Annotated Logic Chapters

from

Boole's Algebra of Logic

as presented in the first 15 chapters
of his 1854 book

AN INVESTIGATION OF
THE LAWS OF THOUGHT
ON WHICH ARE FOUNDED
THE MATHEMATICAL THEORIES
OF LOGIC AND PROBABILITIES

Boole's 1854 key phrase "The Laws of Thought" was likely borrowed from the popular (traditional) logic book "Outline of the Laws of Thought" by William Thomson of Oxford, first published in 1842.

by

Stanley N Burris

Dept. of Pure Mathematics
Univ. of Waterloo

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Chapter I

Nature and Design of this Work.

1. The design of the following treatise is to investigate the fundamental laws of those operations of the mind by which reasoning is performed; to give expression to them in the symbolical language of a Calculus, and upon this foundation to establish the science of Logic and construct its method; to make that method itself the basis of a general method for the application of the mathematical doctrine of Probabilities; and, finally, to collect from the various elements of truth brought to view in the course of these inquiries some probable intimations concerning the nature and constitution of the human mind.

2. That this design is not altogether a novel one it is almost needless to remark, and it is well known that to its two main practical divisions of Logic and Probabilities a very considerable share of the attention of philosophers has been directed. In its ancient and scholastic form, indeed, the subject of Logic stands almost exclusively associated with the great name of Aristotle. As it was presented to ancient Greece in the partly technical, partly metaphysical disquisitions of the Organon, such, with scarcely any essential change, it has continued to the present day. The stream of original inquiry has rather been directed towards questions of general philosophy, which, though they

The subject of Logic had long belonged to Metaphysics. Logicians had named various mental abilities that were needed in logic, such as perception, conception, abstraction, language, judgement, and reasoning.

In LT Boole tried to anchor his algebra of logic in mental processes. Perhaps he was aiming to develop the science of the inner world, the mind, just as Newton had found laws for the outer world.

Recall that both Boole and Newton were sons of Lincolnshire, and as a young man Boole had given a talk on Newton's *Principia*.

Boole offered no significant insight into the workings of the mind.

have arisen among the disputes of the logicians, have outgrown their origin, and given to successive ages of speculation their peculiar bent and character. The eras of Porphyry and Proclus, of Anselm and Abelard, of Ramus, and of Descartes, together with the final protests of Bacon and Locke, rise up before the mind as examples of the remoter influences of the study upon the course of human thought, partly in suggesting topics fertile of discussion, partly in provoking remonstrance against its own undue pretensions. The history of the theory of Probabilities, on the other hand, has presented far more of that character of steady growth which belongs to science. In its origin the early genius of Pascal,—in its maturer stages of development the most recondite of all the mathematical speculations of Laplace,—were directed to its improvement; to omit here the mention of other names scarcely less distinguished than these. As the study of Logic has been remarkable for the kindred questions of Metaphysics to which it has given occasion, so that of Probabilities also has been remarkable for the impulse which it has bestowed upon the higher departments of mathematical science. Each of these subjects has, moreover, been justly regarded as having relation to a speculative as well as to a practical end. To enable us to deduce correct inferences from given premises is not the only object of Logic; nor is it the sole claim of the theory of Probabilities that it teaches us how to establish the business of life assurance on a secure basis; and how to condense whatever is valuable in the records of innumerable observations in astronomy, in physics, or in that field of social inquiry which is fast assuming a character of great importance. Both these studies have also an interest of another kind, derived from the light which they shed upon the intellectual powers. They instruct us concerning the mode in which language and number serve as instrumental aids to the processes of reasoning; they reveal to us in some degree the connexion between different powers of our common intellect; they set before us what, in the two domains of demonstrative and of probable knowledge, are the essential standards of truth and correctness,—standards not derived from without, but deeply founded in the constitution of the human faculties. These ends of speculation yield neither in interest nor in dignity, nor yet, it may be added, in importance, to the practical objects, with the pursuit of which they have been historically associated. To unfold the secret laws and relations of those high faculties of thought by which all beyond the merely perceptive knowledge of the world and of ourselves is attained or matured, is an object which does not stand in need of commendation to a rational mind.

3. But although certain parts of the design of this work have been entertained by others, its general conception, its method, and, to a considerable extent, its results, are believed to be original. For this reason I shall offer, in the present chapter, some preparatory statements and explanations, in order that the real aim of this treatise may be understood, and the treatment of its subject facilitated.

What LT actually achieved in logic was a continuation, with some quietly introduced improvements and corrections, of his book MAL: the expression of premiss propositions as equations, algorithms to derive conclusion equations from them, and the interpretation of these equations as propositional conclusions to the initial premisses.

It is designed, in the first place, to investigate the fundamental laws of those operations of the mind by which reasoning is performed. It is unnecessary to enter here into any argument to prove that the operations of the mind are in a certain real sense subject to laws, and that a science of the mind is therefore *possible*. If these are questions which admit of doubt, that doubt is not to be met by an endeavour to settle the point of dispute *à priori*, but by directing the attention of the objector to the evidence of actual laws, by referring him to an actual science. And thus the solution of that doubt would belong not to the introduction to this treatise, but to the treatise itself. Let the assumption be granted, that a science of the intellectual powers is possible, and let us for a moment consider how the knowledge of it is to be obtained.

4. Like all other sciences, that of the intellectual operations must primarily rest upon observation,—the subject of such observation being the very operations and processes of which we desire to determine the laws. But while the necessity of a foundation in experience is thus a condition common to all sciences, there are some special differences between the modes in which this principle becomes available for the determination of general truths when the subject of inquiry is the mind, and when the subject is external nature.To these it is necessary to direct attention.

The general laws of Nature are not, for the most part, immediate objects of perception. They are either inductive inferences from a large body of facts, the common truth in which they express, or, in their origin at least, physical hypotheses of a causal nature serving to explain phænomena with undeviating precision, and to enable us to predict new combinations of them. They are in all cases, and in the strictest sense of the term, *probable* conclusions, approaching, indeed, ever and ever nearer to certainty, as they receive more and more of the confirmation of experience. But of the character of probability, in the strict and proper sense of that term, they are never wholly divested. On the other hand, the knowledge of the laws of the mind does not require as its basis any extensive collection of observations. The general truth is seen in the particular instance, and it is not confirmed by the repetition of instances. We may illustrate this position by an obvious example. It may be a question whether that formula of reasoning, which is called the *dictum* of Aristotle, *de omni et nullo*, expresses a primary law of human reasoning or not; but it is no question that it expresses a general truth in Logic. Now that truth is made manifest in all its generality by reflection upon a single instance of its application. And this is both an evidence that the particular principle or formula in question is founded upon some general law or laws of the mind, and an illustration of the doctrine that the perception of such general truths is not derived from an induction from many instances, but is involved in the clear apprehension of a single instance. In connexion with this truth is seen the not less important one that our knowledge of the laws upon which the science of the intellectual powers rests, whatever may be its extent or its deficiency, is not probable knowledge. For we not only see in the particular example the general truth, but we see it also as a certain truth,—a truth, our confidence in which will not continue to increase with increasing experience of its practical verifications.

The laws for an equational calculus of classes are not complicated, and easily command assent. But the difficulty of determining correct laws for more powerful symbolic logic systems is well-known. Frege's system seemed quite correct when applied to common examples, but it was discovered to be inconsistent (by Russell).

5. But if the general truths of Logic are of such a nature that when presented to the mind they at once command assent, wherein consists the difficulty of constructing the Science of Logic? Not, it may be answered, in collecting the materials of knowledge, but in discriminating their nature, and determining their mutual place and relation. All sciences consist of general truths, but of those truth some only are primary and fundamental, others are secondary and derived. The laws of elliptic motion, discovered by Kepler, are general truths in astronomy, but they are not its fundamental truths. And it is so also in the purely mathematical sciences. An almost boundless diversity of theorems, which are known, and an infinite possibility of others, as yet unknown, rest together upon the foundation of a few simple axioms; and yet these are all *general* truths. It may be added, that they are truths which to an intelligence sufficiently refined would shine forth in their own unborrowed light, without the need of those connecting links of thought, those steps of wearisome and often painful deduction, by which the knowledge of them is actually acquired. Let us define as fundamental those laws and principles from which all other general truths of science may be deduced, and into which they may all be again resolved. Shall we then err in regarding that as the true science of Logic which, laying down certain elementary laws, confirmed by the very testimony of the mind, permits us thence to deduce, by uniform processes, the entire chain of its secondary consequences, and furnishes, for its practical applications, methods of perfect generality? Let it be considered whether in any science, viewed either as a system of truth or as the foundation of a practical art, there can properly be any other test of the completeness and the fundamental character of its laws, than the completeness of its system of derived truths, and the generality of the methods which it serves to establish. Other questions may indeed present themselves. Convenience, prescription, individual preference, may urge their claims and deserve attention. But as respects the question of what constitutes science in its abstract integrity, I apprehend that no other considerations than the above are properly of any value.

6. It is designed, in the next place, to give expression in this treatise to the fundamental laws of reasoning in the symbolical language of a Calculus. Upon this head it will suffice to say, that those laws are such as to suggest this mode of expression, and to give to it a peculiar and exclusive fitness for the ends in view.

Do biology and physics consist only of general truths? Are general truths the same as laws? What is the difference between primary and fundamental truths?

What is the difference between 'from which they may be deduced' and 'to which they may be resolved'? Does Boole think there is a unique collection of fundamental laws for the science of logic?

Boole did not offer a definition of "is true" in his algebra of logic. A natural choice would be the equations and equational arguments that hold in his partial algebra of classes. But not all of these, e.g., $(x + y)^2 = x + y$, can be derived from his fundamental laws, nor do they hold in $\mathbb{Z}_{01}$ (described on page 37).

Boole surely used the algebra of numbers as the starting point for his algebra of logic—this hypothesis explains his curious partial algebra models.

There is not only a close analogy between the operations of the mind in general reasoning and its operations in the particular science of Algebra, but there is to a considerable extent an exact agreement in the laws by which the two classes of operations are conducted. Of course the laws must in both cases be determined independently; any formal agreement between them can only be established *à posteriori* by actual comparison. To borrow the notation of the science of Number, and then assume that in its new application the laws by which its use is governed will remain unchanged, would be mere hypothesis. There exist, indeed, certain general principles founded in the very nature of language, by which the use of symbols, which are but the elements of scientific language, is determined. To a certain extent these elements are arbitrary. Their interpretation is purely conventional: we are permitted to employ them in whatever sense we please. But this permission is limited by two indispensable conditions,—first, that from the sense once conventionally established we never, in the same process of reasoning, depart; secondly, that the laws by which the process is conducted be founded exclusively upon the above fixed sense or meaning of the symbols employed. In accordance with these principles, any agreement which may be established between the laws of the symbols of Logic and those of Algebra can but issue in an agreement of processes. The two provinces of interpretation remain apart and independent, each subject to its own laws and conditions.

Now the actual investigations of the following pages exhibit Logic, in its practical aspect, as a system of processes carried on by the aid of symbols having a definite interpretation, and subject to laws founded upon that interpretation alone. But at the same time they exhibit those laws as identical in form with the laws of the general symbols of algebra, with this single addition, viz., that the symbols of Logic are further subject to a special law (Chap, II.), to which the symbols of quantity, as such, are not subject. Upon the nature and the evidence of this law it is not purposed here to dwell. These questions will be fully discussed in a future page. But as constituting the essential ground of difference between those forms of inference with which Logic is

The (equational) laws of Boole's algebra of logic are almost the same as the laws of the algebra of numbers.

The algebra of logic laws were some of the equations that held in his partial algebras—he sidestepped those that caused problems, e.g., $(x+y)^2 = x+y$.

Perhaps he meant to include rules of inference, which he called 'axioms', in his laws.

"Processes" were the steps in the derivation of equations, often algorithms to derive desired conclusions from given equations, like his expansion, reduction, elimination and solution processes.

Laws of the algebra of logic = laws of the algebra of numbers plus a 'special law', the law of idempotent variables.

conversant, and those which present themselves in the particular science of Number, the law in question is deserving of more than a passing notice. It may be said that it lies at the very foundation of general reasoning,—that it governs those intellectual acts of conception or of imagination which are preliminary to the processes of logical deduction, and that it gives to the processes themselves much of their actual form and expression. It may hence be affirmed that this law constitutes the germ or seminal principle, of which every approximation to a general method in Logic is the more or less perfect development.

The idempotent law for variables.

7. The principle has already been laid down (5) that the sufficiency and truly fundamental character of any assumed system of laws in the science of Logic must partly be seen in the perfection of the methods to which they conduct us. It remains, then, to consider what the requirements of a general method in Logic are, and how far they are fulfilled in the system of the present work.

As part of the 'perfection' that Boole claimed for his approach to logic, he gave what he considered to be a comprehensive view of what constituted *propositions*. They had to express either

(1) a relation among *things*; this soon became a relation between two *classes*,
or
(2) a relation among *facts*, which soon became a relation among *propositions*.

