi_i(\vec{u}). \qquad (*)$$

THEOREM 3. *Let K be a class of structures closed under finite direct products. Then $A \in K^{ec}$ implies A satisfies $(*)$ for all primitive formulas $\Phi(\vec{u})$.*

Proof. Certainly $A \models \Phi(\vec{a})$ implies $A \models \Phi_i(\vec{a})$, $0 \le i \le n$. Conversely suppose $A \models \Phi_i(\vec{a})$, $0 \le i \le n$. Then using the natural embedding α of A into A^{n+1} we see that $A^{n+1} \models \Phi(\alpha\vec{a})$, and thus $A \models \Phi(\vec{a})$. □

Thus we see that for K closed under finite products the members of K^{ec} lie in the subclass of K defined by the $\forall\exists$ sentences from $(*)$. Within this subclass it suffices to look at primitive formulas involving at most one negated atomic formula to test for being existentially closed.

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University of Waterloo
Waterloo, Ontario
Canada