Logic is conversant with two kinds of relations,—relations among things, and relations among facts. But as facts are expressed by propositions, the latter species of relation may, at least for the purposes of Logic, be resolved into a relation among propositions. The assertion that the fact or event A is an invariable consequent of the fact or event B may, to this extent at least, be regarded as equivalent to the assertion, that the truth of the proposition affirming the occurrence of the event B always implies the truth of the proposition affirming the occurrence of the event A. Instead, then, of saying that Logic is conversant with relations among things and relations among facts, we are permitted to say that it is concerned with relations among things and relations among propositions. Of the former kind of relations we have an example in the proposition—"All men are mortal;" of the latter kind in the proposition—"If the sun is totally eclipsed, the stars will become visible." The one expresses a relation between "men" and "mortal beings," the other between the elementary propositions—"The sun is totally eclipsed;" "The stars will become visible." Among such relations I suppose to be included those which affirm or deny existence with respect to things, and those which affirm or deny truth with respect

Modulo specifying what kinds of relations are allowed, Item (1) defined *primary* propositions, (2) defined *secondary* propositions.

For details on primary propositions, see pages 34-35, 52-53, 58–64, 152, 162.

For secondary propositions see page 160.

For the subject and predicate terms of primary propositions see pages 54–57.

to propositions. Now let those things or those propositions among which relation is expressed be termed the elements of the propositions by which such relation is expressed. Proceeding from this definition, we may then say that the *premises* of any logical argument express *given* relations among certain elements, and that the *conclusion* must express an *implied* relation among those elements, or among a part of them, i.e. a relation implied by or inferentially involved in the premises.

The elements of propositions were expressed in symbolic form by single letters $x, y, \ldots$.

8. Now this being premised, the requirements of a general method in Logic seem to be the following:—

General method = algorithms for solving main problems.

1st. As the conclusion must express a relation among the whole or among a part of the elements involved in the premises, it is requisite that we should possess the means of eliminating those elements which we desire not to appear in the conclusion, and of determining the whole amount of relation implied by the premises among the elements which we wish to retain. Those elements which do not present themselves in the conclusion are, in the language of the common Logic, called middle terms; and the species of elimination exemplified in treatises on Logic consists in deducing from two propositions, containing a common element or middle term, a conclusion connecting the two remaining terms. But the problem of elimination, as contemplated in this work, possesses a much wider scope. It proposes not merely the elimination of one middle term from two propositions, but the elimination generally of middle terms from propositions, without regard to the number of either of them, or to the nature of their connexion. To this object neither the processes of Logic nor those of Algebra, in their actual state, present any strict parallel. In the latter science the problem of elimination is known to be limited in the following manner:—From two equations we can eliminate one symbol of quantity; from three equations two symbols; and, generally, from n equations $n-1$ symbols. But though this condition, necessary in Algebra, seems to prevail in the existing Logic also, it has no essential place in Logic as a science. There, no relation whatever can be proved to prevail between the number of terms to be eliminated and the number of propositions from which the elimination is to be effected. From the equation representing a single proposition, any number

Boole gave no proof that his elimination procedure (see Chapter VII) determined "the whole amount of relation ..."

Aristotelian logic only had the elimination of one term from two categorical propositions (by syllogisms). Expressed in Boole's equational algebra, this meant the elimination of one variable from two equations, just as the algebra of numbers had the elimination of one variable from two equations. Boole said that his algebra of logic was different from numerical algebra in that one could eliminate any number of variables from any number of equations, and hence any number of 'middle' terms from any number of propositions. (See page 100.)

of symbols representing terms or elements in Logic may be eliminated; and from any number of equations representing propositions, one or any other number of symbols of this kind may be eliminated in a similar manner. For such elimination there exists one general process applicable to all cases. This is one of the many remarkable consequences of that distinguishing law of the symbols of Logic, to which attention has been already directed.

2ndly. It should be within the province of a general method in Logic to express the final relation among the elements of the conclusion by any admissible *kind* of proposition, or in any selected *order* of terms. Among varieties of kind we may reckon those which logicians have designated by the terms categorical, hypothetical, disjunctive, &c. To a choice or selection in the order of the terms, we may refer whatsoever is dependent upon the appearance of particular elements in the subject or in the predicate, in the antecedent or in the consequent, of that proposition which forms the "conclusion." But waiving the language of the schools, let us consider what really distinct species of problems may present themselves to our notice. We have seen that the elements of the final or inferred relation may either be *things* or *propositions.* Suppose the former case; then it might be required to deduce from the premises a definition or description of some one thing, or class of things, constituting an element of the conclusion in terms of the other things involved in it. Or we might form the conception of some thing or class of things, involving more than one of the elements of the conclusion, and require its expression in terms of the other elements. Again, suppose the elements retained in the conclusion to be propositions, we might desire to ascertain such points as the following, viz., Whether, in virtue of the premises, any of those propositions, taken singly, are true or false?—Whether particular combinations of them are true or false?—Whether, assuming a particular proposition to be true, any consequences will follow, and if so, what consequences, with respect to the other propositions?—Whether any particular condition being assumed with reference to certain of the propositions, any consequences, and what consequences, will follow with respect to the others? and so on. I say that these are general questions, which it should fall within the scope or province of a general method in Logic to solve. Perhaps we might include them all under this one statement of the final problem of practical Logic. Given a set of premises expressing relations among certain elements, whether things or propositions: required explicitly the whole relation consequent among *any* of those elements under any proposed conditions, and in any proposed form. That this problem, under all its aspects, is resolvable, will hereafter appear. But it is not for the sake of noticing this fact, that the above inquiry into the nature and the functions of a general method in Logic has been introduced. It is necessary that the reader should apprehend what are the specific ends of the investigation upon which we are entering, as well as the principles which are to guide us to the attainment of them.

Boole's description of the ultimate goal of general method(s) in the applications of Logic.

9. Possibly it may here be said that the Logic of Aristotle, in its rules of syllogism and conversion, sets forth the elementary processes of which all reasoning consists, and that beyond these there is neither scope nor occasion for a general method. I have no desire to point out the defects of the common Logic, nor do I wish to refer to it any further than is necessary, in order to place in its true light the nature of the present treatise. With this end alone in view, I would remark:—1st. That syllogism, conversion, &c., are not the ultimate processes of Logic. It will be shown in this treatise that they are founded upon, and are resolvable into, ulterior and more simple processes which constitute the real elements of method in Logic. Nor is it true in fact that all inference is reducible to the particular forms of syllogism and conversion.—*Vide* Chap. xv. 2ndly. If all inference were reducible to these two processes (and it has been maintained that it is reducible to syllogism alone), there would still exist the same necessity for a general method. For it would still be requisite to determine in what order the processes should succeed each other, as well as their particular nature, in order that the desired relation should be obtained. By the desired relation I mean that full relation which, in virtue of the premises, connects any elements selected out of the premises at will, and which, moreover, expresses that relation in any desired form and order.

The sufficiency of conversions and syllogisms was the standard view of logicians at that time. See pages 238–240 .

Boole noted that even if every valid logical argument could be verified by a suitable collection of syllogisms and conversions, the prevailing logic did not offer an algorithm to find and arrange the desired collection of syllogisms.

If we may judge from the mathematical sciences, which are the most perfect examples of method known, this *directive* function of Method constitutes its chief office and distinction. The fundamental processes of arithmetic, for instance, are in themselves but the elements of a possible science. To assign their nature is the first business of its method, but to arrange their succession is its subsequent and higher function. In the more complex examples of logical deduction, and especially in those which form a basis for the solution of difficult questions in the theory of Probabilities, the aid of a directive method, such as a Calculus alone can supply, is indispensable.

Method = algorithm(s)!

10. Whence it is that the ultimate laws of Logic are mathematical in their form; why they are, except in a single point, identical with the general laws of Number; and why in that particular point they differ;—are questions upon which it might not be very remote from presumption to endeavour to pronounce a positive judgment. Probably they lie beyond the reach of our limited faculties. It may, perhaps, be permitted to the mind to attain a knowledge of the laws to which it is itself subject, without its being also given to it to understand their ground and origin, or even, except in a very limited degree, to comprehend their fitness for their end, as compared with other and conceivable systems of law. Such knowledge is, indeed, unnecessary for the ends of science, which properly concerns itself with what is, and seeks not for grounds of preference or reasons of appointment. These considerations furnish a sufficient answer to all protests against the exhibition of Logic in the form of a Calculus. It is not because we choose to assign to it such a mode of manifestation, but because the ultimate laws of thought render that mode possible, and prescribe its character, and forbid, as it would seem, the perfect manifestation of the science in any other form, that such a mode demands adoption. It is to be remembered that it is the business of science not to create laws, but to discover them. We do not originate the constitution of our own minds, greatly as it may be in our power to modify their character. And as the laws of the human intellect do not depend upon our will, so the forms of the science, of which they constitute the basis, are in all essential regards independent of individual choice.

Boole said, again, that the fundamental laws of logic = the fundamental laws of numbers augmented by the idempotent law for variables. He said it may be beyond the capability of man to understand why.

But, we do understand why! Boole was able to design his algebra of logic, using partial operations, to satisfy the laws of the algebra of numbers, and discovered he could add the idempotent law for variables. Then he found algorithms (methods) for this algebra to answer what he considered to be the only problems of interest for logic.

It seems Boole was claiming to have found the unique symbolic system for logic.

11. Beside the general statement of the principles of the above method, this treatise will exhibit its application to the analysis of a considerable variety of propositions, and of trains of propositions constituting the premises of demonstrative arguments. These examples have been selected from various writers, they differ greatly in complexity, and they embrace a wide range of subjects. Though in this particular respect it may appear to some that too great a latitude of choice has been exercised, I do not deem it necessary to offer any apology upon this account. That Logic, as a science, is susceptible of very wide applications is admitted; but it is equally certain that its ultimate forms and processes are mathematical. Any objection *à priori* which may therefore be supposed to lie against the adoption of such forms and processes in the discussion of a problem of morals or of general philosophy must be founded upon misapprehension or false analogy. It is not of the essence of mathematics to be conversant with the ideas of number and quantity. Whether as a general habit of mind it would be desirable to apply symbolical processes to moral argument, is another question. Possibly, as I have elsewhere observed,[1] the perfection of the method of Logic may be chiefly valuable as an evidence of the speculative truth of its principles. To supersede the employment of common reasoning, or to subject it to the rigour of technical forms, would be the last desire of one who knows the value of that intellectual toil and warfare which imparts to the mind an athletic vigour, and teaches it to contend with difficulties, and to rely upon itself in emergencies. Nevertheless, cases may arise in which the value of a scientific procedure, even in those things which fall confessedly under the ordinary dominion of the reason, may be felt and acknowledged. Some examples of this kind will be found in the present work.

If by "mathematical" he meant 'equational algebra', then it is not "equally certain".

The "perfection" of his method depended on the limits he set on the nature of propositions, and the goals he set for a method in logic. In De Morgan's "Formal Logic" of 1847, it was noted that the Aristotelian logic was not able to handle a simple argument like "All men are animals", therefore "All heads of men are heads of animals". Boole's version of logic also could not handle such arguments.

The rest of this chapter, except the last item (20), is about probability. There are no further comments on pages 14–20.

12. The general doctrine and method of Logic above explained form also the basis of a theory and corresponding method of Probabilities. Accordingly, the development of such a theory and method, upon the above principles, will constitute a distinct object of the present treatise. Of the nature of this application it may be desirable to give here some account, more especially as

[1] Mathematical Analysis of Logic. London : G. Bell. 1847.

regards the character of the solutions to which it leads. In connexion with this object some further detail will be requisite concerning the forms in which the results of the logical analysis are presented.

The ground of this necessity of a prior method in Logic, as the basis of a theory of Probabilities, may be stated in a few words. Before we can determine the mode in which the expected frequency of occurrence of a particular event is dependent upon the known frequency of occurrence of any other events, we must be acquainted with the mutual dependence of the events themselves. Speaking technically, we must be able to express the event whose probability is sought, as a function of the events whose probabilities are given. Now this explicit determination belongs in all instances to the department of Logic. Probability, however, in its mathematical acceptation, admits of numerical measurement. Hence the subject of Probabilities belongs equally to the science of Number and to that of Logic. In recognising the co-ordinate existence of both these elements, the present treatise differs from all previous ones; and as this difference not only affects the question of the possibility of the solution of problems in a large number of instances, but also introduces new and important elements into the solutions obtained, I deem it necessary to state here, at some length, the peculiar consequences of the theory developed in the following pages.

Boole viewed Logic as a prerequisite for Probability Theory.

13. The measure of the probability of an event is usually defined as a fraction, of which the numerator represents the number of cases favourable to the event, and the denominator the whole number of cases favourable and unfavourable; all cases being supposed equally likely to happen. That definition is adopted in the present work. At the same time it is shown that there is another aspect of the subject (shortly to be referred to) which might equally be regarded as fundamental, and which would actually lead to the same system of methods and conclusions. It may be added, that so far as the received conclusions of the theory of Probabilities extend, and so far as they are consequences of its fundamental definitions, they do not differ from the results (supposed to be equally correct in inference) of the method of this work.

Again, although questions in the theory of Probabilities present themselves under various aspects, and may be variously modified by algebraical and other conditions, there seems to be one general type to which all such questions, or so much of each of them as truly belongs to the theory of Probabilities, may be referred. Considered with reference to the *data* and the *quæsitum*, that type may be described as follows:—1st. The data are the probabilities of one or more given events, each probability being either that of the absolute fulfilment of the event to which it relates, or the probability of its fulfilment under given supposed conditions. 2ndly. The *quæsitum*, or object sought, is the probability of the fulfilment, absolutely or conditionally, of some other event differing in expression from those in the data, but more or less involving the same elements. As concerns the data, they are either *causally given*,—as when the probability of a particular throw of a die is deduced from a knowledge of the constitution of the piece,—or they are derived from observation of repeated instances of the success or failure of events. In the latter case the probability of an event may be defined as the limit toward which the ratio of the favourable to the whole number of observed cases approaches (the uniformity of nature being presupposed) as the observations are indefinitely continued. Lastly, as concerns the nature or relation of the events in question, an important distinction remains. Those events are either *simple* or *compound*. By a compound event is meant one of which the expression in language, or the conception in thought, depends upon the expression or the conception of other events, which, in relation to it, may be regarded as *simple* events. To say "it rains," or to say "it thunders," is to express the occurrence of a simple event; but to say "it rains and thunders," or to say "it either rains or thunders," is to express that of a compound event. For the expression of that event depends upon the elementary expressions, "it rains," "it thunders." The criterion of simple events is not, therefore, any supposed simplicity in their nature. It is founded solely on the mode of their expression in language or conception in thought.

14. Now one general problem, which the existing theory of Probabilities enables us to solve, is the following, viz.:—Given the probabilities of any simple events: required the probability of a given compound event, i.e. of an event compounded in a given manner out of the given simple events. The problem can also be solved when the compound event, whose probability is required, is subjected to given conditions, i.e. to conditions dependent also in a given manner on the given simple events. Beside this general problem, there exist also particular problems of which the principle of solution is known. Various questions relating to *causes* and *effects* can be solved by known methods under the particular hypothesis that the causes are mutually exclusive, but apparently not otherwise. Beyond this it is not clear that any advance has been made toward the solution of what may be regarded as the general problem of the science, viz.: Given the probabilities of any events, simple or compound, conditioned or unconditioned: required the probability of any other event equally arbitrary in expression and conception. In the statement of this question it is not even postulated that the events whose probabilities are given, and the one whose probability is sought, should involve some common elements, because it is the office of a method to determine whether the data of a problem are sufficient for the end in view, and to indicate, when they are not so, wherein the deficiency consists.

This problem, in the most unrestricted form of its statement, is resolvable by the method of the present treatise; or, to speak more precisely, its theoretical solution is completely given, and its practical solution is brought to depend only upon processes purely mathematical, such as the resolution and analysis of equations. The order and character of the general solution may be thus described.

15. In the first place it is always possible, by the preliminary method of the Calculus of Logic, to express the event whose probability is sought as a logical function of the events whose probabilities are given. The result is of the following character: Suppose that X represents the event whose probability is sought, A, B, C, &c. the events whose probabilities are given, those events being either simple or compound. Then the *whole* relation of the event X to the events A, B, C, &c. is deduced in the form of what mathematicians term a *development*, consisting, in the most general case, of four distinct classes of terms. By the first class are expressed those combinations of the events A, B, C, which both necessarily accompany and necessarily indicate the occurrence of the event X; by the second class, those combinations which necessarily accompany, but do not necessarily imply, the occurrence of the event X; by the third class, those combinations whose occurrence in connexion with the event X is impossible, but not otherwise impossible; by the fourth class, those combinations whose occurrence is impossible under any circumstances. I shall not dwell upon this statement of the result of the logical analysis of the problem, further than to remark that the elements which it presents are precisely those by which the expectation of the event X, as dependent upon our knowledge of the events A, B, C, is, or alone can be, affected. General reasoning would verify this conclusion; but general reasoning would not usually avail to disentangle the complicated web of events and circumstances from which the solution above described must be evolved. The attainment of this object constitutes the first step towards the complete solution of the question proposed. It is to be noted that thus far the process of solution is logical, i.e. conducted by symbols of logical significance, and resulting in an equation interpretable into a *proposition*. Let this result be termed the *final logical equation*.

The second step of the process deserves attentive remark. From the final logical equation to which the previous step has conducted us, are deduced, by inspection, a series of algebraic equations implicitly involving the complete solution of the problem proposed. Of the mode in which this transition is effected let it suffice to say, that there exists a definite relation between the laws by which the probabilities of events are expressed as algebraic functions of the probabilities of other events upon which they depend, and the laws by which the logical connexion of the events is itself expressed. This relation, like the other coincidences of formal law which have been referred to, is not founded upon hypothesis, but is made known to us by observation (I.4), and reflection. If, however, its reality were assumed *à priori* as the basis of the very definition of Probability, strict deduction would thence lead us to the received numerical definition as a necessary consequence. The Theory of Probabilities stands, as it has already been remarked (I.12), in equally close relation to Logic and to Arithmetic; and it is indifferent, so far as results are concerned, whether we regard it as springing out of the latter of these sciences, or as founded in the mutual relations which connect the two together.

16. There are some circumstances, interesting perhaps to the mathematician, attending the general solutions deduced by the above method, which it may be desirable to notice.

1st. As the method is independent of the number and the nature of the data, it continues to be applicable when the latter are insufficient to render determinate the value sought. When such is the case, the final expression of the solution will contain terms with arbitrary constant coefficients. To such terms there will correspond terms in the final logical equation (I. 15), the interpretation of which will inform us what new data are requisite in order to determine the values of those constants, and thus render the numerical solution complete. If such data are not to be obtained, we can still, by giving to the constants their limiting values 0 and 1, determine the limits within which the probability sought must lie independently of all further experience. When the event whose probability is sought is *quite* independent of those whose probabilities are given, the limits thus obtained for its value will be 0 and 1, as it is evident that they ought to be, and the interpretation of the constants will only lead to a re-statement of the original problem.

2ndly. The expression of the final solution will in all cases involve a particular element of quantity, determinable by the solution of an algebraic equation. Now when that equation is of an elevated degree, a difficulty may seem to arise as to the selection of the proper root. There are, indeed, cases in which both the elements given and the element sought are so obviously restricted by necessary conditions that no choice remains. But in complex instances the discovery of such conditions, by unassisted force of reasoning, would be hopeless. A distinct method is requisite for this end,—a method which might not appropriately be termed the Calculus of Statistical Conditions, into the nature of this method I shall not here further enter than to say, that, like the previous method, it is based upon the employment of the "final logical equation," and that it definitely assigns, 1st, the conditions which must be fulfilled among the numerical elements of the data, in order that the problem may be real, i.e. derived from a *possible experience*; 2ndly, the numerical limits, within which the probability sought must have been confined, if, instead of being determined by theory, it had been deduced directly by observation from the same system of phænomena from which the data were derived. It is clear that these limits will be actual limits of the probability sought. Now, on supposing the data subject to the conditions above assigned to them, it appears in every instance which I have examined that there exists one root, and only one root, of the final algebraic equation which is subject to the required limitations. Every source of ambiguity is thus removed. It would even seem that new truths relating to the theory of algebraic equations are thus incidentally brought to light. It is remarkable that the special element of quantity, to which the previous discussion relates, depends only upon the *data*, and not at all upon the *quæsitum* of the problem proposed. Hence the solution of each particular problem unties the knot of difficulty for a system of problems, viz., for that system of problems which is marked by the possession of common data, independently of the nature of their *quæsita*. This circumstance is important whenever from a particular system of data it is required to deduce a series of connected conclusions. And it further gives to the solutions of particular problems that character of relationship, derived from their dependence upon a central and fundamental unity, which not unfrequently marks the application of general methods.

17. But though the above considerations, with others of a like nature, justify the assertion that the method of this treatise, for the solution of questions in the theory of Probabilities, is a general method, it does not thence follow that we are relieved in all cases from the necessity of recourse to hypothetical grounds. It has been observed that a solution may consist entirely of terms affected by arbitrary constant coefficients,—may, in fact, be wholly indefinite. The application of the method of this work to some of the most important questions within its range would—

were the data of experience alone employed—present results of this character. To obtain a *definite* solution it is necessary, in such cases, to have recourse to hypotheses possessing more or less of independent probability, but incapable of exact verification. Generally speaking, such hypotheses will differ from the immediate results of experience in partaking of a logical rather than of a numerical character; in prescribing the conditions under which phænomena occur, rather than assigning the relative frequency of their occurrence. This circumstance is, however, unimportant. Whatever their nature may be, the hypotheses assumed must thenceforth be regarded as belonging to the actual data, although tending, as is obvious, to give to the solution itself somewhat of a hypothetical character. With this understanding as to the possible sources of the data actually employed, the method is perfectly general, but for the correctness of the hypothetical elements introduced it is of course no more responsible than for the correctness of the numerical data derived from experience.

In illustration of these remarks we may observe that the theory of the reduction of astronomical observations[2] rests, in part, upon hypothetical grounds. It assumes certain positions as to the nature of error, the equal probabilities of its occurrence in the form of excess or defect, &c., without which it would be impossible to obtain any *definite* conclusions from a system of conflicting observations. But granting such positions as the above, the residue of the investigation falls strictly within the province of the theory of Probabilities. Similar observations apply to the important problem which proposes to deduce from the records of the majorities of a deliberative assembly the mean probability of correct judgment in one of its members. If the method of this treatise be applied to the mere numerical data, the solution obtained is of that wholly indefinite kind above described. And to show in a more eminent degree the insufficiency of those data by themselves, the interpretation of the arbitrary constants (I. 16) which appear in the solution, merely produces

[2]The author designs to treat this subject either in a separate work or in a future Appendix. In the present treatise he avoids the use of the integral calculus.

a re-statement of the original problem. Admitting, however, the hypothesis of the independent formation of opinion in the individual mind, either absolutely, as in the speculations of Laplace and Poisson, or under limitations imposed by the actual data, as will be seen in this treatise, Chap. XXI., the problem assumes a far more definite character. It will be manifest that the ulterior value of the theory of Probabilities must depend very much upon the correct formation of such mediate hypotheses, where the purely experimental data are insufficient for *definite* solution, and where that further experience indicated by the interpretation of the final logical equation is unattainable. Upon the other hand, an undue readiness to form hypotheses in subjects which from their very nature are placed beyond human ken, must re-act upon the credit of the theory of Probabilities, and tend to throw doubt in the general mind over its most legitimate conclusions.

18. It would, perhaps, be premature to speculate here upon the question whether the methods of abstract science are likely at any future day to render service in the investigation of social problems at all commensurate with those which they have rendered in various departments of physical inquiry. An attempt to resolve this question upon pure *à priori* grounds of reasoning would be very likely to mislead us. For example, the consideration of human free-agency would seem at first sight to preclude the idea that the movements of the social system should ever manifest that character of orderly evolution which we are prepared to expect under the reign of a physical necessity. Yet already do the researches of the statist reveal to us facts at variance with such an anticipation. Thus the records of crime and pauperism present a degree of regularity unknown in regions in which the disturbing influence of human wants and passions is unfelt. On the other hand, the distemperature of seasons, the eruption of volcanoes, the spread of blight in the vegetable, or of epidemic maladies in the animal kingdom, things apparently or chiefly the product of natural causes, refuse to be submitted to regular and apprehensible laws. "Fickle as the wind," is a proverbial expression. Reflection upon these points teaches us in some degree to correct our earlier judgments. We learn that we are not to expect, under the dominion of necessity, an order perceptible to human observation, unless the play of its producing causes is sufficiently simple; nor, on the other hand, to deem that free agency in the individual is inconsistent with regularity in the motions of the system of which he forms a component unit. Human freedom stands out as an apparent fact of our consciousness, while it is also, I conceive, a highly probable deduction of analogy (Chap, XXII.) from the nature of that portion of the mind whose scientific constitution we are able to investigate. But whether accepted as a fact reposing on consciousness, or as a conclusion sanctioned by the reason, it must be so interpreted as not to conflict with an established result of observation, viz.: that phænomena, in the production of which large masses of men are concerned, do actually exhibit a very remarkable degree of regularity, enabling us to collect in each succeeding age the elements upon which the estimate of its state and progress, so far as manifested in outward results, must depend. There is thus no sound objection *à priori* against the possibility of that species of data which is requisite for the experimental foundation of a science of social statistics. Again, whatever other object this treatise may accomplish, it is presumed that it will leave no doubt as to the existence of a system of abstract principles and of methods founded upon those principles, by which any collective body of social data may be made to yield, in an explicit form, whatever information they implicitly involve. There may, where the data are exceedingly complex, be very great difficulty in obtaining this information,—difficulty due not to any imperfection of the theory, but to the laborious character of the analytical processes to which it points. It is quite conceivable that in many instances that difficulty may be such as only united effort could overcome. But that we possess theoretically in all cases, and practically, so far as the requisite labour of calculation may be supplied, the means of evolving from statistical records the seeds of general truths which lie buried amid the mass of figures, is a position which may, I conceive, with perfect safety be affirmed.

The possibility of applications of his work to social science are noted on this page.

19. But beyond these general positions I do not venture to speak in terms of assurance. Whether the results which might be expected from the application of scientific methods to statistical records, over and above those the discovery of which requires no such aid, would so far compensate for the labour involved as to render it worth while to institute such investigations upon a proper scale of magnitude, is a point which could, perhaps, only be determined by experience. It is to be desired, and it might without great presumption be expected, that in this, as in other instances, the abstract doctrines of science should minister to more than intellectual gratification. Nor, viewing the apparent order in which the sciences have been evolved, and have successively contributed their aid to the service of mankind, does it seem very improbable that a day may arrive in which similar aid may accrue from departments of the field of knowledge yet more intimately allied with the elements of human welfare. Let the speculations of this treatise, however, rest at present simply upon their claim to be regarded as true.

20. I design, in the last place, to endeavour to educe from the scientific results of the previous inquiries some general intimations respecting the nature and constitution of the human mind. Into the grounds of the possibility of this species of inference it is not necessary to enter here. One or two general observations may serve to indicate the track which I shall endeavour to follow. It cannot but be admitted that our views of the science of Logic must materially influence, perhaps mainly determine, our opinions upon the nature of the intellectual faculties. For example, the question whether reasoning consists merely in the application of certain first or necessary truths, with which the mind has been originally imprinted, or whether the mind is itself a seat of law, whose operation is as manifest and as conclusive in the particular as in the general formula, or whether, as some not undistinguished writers seem to maintain, all reasoning is of particulars; this question, I say, is one which not merely affects the science of Logic, but also concerns the formation of just views of the constitution of the intellectual faculties. Again, if it is concluded that the mind is by original constitution a seat of law, the question of the nature of its subjection to this law,—whether, for instance, it is an obedience founded upon necessity, like that which sustains the revolutions of the heavens, and preserves the order of Nature,—or whether

Not a successful project

it is a subjection of some quite distinct kind, is also a matter of deep speculative interest. Further, if the mind is truly determined to be a subject of law, and if its laws also are truly assigned, the question of their probable or necessary influence upon the course of human thought in different ages is one invested with great importance, and well deserving a patient investigation, as matter both of philosophy and of history. These and other questions I propose, however imperfectly, to discuss in the concluding portion of the present work. They belong, perhaps, to the domain of probable or conjectural, rather than to that of positive, knowledge. But it may happen that where there is not sufficient warrant for the certainties of science, there may be grounds of analogy adequate for the suggestion of highly probable opinions. It has seemed to me better that this discussion should be entirely reserved for the sequel of the main business of this treatise,—which is the investigation of scientific truths and laws. Experience sufficiently instructs us that the proper order of advancement in all inquiries after truth is to proceed from the known to the unknown. There are parts, even of the philosophy and constitution of the human mind, which have been placed fully within the reach of our investigation. To make a due acquaintance with those portions of our nature the basis of all endeavours to penetrate amid the shadows and uncertainties of that conjectural realm which lies beyond and above them, is the course most accordant with the limitations of our present condition.

Chapter II

Of Signs in General, and of the Signs Appropriate to the Science of Logic in Particular; also of the Laws to which that Class of Signs are Subject.

1. That Language is an instrument of human reason, and not merely a medium for the expression of thought, is a truth generally admitted. It is proposed in this chapter to inquire what it is that renders Language thus subservient to the most important of our intellectual faculties. In the various steps of this inquiry we shall be led to consider the constitution of Language, considered as a system adapted to an end or purpose; to investigate its elements; to seek to determine their mutual relation and dependence; and to inquire in what manner they contribute to the attainment of the end to which, as co-ordinate parts of a system, they have respect.

In proceeding to these inquiries, it will not be necessary to enter into the discussion of that famous question of the schools, whether Language is to be regarded as an *essential* instrument of reasoning, or whether, on the other hand, it is possible for us to reason without its aid. I suppose this question to be beside the design of the present treatise, for the following reason, viz., that it is the business of Science to investigate laws; and that, whether we regard signs as the representatives of things and of their relations, or as the representatives of the conceptions and operations of the human intellect, in studying the laws of signs, we are in effect studying the manifested laws of reasoning. If there exists a difference between the two inquiries, it is one which does not affect the scientific expressions of formal law, which are the object of investigation in the present stage of this work, but relates only to the mode in which those results are presented to the mental regard. For though in investigating the laws of signs, *à posteriori*, the immediate subject of examination is Language, with the rules which govern its use; while in making the internal

On Boole's Notation:

The class-symbols used are p,q,r,s,t,u,v,w,x,y,z. They are never subscripted. In Chapter XV one finds the only variation on the use of unadorned letters, namely there are primed symbols, like v'.

Boole used the lower case letter f, starting on page 71, to represent algebraic expressions when variables were indicated, as in $f(x)$ and $f(x, y)$. When no variables were to be indicated, he used the capital letters A,B,C,D,E,P,Q,X,Y,V. V was the favorite, starting on page 79. The one exception to capital letters was the use of t for constituents, starting on page 78, where he also introduced subscripts for the first time, e.g., t_1, t_2.

The symbols for numerical coefficients are lower case letters a, b, c, d at the beginning of the alphabet, with a being the only one that was subscripted.

Boole did not use the summation sign $\sum$ or the product sign $\prod$ in his chapters on logic. (The $\sum$ sign appears in the chapters on probability.)

processes of thought the direct object of inquiry, we appeal in a more immediate way to our personal consciousness,—it will be found that in both cases the results obtained are formally equivalent. Nor could we easily conceive, that the unnumbered tongues and dialects of the earth should have preserved through a long succession of ages so much that is common and universal, were we not assured of the existence of some deep foundation of their agreement in the laws of the mind itself.

2. The elements of which all language consists are signs or symbols. Words are signs. Sometimes they are said to represent things; sometimes the operations by which the mind combines together the simple notions of things into complex conceptions; sometimes they express the relations of action, passion, or mere quality, which we perceive to exist among the objects of our experience; sometimes the emotions of the perceiving mind. But words, although in this and in other ways they fulfil the office of signs, or representative symbols, are not the only signs which we are capable of employing. Arbitrary marks, which speak only to the eye, and arbitrary sounds or actions, which address themselves to some other sense, are equally of the nature of signs, provided that their representative office is defined and understood. In the mathematical sciences, letters, and the symbols $+$, $-$, $=$, &c., are used as signs, although the term "sign" is applied to the latter class of symbols, which represent operations or relations, rather than to the former, which represent the elements of number and quantity. As the real import of a sign does not in any way depend upon its particular form or expression, so neither do the laws which determine its use. In the present treatise, however, it is with written signs that we have to do, and it is with reference to these exclusively that the term "sign" will be employed. The essential properties of signs are enumerated in the following definition.

From now on, 'signs' means 'written signs'.

Definition.—A sign is an arbitrary mark, having a fixed interpretation, and susceptible of combination with other signs in subjection to fixed laws dependent upon their mutual interpretation.

"having a fixed interpretation . . . " belongs in the context of considering a particular model.

3. Let us consider the particulars involved in the above definition separately.

(1.) In the first place, a sign is an *arbitrary* mark. It is clearly indifferent what particular word or token we associate with a given idea, provided that the association once made is permanent. The Romans expressed by the word "civitas" what we designate by the word "state." But both they and we might equally well have employed any other word to represent the same conception. Nothing, indeed, in the nature of Language would prevent us from using a mere letter in the same sense. Were this done, the laws according to which that letter would require to be used would be essentially the same with the laws which govern the use of "civitas" in the Latin, and of "state" in the English language, so far at least as the use of those words is regulated by any general principles common to all languages alike.

(2.) In the second place, it is necessary that each sign should possess, within the limits of the same discourse or process of reasoning, a fixed interpretation. The necessity of this condition is obvious, and seems to be founded in the very nature of the subject. There exists, however, a dispute as to the precise nature of the representative office of words or symbols used as names in the processes of reasoning. By some it is maintained, that they represent the conceptions of the mind alone; by others, that they represent things. The question is not of great importance here, as its decision cannot affect the laws according to which signs are employed. I apprehend, however, that the general answer to this and such like questions is, that in the processes of reasoning, signs stand in the place and fulfil the office of the conceptions and operations of the mind; but that as those conceptions and operations represent things, and the connexions and relations of things, so signs represent things with their connexions and relations; and lastly, that as signs stand in the place of the conceptions and operations of the mind, they are subject to the laws of those conceptions and operations. This view will be more fully elucidated in the next chapter; but it here serves to explain the third of those particulars involved in the definition of a sign, viz., its subjection to fixed laws of combination depending upon the nature of its interpretation.

Boole shifts from (1) dealing with conceptions of things and operations on such conceptions to (2) dealing directly with their (written) signs.

4. The analysis and classification of those signs by which the operations of reasoning are conducted will be considered in the following Proposition:

Proposition I.

All the operations of Language, as an instrument of reasoning, may be conducted by a system of signs composed of the following elements, viz.:

1st. *Literal symbols, as x, y, &c., representing things as subjects of our conceptions.*

2nd. *Signs of operation, as $+$, $-$, $\times$, standing for those operations of the mind by which the conceptions of things are combined or resolved so as to form new conceptions involving the same elements.*

3rd. *The sign of identity, $=$.*

And these symbols of Logic are in their use subject to definite laws, partly agreeing with and partly differing from the laws of the corresponding symbols in the science of Algebra.

Let it be assumed as a criterion of the true elements of rational discourse, that they should be susceptible of combination in the simplest forms and by the simplest laws, and thus combining should generate all other known and conceivable forms of language; and adopting this principle, let the following classification be considered.

CLASS I.

5. *Appellative or descriptive signs, expressing either the name of a thing, or some quality or circumstance belonging to it.*

To this class we may obviously refer the substantive proper or common, and the adjective. These may indeed be regarded as differing only in this respect, that the former expresses the substantive existence of the individual thing or things to which it refers; the latter implies that existence. If we attach to the adjective the universally understood subject "being" or "thing," it becomes virtually a substantive, and may for all the essential purposes of reasoning be replaced by the substantive. Whether or not, in every particular of the mental regard, it is the same thing to say, "Water is a fluid thing," as to say, "Water is fluid;" it is at least equivalent in the expression of the processes of reasoning.

The word 'variable' is not used by Boole for x, y, etc.

Note that 0 and 1 are not included at this point. 1 is needed to express the complement $1 - x$ of x, and 0 is used to express 'No x is y' by an equation, namely by $xy = 0$.

The division sign will appear in Chapter IV, in the context of solving a polynomial equation for a variable.

Class I introduces the letters $x, y, \ldots$ to represent classes (see middle of page 28); and the operation of multiplication is introduced as concatenation (see bottom of page 28) into Boole's algebra of logic.

Boole was slow to borrow the numerical algebra vocabulary, like the word *product*, for his algebra of logic. Instead he initially called xy a *combination*, or simply an *expression*. Further marginal comments on this topic are given on page 29.

It is clear also, that to the above class we must refer any sign which may conventionally be used to express some circumstance or relation, the detailed exposition of which would involve the use of many signs. The epithets of poetic diction are very frequently of this kind. They are usually compounded adjectives, singly fulfilling the office of a many-worded description. Homer's "deep-eddying ocean" embodies a virtual description in the single word βαθυδίνης. And conventionally any other description addressed either to the imagination or to the intellect might equally be represented by a single sign, the use of which would in all essential points be subject to the same laws as the use of the adjective "good" or "great." Combined with the subject "thing," such a sign would virtually become a substantive; and by a single substantive the combined meaning both of thing and quality might be expressed.

6. Now, as it has been defined that a sign is an arbitrary mark, it is permissible to replace all signs of the species above described by letters. Let us then agree to represent the class of individuals to which a particular name or description is applicable, by a single letter, as x. If the name is "men," for instance, let x represent "all men," or the class "men." By a class is usually meant a collection of individuals, to each of which a particular name or description may be applied; but in this work the meaning of the term will be extended so as to include the case in which but a single individual exists, answering to the required name or description, as well as the cases denoted by the terms "nothing" and "universe," which as "classes" should be understood to comprise respectively "no beings," "all beings." Again, if an adjective, as "good," is employed as a term of description, let us represent by a letter, as y, all things to which the description "good" is applicable, i.e. "all good things," or the class "good things." Let it further be agreed, that by the combination xy shall be represented that class of things to which the names or descriptions represented by x and y are simultaneously applicable. Thus, if x alone stands for "white things," and y for "sheep," let xy stand for "white sheep;" and in like manner, if z stand for "horned things," and x and y retain their previous interpretations, let zxy represent

Lower case letters $x, y, \ldots$ name classes. (Did Boole have the concept of an *arbitrary* class?)

Boole expanded the scope of 'class' beyond the traditional usage (that is, a class with at least two members) by including singletons, the empty class and the universe.

MULTIPLICATION is intersection. $xy := x \cap y$.

"horned white sheep," i.e. that collection of things to which the name "sheep," and the descriptions "white" and "horned" are together applicable.

Let us now consider the laws to which the symbols x, y, &c., used in the above sense, are subject.

7. First, it is evident, that according to the above combinations, the order in which two symbols are written is indifferent. The expressions xy and yx equally represent that class of things to the several members of which the names or descriptions x and y are together applicable. Hence we have,

$$\boxed{xy = yx}. \qquad (1)$$

Multiplication is COMMUTATIVE.

Unlike in MAL, Boole was reluctant to use the words 'multiplication', 'multiply' and 'product' (from the algebra of numbers) with this operation in his algebra of logic; but they are used freely in these marginal notes.

Perhaps the reason for this reluctance was the fact that the number of elements of xy was rarely the number of elements of x times the number of elements of y.

The first time 'product' is used in his algebra of logic is on page 49; the first time 'multiply' is used is on page 83; and the first time for 'multiplication' is on page 107.

In the case of x representing white things, and y sheep, either of the members of this equation will represent the class of "white sheep." There may be a difference as to the order in which the conception is formed, but there is none as to the individual things which are comprehended under it. In like manner, if x represent "estuaries," and y "rivers," the expressions xy and yx will indifferently represent "rivers that are estuaries," or "estuaries that are rivers," the combination in this case being in ordinary language that of two substantives, instead of that of a substantive and an adjective as in the previous instance. Let there be a third symbol, as z, representing that class of things to which the term "navigable" is applicable, and any one of the following expressions,

$$zxy,\ zyx,\ xyz,\ \&c.,$$

will represent the class of "navigable rivers that are estuaries."

If one of the descriptive terms should have some implied reference to another, it is only necessary to include that reference expressly in its stated meaning, in order to render the above remarks still applicable. Thus, if x represent "wise" and y "counsellor," we shall have to define whether x implies wisdom in the absolute sense, or only the wisdom of counsel. With such definition the law $xy = yx$ continues to be valid.

We are permitted, therefore, to employ the symbols x, y, z, &c., in the place of the substantives, adjectives, and descriptive phrases subject to the rule of interpretation, that any expression in which several of these symbols are written together shall represent all the objects or

This says
$$xyz\cdots := x \cap y \cap z \cap \cdots,$$
and any rearrangement of the letters gives the same result.

individuals to which their several meanings are together applicable, and to the law that the order in which the symbols succeed each other is indifferent.

As the rule of interpretation has been sufficiently exemplified, I shall deem it unnecessary always to express the subject "things" in defining the interpretation of a symbol used for an adjective. When I say, let x represent "good," it will be understood that x only represents "good" when a subject for that quality is supplied by another symbol, and that, used alone, its interpretation will be "good things."

8. Concerning the law above determined, the following observations, which will also be more or less appropriate to certain other laws to be deduced hereafter, may be added.

First, I would remark, that this law is a law of thought, and not, properly speaking, a law of things. Difference in the order of the qualities or attributes of an object, apart from all questions of causation, is a difference in conception merely. The law (1) expresses as a general truth, that the same thing may be conceived in different ways, and states the nature of that difference; and it does no more than this.

Secondly, As a law of thought, it is actually developed in a law of Language, the product and the instrument of thought. Though the tendency of prose writing is toward uniformity, yet even there the order of sequence of adjectives absolute in their meaning, and applied to the same subject, is indifferent, but poetic diction borrows much of its rich diversity from the extension of the same lawful freedom to the substantive also. The language of Milton is peculiarly distinguished by this species of variety. Not only does the substantive often precede the adjectives by which it is qualified, but it is frequently placed in their midst. In the first few lines of the invocation to Light, we meet with such examples as the following:

> "*Offspring of heaven first-born.*"
> "The rising world of *waters dark and deep.*"
> "Bright effluence of *bright essence increate.*"

Now these inverted forms are not simply the fruits of a poetic license. They are the natural expressions of a freedom sanctioned by the intimate laws of thought, but for reasons of convenience not exercised in the ordinary use of language.

Thirdly, The law expressed by (1) may be characterized by saying that the literal symbols x, y, z, are *commutative, like the symbols of Algebra*. In saying this, it is not affirmed that the process of multiplication in Algebra, of which the fundamental law is expressed by the equation

$$xy = yx,$$

possesses in itself any analogy with that process of logical combination which xy has been made to represent above; but only that if the arithmetical and the logical are expressed in the same manner, their symbolical expressions will be subject to the same formal law. The evidence of that subjection is in the two cases quite distinct.

9. As the combination of two literal symbols in the form xy expresses the whole of that class of objects to which the names or qualities represented by x and y are together applicable, it follows that if the two symbols have exactly the same signification, their combination expresses no more than either of the symbols taken alone would do. In such case we should therefore have

$$xy = x.$$

As y is, however, supposed to have the same meaning as x, we may replace it in the above equation by x, and we thus get

$$xx = x.$$

Now in common Algebra the combination xx is more briefly represented by x^2. Let us adopt the same principle of notation here; for the mode of expressing a particular succession of mental operations is a thing in itself quite as arbitrary as the mode of expressing a single idea or operation (II. 3). In accordance with this notation, then, the above equation assumes the form

$$\boxed{x^2 = x,} \qquad (2)$$

and is, in fact, the expression of a second general law of those symbols by which names, qualities, or descriptions, are symbolically represented.

Multiplication is IDEMPOTENT.
One eventually realizes this law only applies to variables—see, e.g., pp. 55, 56. There are non-variable idempotent algebraic terms, but they are not covered by this law—one needs to prove that they are idempotent. (Note: Boole did not use the word *idempotent*.)

The reader must bear in mind that although the symbols x and y in the examples previously formed received significations distinct from each other, nothing prevents us from attributing to them precisely the same signification. It is evident that the more nearly their actual significations approach to each other, the more nearly does the class of things denoted by the combination xy approach to identity with the class denoted by x, as well as with that denoted by y. The case supposed in the demonstration of the equation (2) is that of *absolute* identity of meaning. The law which it expresses is practically exemplified in language. To say "good, good," in relation to any subject, though a cumbrous and useless pleonasm, is the same as to say "good." Thus "good, good" men, is equivalent to "good" men. Such repetitions of words are indeed sometimes employed to heighten a quality or strengthen an affirmation. But this effect is merely secondary and conventional; it is not founded in the intrinsic relations of language and thought. Most of the operations which we observe in nature, or perform ourselves, are of such a kind that their effect is augmented by repetition, and this circumstance prepares us to expect the same thing in language, and even to use repetition when we design to speak with emphasis. But neither in strict reasoning nor in exact discourse is there any just ground for such a practice.

It was easier to justify the idempotent law in MAL when Boole used selection operators

$$\sigma_X : Y \mapsto X \cap Y$$

as the elements of his algebra of logic, and multiplication was composition: clearly

$$\sigma_X \circ \sigma_X(Y) = \sigma_X(Y).$$

10. We pass now to the consideration of another class of the signs of speech, and of the laws connected with their use.

CLASS II.

11. *Signs of those mental operations whereby we collect parts into a whole, or separate a whole into its parts.*

Class II introduced the operations of addition (+) and subtraction (−) into Boole's algebra of logic as partially defined operations.

We are not only capable of entertaining the conceptions of objects, as characterized by names, qualities, or circumstances, applicable to each individual of the group under consideration, but also of forming the aggregate conception of a group of objects consisting of partial groups, each of which is separately named or described. For this purpose we use the conjunctions "and," "or," &c. "Trees and minerals," "barren mountains, or fertile vales," are examples of this kind. In strictness, the words "and," "or," interposed between the terms descriptive of two or more classes of objects, imply that those classes are quite distinct, so that no member of one is found in another. In this and in all other respects the words "and" "or" are analogous with the sign + in algebra, and their laws are identical. Thus the expression "men and women" is, conventional meanings set aside, equivalent with the expression "women and men." Let x represent "men," y, "women;" and let + stand for "*and*" and "*or*," then we have

$$\boxed{x + y \;=\; y + x,} \tag{3}$$

an equation which would equally hold true if x and y represented *numbers*, and + were the sign of arithmetical addition.

Let the symbol z stand for the adjective "European," then since it is, in effect, the same thing to say "European men and women," as to say "European men and European women," we have

$$\boxed{z\,(x + y) \;=\; zx + zy.} \tag{4}$$

And this equation also would be equally true were x, y, and z symbols of number, and were the juxtaposition of two literal symbols to represent their algebraic product, just as in the logical signification previously given, it represents the class of objects to which both the epithets conjoined belong.

The above are the laws which govern the use of the sign +, here used to denote the positive operation of aggregating parts into a whole. But the very idea of an operation effecting some positive change seems to suggest to us the idea of an opposite or negative operation, having the effect of undoing what the former one has done. Thus we cannot conceive it possible to collect parts into a whole, and not conceive it also possible to separate a part from a whole. This operation we express in common language by the sign *except*, as, "All men *except* Asiatics," "All states *except* those which are monarchical." Here it is implied that the things excepted form a part of the things from which they are excepted. As we have expressed the operation of aggregation by the sign +, so we may express the negative operation above described by − minus. Thus if x be taken to represent men, and y, Asiatics, i.e. Asiatic men,

A hint that $x + y$ was only defined for disjoint classes x and y. For a clearer statement of this restriction, see page 66.

Addition is COMMUTATIVE.
One eventually realizes that numerical laws, like this one, viewed as algebra of logic laws, are *not* restricted to variables.

Multiplication is DISTRIBUTIVE over addition.

Margin Notes Terminology:
An equation $s(\vec{x}) = t(\vec{x})$ will be said to **hold** in Boole's partial algebras if it holds whenever both sides are defined.

SUBTRACTION:
For $y \subseteq x$, let $x - y := x \setminus y$. Boole did not give the operation "−" a name in his algebra of logic; the word 'minus' used here never appears again in his chapters on the algebra of logic. He once mentioned 'subtract' (p. 36) and once 'subtracting' (p. 91), referring to the operation being applied to equations in the algebra of logic.

then the conception of "All men except Asiatics" will be expressed by $x - y$. And if we represent by x, "states," and by y the descriptive property "having a monarchical form," then the conception of "All states except those which are monarchical" will be expressed by $x - xy$.

As it is indifferent for all the *essential* purposes of reasoning whether we express excepted cases first or last in the order of speech, it is also indifferent in what order we write any series of terms, some of which are affected by the sign $-$. Thus we have, as in the common algebra,

$$\boxed{x - y \;=\; -y + x.} \tag{5}$$

Still representing by x the class "men," and by y "Asiatics," let z represent the adjective "white." Now to apply the adjective "white" to the collection of men expressed by the phrase "Men except Asiatics," is the same as to say, "White men, except white Asiatics." Hence we have

$$\boxed{z\,(x - y) \;=\; zx - zy.} \tag{6}$$

This is also in accordance with the laws of ordinary algebra.

The equations (4) and (6) may be considered as exemplification of a single general law, which may be stated by saying, *that the literal symbols, x, y, z, &c. are distributive in their operation.* The general fact which that law expresses is this, viz.:—If any quality or circumstance is ascribed to all the members of a group, formed either by aggregation or exclusion of partial groups, the resulting conception is the same as if the quality or circumstance were first ascribed to each member of the partial groups, and the aggregation or exclusion effected afterwards. That which is ascribed to the members of the whole is ascribed to the members of all its parts, howsoever those parts are connected together.

CLASS III.

12. *Signs by which relation is expressed, and by which we form propositions.*

Though all verbs may with propriety be referred to this class, it is sufficient for the purposes of Logic to consider it as including only the substantive verb *is* or *are*, since every other verb

Equation (5) is dubious way of introducing the unary minus. Better: $-x := 0 - x$, as far as the algebra goes, but not so good for the semantics since $0 - x$ is only defined for $x = 0$.

Multiplication is
DISTRIBUTIVE
over subtraction.

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(6) is the last law that Boole explicitly stated. Missing are the associative laws for $\cdot$ and $+$ (and some other laws).

Boole did not view an algebraic term as a *string of symbols*, like $(x + (y - z)) \cdot ((y \cdot w) \cdot z)$. Instead Boole used standard abbreviated expressions like $(x + y - z)ywz$.

Note: the domain of definition of $x + y - z$ depends on where parentheses are inserted: does this represent $(x + y) - z$ or $x + (y - z)$?

Modern algebraic terms are recursively defined, making it possible to give a recursive definition of the domain of definition (i.e., of interpretation).

Class III is $\{=\}$. This is based on noting that the only verb needed in primary propositions is "is".

may be resolved into this element, and one of the signs included under Class I. For as those signs are used to express quality or circumstance of every kind, they may be employed to express the active or passive relation of the subject of the verb, considered with reference either to past, to present, or to future time. Thus the Proposition, "Cæsar conquered the Gauls," may be resolved into "Cæsar is he who conquered the Gauls." The ground of this analysis I conceive to be the following:—Unless we understand what is meant by having conquered the Gauls, i.e. by the expression "One who conquered the Gauls," we cannot understand the sentence in question. It is, therefore, truly an element of that sentence; another element is "Cæsar," and there is yet another required, the copula *is*, to show the connexion of these two. I do not, however, affirm that there is no other mode than the above of contemplating the relation expressed by the proposition, "Cæsar conquered the Gauls;" but only that the analysis here given is a correct one for the particular point of view which has been taken, and that it suffices for the purposes of logical deduction. It may be remarked that the passive and future participles of the Greek language imply the existence of the principle which has been asserted, viz.: that the sign *is* or *are* may be regarded as an element of every personal verb.

This is an example of how any verb connecting two classes can be converted to "is".

13. The above sign, *is* or *are* may be expressed by the symbol =. The laws, or as would usually be said, the axioms which the symbol introduces, are next to be considered.

The verb 'is' will be translated as '='. Boole's "axioms" for '=' would, in equational logic, now be called 'rules of inference'.

Let us take the Proposition, "The stars are the suns and the planets," and let us represent stars by x, suns by y, and planets by z; we have then

$$x = y + z. \quad (7)$$

Now if it be true that the stars are the suns and the planets, it will follow that the stars, except the planets, are suns. This would give the equation

$$x - z = y, \quad (8)$$

which must therefore be a deduction from (7). Thus a term z has been removed from one side of an equation to the other by

The algebraic rule of TRANSPOSITION holds: $x = y + z$ iff $x - z = y$.

changing its sign. This is in accordance with the algebraic rule of transposition.

But instead of dwelling upon particular cases, we may at once affirm the general axioms :—

1st. If equal things are added to equal things, the wholes are equal.

2nd. If equal things are taken from equal things, the remainders are equal.

And it hence appears that we may add or subtract equations, and employ the rule of transposition above given just as in common algebra.

Again: If two classes of things, x and y, be identical, that is, if all the members of the one are members of the other, then those members of the one class which possess a given property z will be identical with those members of the other which possess the same property z. Hence if we have the equation

$$x = y;$$

then whatever class or property z may represent, we have also

$$zx = zy.$$

This is formally the same as the algebraic law:—If both members of an equation are multiplied by the same quantity, the products are equal.

In like manner it may be shown that if the corresponding members of two equations are multiplied together, the resulting equation is true.

14. Here, however, the analogy of the present system with that of algebra, as commonly stated, appears to stop. Suppose it true that those members of a class x which possess a certain property z are identical with those members of a class y which possess the same property z, it does not follow that the members of the class x universally are identical with the members of the class y. Hence it cannot be inferred from the equation

$$zx = zy,$$

that the equation

$$x = y$$

is also true. In other words, the axiom of algebraists, that both

On this page:
AXIOMS = Rules of Inference.

First two axioms are:
Equals ± Equals gives Equals.

Third axiom:
Equals × Equals gives Equals

Note: Axioms not stated:
Equals is reflexive, symmetric and transitive.

The 'cancellation law' from the algebra of numbers is
$zx = zy \therefore z = 0$ or $x = y$.
This is not an equational argument; disjunctions of equations are not equations.

Another valid argument from the algebra of numbers that does not carry over to the algebra of logic is:
$x^2 = x \therefore x = 0$ or $x = 1$.
It is also not an equational argument.

sides of an equation may be divided by the same quantity, has no formal equivalent here. I say no *formal equivalent*, because, in accordance with the general spirit of these inquiries, it is not even sought to determine whether the mental operation which is represented by removing a logical symbol, z, from a combination zx, is in itself analogous with the operation of division in Arithmetic. That mental operation is indeed identical with what is commonly termed Abstraction, and it will hereafter appear that its laws are dependent upon the laws already deduced in this chapter. What has now been shown is, that there does not exist among those laws anything analogous in *form* with a commonly received axiom of Algebra.

But a little consideration will show that even in common algebra that axiom does not possess the generality of those other axioms which have been considered. The deduction of the equation $x = y$ from the equation $zx = zy$ is only valid when it is known that z is not equal to 0. If then the value $z = 0$ is supposed to be admissible in the algebraic system, the axiom above stated ceases to be applicable, and the analogy before exemplified remains at least unbroken.

15. However, it is not with the symbols of quantity generally that it is of any importance, except as a matter of speculation, to trace such affinities. We have seen (II. 9) that the symbols of Logic are subject to the special law,

$$x^2 = x.$$

Now of the symbols of Number there are but two, viz. 0 and 1, which are subject to the same formal law. We know that $0^2 = 0$, and that $1^2 = 1$; and the equation $x^2 = x$, considered as algebraic, has no other roots than 0 and 1. Hence, instead of determining the measure of formal agreement of the symbols of Logic with those of Number generally, it is more immediately suggested to us to compare them with symbols of quantity *admitting only of the values* 0 *and* 1. Let us conceive, then, of an Algebra in which the symbols x, y, z, etc. admit indifferently of the values 0 and 1, and of these values alone. The laws, the axioms, and the processes, of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra

Boole did not explicitly state (or realize?) that his algebra of logic was an *equational logic.* Arguments in equational logic have the form

$$E_1, \ldots, E_k \;\therefore E$$

where the E_i and E are equations.

The last paragraph below has Boole's brilliant observation, the **Rule of 0 and 1**, called simply **R01**. For "laws" read 'equations', for "axioms" read 'rules of inference', and for "processes" read 'equational arguments'.

R01 says that an equational argument

$$E_1, \ldots, E_k \;\therefore E$$

is **valid** in Boole's algebra of logic iff it holds in $\mathbb{Z}_{01}$, that is, in the algebra of the integers $\mathbb{Z}$ when the variables are restricted to the values 0 and 1.

Thus *every* equational argument that holds in the algebra of numbers is valid in Boole's algebra of logic, and holds in Boole's partial algebras.

In the modern terminology of Universal Algebra, **R01** says: a quasi-equation $q(\mathbf{x})$ is **valid** in Boole's algebra iff $\mathbb{Z} \models_{01} q(\mathbf{x})$, that is, $q(\mathbf{x})$ holds in $\mathbb{Z}_{01}$.

These quasi-equations are precisely the ones that hold in commutative rings with unity, and without additive torsion, when the variables are restricted to idempotent elements of the rings.

of Logic. Difference of interpretation will alone divide them. Upon this principle the method of the following work is established.

16. It now remains to show that those constituent parts of ordinary language which have not been considered in the previous sections of this chapter are either resolvable into the same elements as those which have been considered, or are subsidiary to those elements by contributing to their more precise definition.

The substantive, the adjective, and the verb, together with the particles *and*, *except*, we have already considered. The pronoun may be regarded as a particular form of the substantive or the adjective. The adverb modifies the meaning of the verb, but does not affect its nature. Prepositions contribute to the expression of circumstance or relation, and thus tend to give precision and detail to the meaning of the literal symbols. The conjunctions *if*, *either*, *or*, are used chiefly in the expression of relation among propositions, and it will hereafter be shown that the same relations can be completely expressed by elementary symbols analogous in interpretation, and identical in form and law with the symbols whose use and meaning have been explained in this Chapter. As to any remaining elements of speech, it will, upon examination, be found that they are used either to give a more definite significance to the terms of discourse, and thus enter into the interpretation of the literal symbols already considered, or to express some emotion or state of feeling accompanying the utterance of a proposition, and thus do not belong to the province of the understanding, with which alone our present concern lies. Experience of its use will testify to the sufficiency of the classification which has been adopted.

R01 was the very foundation of Boole's algebra of logic.

To see how **R01** was frequently used see pages 70, 73, 80.

Boole would have made this foundation much clearer, and received credit for a truth-table like method, if he had given some detailed simple examples. For example, to verify the equation $x + y = y + x$:

x	y	$x+y$	$y+x$	$x+y=y+x$
1	1	2	2	T
1	0	1	1	T
0	1	1	1	T
0	0	1	1	T

And to reject $(x + y)^2 = x + y$:

x	y	$x+y$	$(x+y)^2$	$(x+y)^2=x+y$
1	1	2	4	F
1	0	1	1	T
0	1	1	1	T
0	0	0	0	T

A possible source of confusion for those accustomed to working with total algebras is that an equation, or equational argument, can **hold** in Boole's partial algebras, but not be **valid** in Boole's algebra of logic. Boole did not comment on this distinction.

A good example for this is

$$(x + y)^2 = x + y,$$

which holds (whenever both sides are defined) in Boole's partial algebras, but it is not valid in Boole's algebra of logic (see the last table above).

Boole did not have the concept of *term* as it is used in modern universal algebra and logic. Such terms are defined recursively—here is the definition for Boole's algebra of logic:

- variables are terms
- 0, 1 are terms
- if s and t are terms then so are $(s+t)$, $(s-t)$ and $(s \cdot t)$.

In the marginal notes these will be called *algebraic terms*.

Although Boole started his algebra of logic by defining his operations and by introducing several laws and rules of inference for his algebra of logic, it was really his Rule of 0 and 1 that justified most of his results. Unfortunately he did not explain this Rule very well, nor did he give it sufficient emphasis, with the result that it took nearly 150 years before it was deciphered and its central role understood.

The Rule of 0 and 1 simply says that a (polynomial) equation or equational argument φ is **valid** (i.e., accepted) in his algebra of logic iff it holds in the integers when the variables were restricted to the values 0 and 1. We express this condition by saying φ **holds** in $\mathbb{Z}_{01}$, or $\mathbb{Z} \models_{01} \varphi$. Given a φ that is constructed from algebraic terms, we will say that φ **holds** in Boole's algebraic structures (they are partial algebras) if φ is true whenever the variables take on values for which all the terms in φ are defined. We say simply: φ is true when defined.

Boole works with polynomials with integer coefficients except in his method of solving a polynomial equation $f(\vec{x}, w) = 0$ for w. He rewrites this as a polynomial equation $q(\vec{x})w = p(\vec{x})$, applies formal division to obtain $w = p(\vec{x})/q(\vec{x})$, applies a formal expansion to the right side, and then shows how it is to be interpreted. His work would have been much clearer if he had put all details of his method to find w, in particular the use of rational functions, in a separate section. As it stands, it is often not obvious if Boole is talking about rational functions, or just polynomial functions; quite often it is the latter.

Chapter III

Derivation of the Laws of the Symbols of Logic from the Laws of the Operations of the Human Mind.

1. The object of science, properly so called, is the knowledge of laws and relations. To be able to distinguish what is essential to this end, from what is only accidentally associated with it, is one of the most important conditions of scientific progress. I say, to *distinguish* between these elements, because a consistent devotion to science does not require that the attention should be altogether withdrawn from other speculations, often of a metaphysical nature, with which it is not unfrequently connected. Such questions, for instance, as the existence of a sustaining ground of phænomena, the reality of cause, the propriety of forms of speech implying that the successive states of things are connected by *operations*, and others of a like nature, may possess a deep interest and significance in relation to science, without being essentially scientific. It is indeed scarcely possible to express the conclusions of natural science without borrowing the language of these conceptions. Nor is there necessarily any practical inconvenience arising from this source. They who believe, and they who refuse to believe, that there is more in the relation of cause and effect than an invariable order of succession, agree in their interpretation of the conclusions of physical astronomy. But they only agree because they recognise a common element of scientific truth, which is independent of their particular views of the nature of causation.

Logic texts before Boole, and for at least 50 years after his work, considered analyzing the workings of the mind, when thinking correctly, to be one of their objectives. This has not led to any great insights, only terminology and discussion. There is little of value in the first eleven sections of this chapter.

In sections 12–14, pages 46–48, Boole introduced 0 and 1 as names of classes, and noted that $1 - x$ is the complement of x. This material belongs in Chapter II, before section 15, so that the Rule of 0 and 1 has the desired scope..

2. If this distinction is important in physical science, much more does it deserve attention in connexion with the science of the intellectual powers. For the questions which this science presents become, in expression at least, almost necessarily mixed up with modes of thought and language, which betray a metaphysical origin. The idealist would give to the laws of reasoning one form of expression; the sceptic, if true to his principles, another. They who regard the phænomena with which we are concerned in this inquiry as the mere successive *states* of the thinking subject devoid of any causal connexion, and they who refer them to the *operations* of an active intelligence, would, if consistent, equally differ in their modes of statement. Like difference would also result from a difference of classification of the mental faculties. Now the principle which I would here assert, as affording us the only ground of confidence and stability amid so much of seeming and of real diversity, is the following, viz., that if the laws in question are really deduced from observation, they have a real existence as laws of the human mind, independently of any metaphysical theory which may seem to be involved in the mode of their statement. They contain an element of truth which no ulterior criticism upon the nature, or even upon the reality, of the mind's operations, can essentially affect. Let it even be granted that the mind is but a succession of states of consciousness, a series of fleeting impressions uncaused from without or from within, emerging out of nothing, and returning into nothing again,—the last refinement of the sceptic intellect,—still, as laws of succession, or at least of a past succession, the results to which observation had led would remain true. They would require to be interpreted into a language from whose vocabulary all such terms as cause and effect, operation and subject, substance and attribute, had been banished; but they would still be valid as scientific truths.

Moreover, as any statement of the laws of thought, founded upon actual observation, must thus contain scientific elements which are independent of metaphysical theories of the nature of the mind, the practical application of such elements to the construction of a system or method of reasoning must also be independent of metaphysical distinctions. For it is upon the scientific elements involved in the statement of the laws, that any practical application will rest, just as the practical conclusions of physical astronomy are independent of any theory of the cause of gravitation, but rest only on the knowledge of its phænomenal effects. And, therefore, as respects both the determination of the laws of thought, and the practical use of them when discovered, we are, for all really scientific ends, unconcerned with the truth or falsehood of any metaphysical speculations whatever.

3. The course which it appears to me to be expedient, under these circumstances, to adopt, is to avail myself as far as possible of the language of common discourse, without regard to any theory of the nature and powers of the mind which it may be thought to embody. For instance, it is agreeable to common usage to say that we converse with each other by the communication of ideas, or conceptions, such communication being the office of words; and that with reference to any particular ideas or conceptions presented to it, the mind possesses certain powers or faculties by which the mental regard maybe fixed upon some ideas, to the exclusion of others, or by which the given conceptions or ideas may, in various ways, be combined together. To those faculties or powers different names, as Attention, Simple Apprehension, Conception or Imagination, Abstraction, &c., have been given,—names which have not only furnished the titles of distinct divisions of the philosophy of the human mind, but passed into the common language of men. Whenever, then, occasion shall occur to use these terms, I shall do so without implying thereby that I accept the theory that the mind possesses such and such powers and faculties as distinct elements of its activity. Nor is it indeed necessary to inquire whether such powers of the understanding have a distinct existence or not. We may merge these different titles under the one generic name of *Operations* of the human mind, define these operations so far as is necessary for the purposes of this work, and then seek to express their ultimate laws. Such will be the general order of the course which I shall pursue, though reference will occasionally be made to the names which common agreement has assigned to the particular states or operations of the mind which may fall under our notice.

It will be most convenient to distribute the more definite results of the following investigation into distinct Propositions.

Proposition I.

Boole often used the heading 'Proposition' for 'Proposal' or 'Project'

4. *To deduce the laws of the symbols of Logic from a consideration of those operations of the mind which are implied in the strict use of language as an instrument of reasoning.*

In every discourse, whether of the mind conversing with its own thoughts, or of the individual in his intercourse with others, there is an assumed or expressed limit within which the subjects of its operation are confined. The most unfettered discourse is that in which the words we use are understood in the widest possible application, and for them the limits of discourse are co-extensive with those of the universe itself. But more usually we confine ourselves to a less spacious field. Sometimes, in discoursing of men we imply (without expressing the limitation) that it is of men only under certain circumstances and conditions that we speak, as of civilized men, or of men in the vigour of life, or of men under some other condition or relation. Now, whatever may be the extent of the field within which all the objects of our discourse are found, that field may properly be termed the universe of discourse.

5. Furthermore, this universe of discourse is in the strictest sense the ultimate *subject* of the discourse. The office of any name or descriptive term employed under the limitations supposed is not to raise in the mind the conception of all the beings or objects to which that name or description is applicable, but only of those which exist within the supposed universe of discourse. If that universe of discourse is the actual universe of things, which it always is when our words are taken in their real and literal sense, then by men we mean *all men that exist*; but if the universe of discourse is limited by any antecedent implied understanding, then it is of men under the limitation thus introduced that we speak. It is in both cases the business of the word *men* to direct a certain operation of the mind, by which, from the proper universe of discourse, we select or fix upon the individuals signified.

6. Exactly of the same kind is the mental operation implied by the use of an adjective. Let, for instance, the universe of discourse be the actual Universe. Then, as the word *men* directs us to select mentally from that Universe all the beings to which the term "men" is applicable; so the adjective "good," in the combination "good men," directs us still further to select mentally from the class of *men* all those who possess the further quality "good;" and if another adjective were prefixed to the combination "good men," it would direct a further operation of the same nature, having reference to that further quality which it might be chosen to express.

It is important to notice carefully the real nature of the operation here described, for it is conceivable, that it might have been different from what it is. Were the adjective simply *attributive* in its character, it would seem, that when a particular set of beings is designated by *men*, the prefixing of the adjective *good* would direct us to attach mentally to all those beings the quality of goodness. But this is not the real office of the adjective. The operation which we really perform is one of *selection according to a prescribed principle or idea.* To what faculties of the mind such an operation would be referred, according to the received classification of its powers, it is not important to inquire, but I suppose that it would be considered as dependent upon the two faculties of Conception or Imagination, and Attention. To the one of these faculties might be referred the formation of the general conception; to the other the fixing of the mental regard upon those individuals within the prescribed universe of discourse which answer to the conception. If, however, as seems not improbable, the power of Attention is nothing more than the power of continuing the exercise of any other faculty of the mind, we might properly regard the whole of the mental process above described as referrible to the mental faculty of Imagination or Conception, the first step of the process being the conception of the Universe itself, and each succeeding step limiting in a definite manner the conception thus formed. Adopting this view, I shall describe each such step, or any definite combination of such steps, as a *definite act of conception.* And the use of this term I shall extend so as to include in its meaning not only the conception of classes of objects represented by particular names or simple attributes of quality, but also the combination of such conceptions in any manner consistent with the powers and limitations

This goes back to Boole's selection operators in MAL.

of the human mind; indeed, any intellectual operation short of that which is involved in the structure of a sentence or proposition. The general laws to which such operations of the mind are subject are now to be considered.

7. Now it will be shown that the laws which in the preceding chapter have been determined *à posteriori* from the constitution of language, for the use of the literal symbols of Logic, are in reality the laws of that definite mental operation which has just been described. We commence our discourse with a certain understanding as to the limits of its subject, i.e. as to the limits of its Universe. Every name, every term of description that we employ, directs him whom we address to the performance of a certain mental operation upon that subject. And thus is thought communicated. But as each name or descriptive term is in this view but the representative of an intellectual operation, that operation being also prior in the order of nature, it is clear that the laws of the name or symbol must be of a derivative character,—must, in fact, originate in those of the operation which they represent. That the laws of the symbol and of the mental process are identical in expression will now be shown.

The laws of logic, determined from observing language use, were viewed as revealing laws of the operations of the mind.

8. Let us then suppose that the universe of our discourse is the actual universe, so that words are to be used in the full extent of their meaning, and let us consider the two mental operations implied by the words "white" and "men." The word "men" implies the operation of selecting in thought from its subject, the universe, all men; and the resulting conception, *men*, becomes the subject of the next operation. The operation implied by the word "white" is that of selecting from its subject, "men," all of that class which are white. The final resulting conception is that of "white men." Now it is perfectly apparent that if the operations above described had been performed in a converse order, the result would have been the same. Whether we begin by forming the conception of "*men*," and then by a second intellectual act limit that conception to "white men," or whether we begin by forming the conception of "white objects," and then limit it to such of that class as are "men," is perfectly indifferent so far as the result is concerned. It is obvious that the order of the mental processes would be equally

In LT Boole only used the 'actual universe'. (A concept that later led to difficulties in Cantor's set theory.) A 'limited universe' was introduced by De Morgan in his 1847 book *Formal Logic.* In 1854 Boole introduced the phrase 'universe of discourse' to describe either a limited universe or the actual universe.

indifferent if for the words "white" and "men" we substituted any other descriptive or appellative terms whatever, provided only that their meaning was fixed and absolute. And thus the indifference of the order of two successive acts of the faculty of Conception, the one of which furnishes the subject upon which the other is supposed to operate, is a general condition of the exercise of that faculty. It is a law of the mind, and it is the real origin of that law of the literal symbols of Logic which constitutes its formal expression (1) Chap. II.

9. It is equally clear that the mental operation above described is of such a nature that its effect is not altered by repetition. Suppose that by a definite act of conception the attention has been fixed upon men, and that by another exercise of the same faculty we limit it to those of the race who are white. Then any further repetition of the latter mental act, by which the attention is limited to white objects, does not in any way modify the conception arrived at, viz., that of white men. This is also an example of a general law of the mind, and it has its formal expression in the law ((2) Chap, II.) of the literal symbols.

10 . Again, it is manifest that from the conceptions of two distinct classes of things we can form the conception of that collection of things which the two classes taken together compose; and it is obviously indifferent in what order of position or of priority those classes are presented to the mental view. This is another general law of the mind, and its expression is found in (3) Chap. II.

11. It is not necessary to pursue this course of inquiry and comparison. Sufficient illustration has been given to render manifest the two following positions, viz.:

First, That the operations of the mind, by which, in the exercise of its power of imagination or conception, it combines and modifies the simple ideas of things or qualities, not less than those operations of the reason which are exercised upon truths and propositions, are subject to general laws.

Secondly, That those laws are mathematical in their form, and that they are actually developed in the essential laws of human language. Wherefore the laws of the symbols of Logic are deducible from a consideration of the operations of the mind in reasoning.

12. The remainder of this chapter will be occupied with questions relating to that law of thought whose expression is $x^2 = x$ (II. 9), a law which, as has been implied (II. 15), forms the characteristic distinction of the operations of the mind in its ordinary discourse and reasoning, as compared with its operations when occupied with the general algebra of quantity. An important part of the following inquiry will consist in proving that the symbols 0 and 1 occupy a place, and are susceptible of an interpretation, among the symbols of Logic; and it may first be necessary to show how particular symbols, such as the above, may with propriety and advantage be employed in the representation of distinct systems of thought.

II.15 is the section on **R01**.

Boole did not include 0 ,1 in his original symbolic language. He 'proved' that they were the only numbers n which could be used as names of classes, and that they would name unique classes.

His proof depended on assuming that the algebra of numbers laws satisfied by the numbers n would hold in the algebra of logic, and vice-versa.

The ground of this propriety cannot consist in any community of interpretation. For in systems of thought so truly distinct as those of Logic and Arithmetic (I use the latter term in its widest sense as the science of Number), there is, properly speaking, no community of subject. The one of them is conversant with the very conceptions of things, the other takes account solely of their numerical relations. But inasmuch as the forms and methods of any system of reasoning depend immediately upon the laws to which the symbols are subject, and only mediately, through the above link of connexion, upon their interpretation, there may be both propriety and advantage in employing the same symbols in different systems of thought, provided that such interpretations can be assigned to them as shall render their formal laws identical, and their use consistent. The ground of that employment will not then be community of interpretation, but the community of the formal laws, to which in their respective systems they are subject. Nor must that community of formal laws be established upon any other ground than that of a careful observation and comparison of those results which are seen to flow independently from the interpretations of the systems under consideration.

Boole failed to mention that the real advantage of using the algebra of numbers, at which he was an expert, was the facility it would give him in trying out ideas, examples, etc.

These observations will explain the process of inquiry adopted in the following Proposition. The literal symbols of Logic are universally subject to the law whose expression is $x^2 = x$. Of the symbols of Number there are two only, 0 and 1, which satisfy this law. But each of these symbols is also subject to a law peculiar to itself in the system of numerical magnitude, and this suggests the inquiry, what interpretations must be given to the literal symbols of Logic, in order that the same peculiar and formal laws may be realized in the logical system also.

Boole said that at most two numbers qualified to be used as names of classes, namely 0 and 1, because they must satisfy $x^2 = x$. Then he proceeded, using the two numerical laws below, to show each of these numbers could only be used to name a certain class, namely 0 must designate the empty class and 1 the universe. (The second equation below deserves to be numbered as equation (2).)

Proposition II

13. *To determine the logical value and significance of the symbols 0 and 1.*

The symbol 0, as used in Algebra, satisfies the following formal law,

$$0 \times y = 0, \qquad \text{or} \qquad 0y = 0, \tag{1}$$

whatever *number* y may represent. That this formal law may be obeyed in the system of Logic, we must assign to the symbol 0 such an interpretation that the class represented by $0y$ may be identical with the class represented by 0, whatever the class y may be. A little consideration will show that this condition is satisfied if the symbol 0 represent Nothing. In accordance with a previous definition, we may term Nothing a class. In fact, Nothing and Universe are the two limits of class extension, for they are the limits of the possible interpretations of general names, none of which can relate to fewer individuals than are comprised in Nothing, or to more than are comprised in the Universe. Now whatever the class y may be, the individuals which are common to it and to the class "Nothing" are identical with those comprised in the class "Nothing," for they are none. And thus by assigning to 0 the interpretation Nothing, the law (1) is satisfied; and it is not otherwise satisfied consistently with the perfectly general character of the class y.

0 must denote Nothing.
Note: Boole did not use the phrase 'empty class'.

Secondly, The symbol 1 satisfies in the system of Number the following law, viz.,

$$1 \times y = y, \qquad \text{or} \qquad 1y = y,$$

whatever number y may represent. And this formal equation being assumed as equally valid in the system of this work, in which 1 and y represent classes, it appears that the symbol 1 must represent such a class that all the individuals which are found in *any* proposed class y are also all the individuals $1y$ that are common to that class y and the class represented by 1. A little consideration will here show that the class represented by 1 must be "the Universe," since this is the only class in which are found *all* the individuals that exist in *any* class. Hence the respective interpretations of the symbols 0 and 1 in the system of Logic are *Nothing* and *Universe*.

1 must denote the Universe. In modern times one also likes to use letters like U to denote the Universe, but the only symbol Boole used was 1.

14. As with the idea of any class of objects as "men," there is suggested to the mind the idea of the contrary class of beings which are not men; and as the whole Universe is made up of these two classes together, since of every individual which it comprehends we may affirm either that it is a man, or that it is not a man, it becomes important to inquire how such contrary names are to be expressed. Such is the object of the following Proposition.

Proposition III.

If x represent any class of objects, then will $1-x$ represent the contrary or supplementary class of objects., i.e. *the class including all objects which are not comprehended in the class x.*

$1-x$ is the complement of x.

For greater distinctness of conception let x represent the class men, and let us express, according to the last Proposition, the Universe by 1; now if from the conception of the Universe, as consisting of "men" and "not-men," we exclude the conception of "men," the resulting conception is that of the contrary class, "not-men." Hence the class "not-men" will be represented by $1-x$. And, in general, whatever class of objects is represented by the symbol x, the contrary class will be expressed by $1-x$.

15. Although the following Proposition belongs in strictness to a future chapter of this work, devoted to the subject of *maxims* or *necessary truths*, yet, on account of the great importance of that law of thought to which it relates, it has been thought proper to introduce it here.

Proposition IV.

That axiom of metaphysicians which is termed the principle of contradiction, *and which affirms that it is impossible for any being to possess a quality, and at the same time not to possess it, is a consequence of the fundamental law of thought, whose expression is* $x^2 = x$.

Boole regarded $x^2 = x$ as being more fundamental than Aristotle's 'fundamental axiom', the principle of contradiction, discussed below.

Let us write this equation in the form

$$x - x^2 = 0,$$

whence we have

$$x(1 - x) = 0; \tag{1}$$

both these transformations being justified by the axiomatic laws of combination and transposition (II. 13). Let us, for simplicity of conception, give to the symbol x the particular interpretation of *men*, then $1-x$ will represent the class: of "not-men" (Prop. III.) Now the formal product of the expressions of two classes represents that class of individuals which is common to them both (II. 6). Hence $x(1 - x)$ will represent the class whose members are at once "men," and "not men," and the equation (1) thus express the principle, *that a class whose members are at the same time men and not men does not exist.* In other words, that *it is impossible for the same individual to be at the same time a man and not a man.* Now let the meaning of the symbol x be extended from the representing of "men," to that of any class of beings characterized by the possession of any quality whatever; and the equation (1) will then express that it is impossible for a being to possess a quality and not to possess that quality at the same time. But this is identically that "principle of contradiction" which Aristotle has described as the fundamental axiom of all philosophy. "It is impossible that the same quality should both belong and not belong to the same thing.... This is the most certain of all principles.... Wherefore they who demonstrate refer to this as an ultimate opinion. For it is by nature the source of all the other axioms."[1]

Two alternate forms of $x^2 = x$ are given.
Note: Boole avoided the form $x^2 - x = 0$, which factors as $x(x-1) = 0$, perhaps because $x - 1$ is only defined for $x = 1$.
Presumably axiomatic laws are fundamental laws and axioms.

The first time Boole mentioned the 'product' of (the expressions of) two classes. Not clear what the role of the word 'formal' is.

In modern terminology, the class exists and is empty.

[1] Τὸ γὰρ αὐτὸ ἅμα ὑπάρχειν τε καὶ μὴ ὑπάρχειν ἀδύνατον τῷ αὐτῷ καὶ κατὰ τὸ αὐτό...Αὕτη δὴ πασῶν ἐστὶ βεβαιοτάτη τῶν ἀρχῶν...Διὸ πάντες οἱ ἀποδεικνύντες εἰς ταύτην ἀνάγουσιν ἐσχάτην δόξαν· φύσει γὰρ ἀρξὴ καὶ τῶν ἄλλων ἀξεωμάτων αὕτη πάντων.—*Metaphysica*, III, 3.

The above interpretation has been introduced not on account of its immediate value in the present system, but as an illustration of a significant fact in the philosophy of the intellectual powers, viz., that what has been commonly regarded as the fundamental axiom of metaphysics is but the consequence of a law of thought, mathematical in its form. I desire to direct attention also to the circumstance that the equation (1) in which that fundamental law of thought is expressed is an equation of the second degree. [2] Without speculating at all in this chapter upon the question, whether that circumstance is necessary in its own nature, we may venture to assert that if it had not existed, the whole procedure of the understanding would have been different from what it is. Thus it is a consequence of the fact that the fundamental equation of thought is of the second degree, that we perform the operation of analysis and classification, by division into pairs of

Evidently footnote 2 was inspired by a query of De Morgan that was not included in the book on the Boole–De Morgan correspondence.

[2] Should it here be said that the existence of the equation $x^2 = x$ necessitates also the existence of the equation $x^3 = x$, which is of the third degree, and then inquired whether that equation does not indicate a process of *trichotomy*; the answer is, that the equation $x^3 = x$ is not interpretable in the system of logic. For writing it in either of the forms

$$x\,(1-x)\,(1+x) = 0\,, \qquad (2)$$
$$x\,(1-x)\,(-1-x) = 0\,, \qquad (3)$$

we see that its interpretation, if possible at all, must involve that of the factor $1+x$, or of the factor $-1-x$. The former is not interpretable, because we cannot conceive of the addition of any class x to the universe 1; the latter is not interpretable, because the symbol -1 is not subject to the law $x(1-x) = 0$, to which all class symbols are subject. Hence the equation $x^3 = x$ admits of no interpretation analogous to that of the equation $x^2 = x$. Were the former equation, however, true independently of the latter, i.e. were that act of the mind which is denoted by the symbol x, such that its second repetition should reproduce the result of a single operation, but not its first or mere repetition, it is presumable that we should be able to interpret one of the forms (1), (2), which under the actual conditions of thought we cannot do. There exist operations, known to the mathematician, the law of which may be adequately expressed by the equation $x^3 = x$. But they are of a nature altogether foreign to the province of general reasoning. In saying that it is conceivable that the law of thought might have been different from what it is, I mean only that we can frame such an hypothesis, and study its consequences. The possibility of doing this involves no such doctrine as that the actual law of human reason is the product either of chance or of arbitrary will.

In this footnote one finds the first time that Boole called the algebra of logic operation '+' by the numerical algebra name 'addition'.

opposites, or, as it is technically said, by *dichotomy.* Now if the equation in question had been of the third degree, still admitting of interpretation as such, the mental division must have been threefold in character, and we must have proceeded by a species of *trichotomy*, the real nature of which it is impossible for us, with our existing faculties, adequately to conceive, but the laws of which we might still investigate as an object of intellectual speculation.

16. The law of thought expressed by the equation (1) will, for reasons which are made apparent by the above discussion, be occasionally referred to as the "law of duality."

Boole did not have the concept of *term* as it is used in modern universal algebra and logic. Such terms are defined recursively—here is the definition for Boole's algebra of logic:

- variables are terms
- 0, 1 are terms
- if s and t are terms then so are $(s+t)$, $(s-t)$ and $(s \cdot t)$.

In the marginal notes these will be called *algebraic terms*.

Although Boole started his algebra of logic by defining his operations and by introducing several laws and rules of inference for his algebra of logic, it was really his Rule of 0 and 1 that justified most of his results. Unfortunately he did not explain this Rule very well, nor did he give it sufficient emphasis, with the result that it took nearly 150 years before it was deciphered and its central role understood.

The Rule of 0 and 1 simply says that a (polynomial) equation or equational argument φ is **valid** (i.e., accepted) in his algebra of logic iff it holds in the integers when the variables were restricted to the values 0 and 1. We express this condition by saying φ **holds** in $\mathbb{Z}_{01}$, or $\mathbb{Z} \models_{01} \varphi$. Given a φ that is constructed from algebraic terms, we will say that φ **holds** in Boole's algebraic structures (they are partial algebras) if φ is true whenever the variables take on values for which all the terms in φ are defined. We say simply: φ is true when defined.

Boole works with polynomials with integer coefficients except in his method of solving a polynomial equation $f(\vec{x}, w) = 0$ for w. He rewrites this as a polynomial equation $q(\vec{x})w = p(\vec{x})$, applies formal division to obtain $w = p(\vec{x})/q(\vec{x})$, applies a formal expansion to the right side, and then shows how it is to be interpreted. His work would have been much clearer if he had put all details of his method to find w, in particular the use of rational functions, in a separate section. As it stands, it is often not obvious if Boole is talking about rational functions, or just polynomial functions; quite often it is the latter.

END NOTES for CHAPTER III

III.B

Chapter IV

Of the Division of Propositions into the Two Classes of "Primary" and "Secondary;" of the Characteristic Properties of those Classes, and of the Laws of the Expression of Primary Propositions.

Boole replaced Aristotelian categorical propositions with **primary** propositions, and the hypothetical, etc., propositions with **secondary** propositions.

CATEGORICAL PROPOSITIONS: The four Aristotelian categorical proposition forms are:

- (All/Some) S (is/is not) P,

where S and P denote simple terms which are encoded in Boole's algebra as variables $x, y, \ldots$. (See page 227.)

PRIMARY PROPOSITIONS: Boole's three 'leading types' of primary propositions (see pages 63,64):

- All S is (all/some) P
- Some S is some P,

where S and P could be complex terms, that is, Boolean combinations of simple terms.

SECONDARY PROPOSITIONS: Let PP be the set of primary propositions, SP the set of secondary propositions. Let BC be the Boolean combinations operator. Let IT be the operator '$\cdots$ is True' and IF the operator '$\cdots$ is False'. Then a good guess as to what Boole meant by secondary propositions is: SP = (closure of PP under BC, IT and IF) $\setminus$ PP.

1. The laws of those mental operations which are concerned in the processes of Conception or Imagination having been investigated, and the corresponding laws of the symbols by which they are represented explained, we are led to consider the practical application of the results obtained: first, in the expression of the complex terms of propositions; secondly, in the expression of propositions; and lastly, in the construction of a general method of deductive analysis. In the present chapter we shall be chiefly concerned with the first of these objects, as an introduction to which it is necessary to establish the following Proposition:

Proposition I.

All logical propositions may be considered as belonging to one or the other of two great classes, to which the respective names of "Primary" or "Concrete Propositions," and "Secondary" or "Abstract Propositions," may be given.

Every assertion that we make may be referred to one or the other of the two following kinds. Either it expresses a relation among *things,* or it expresses, or is equivalent to the expression of, a relation among *propositions.* An assertion respecting the properties of things, or the phænomena which they manifest, or the circumstances in which they are placed, is, properly speaking, the assertion of a relation among things. To say that "snow is white," is for the ends of logic equivalent to saying, that "snow is a white thing." An assertion respecting facts or events, their mutual connexion and dependence, is, for the same ends, generally equivalent to the assertion, that such and such propositions concerning those events have a certain relation to each other as respects their mutual truth or falsehood. The former class of propositions, relating to *things*, I call "Primary;" the latter class, relating to *propositions*, I call "Secondary." The distinction is in practice nearly but not quite co-extensive with the common logical distinction of propositions as categorical or hypothetical.

For instance, the propositions, "The sun shines," "The earth is warmed," are primary; the proposition, "If the sun shines the earth is warmed," is secondary. To say, "The sun shines," is to say, "The sun is that which shines," and it expresses a relation between two classes of things, viz., "the sun" and "things which shine." The secondary proposition, however, given above, expresses a relation of dependence between the two primary propositions, "The sun shines," and "The earth is warmed." I do not hereby affirm that the relation between these propositions is, like that which exists between the facts which they express, a relation of causality, but only that the relation among the propositions so implies, and is so implied by, the relation among the facts, that it may for the ends of logic be used as a fit representative of that relation.

The operators '. . . is True' and '. . . is False' convert a primary proposition into a secondary proposition.

2. If instead of the proposition, "The sun shines," we say, "It is true that the sun shines," we then speak not directly of things, but of a proposition concerning things, viz., of the proposition, "The sun shines." And, therefore, the proposition in which we thus speak is a secondary one. Every primary proposition may thus give rise to a secondary proposition, viz., to that secondary proposition which asserts its truth, or declares its falsehood.

It will usually happen, that the particles *if*, *either*, *or*, will indicate that a proposition is secondary; but they do not necessarily imply that such is the case. The proposition, "Animals are either rational or irrational," is primary. It cannot be resolved into "Either animals are rational or animals are irrational," and it does not therefore express a relation of dependence between the two propositions connected together in the latter disjunctive sentence. The particles, *either*, *or*, are in fact no *criterion* of the nature of propositions, although it happens that they are more frequently found in secondary propositions. Even the conjunction *if* may be found in primary propositions. "Men are, if wise, then temperate," is an example of the kind. It cannot be resolved into "If all men are wise, then all men are temperate."

Boole did not catalog the possible written forms of propositions since the English language offers so many ways to express propositions. However Boole claimed that any primary proposition could be reduced to one of the three 'great leading types' (see page 63).

3. As it is not my design to discuss the merits or defects of the ordinary division of propositions, I shall simply remark here, that the principle upon which the present classification is founded is clear and definite in its application, that it involves a real and fundamental distinction in propositions, and that it is of essential importance to the development of a general method of reasoning. Nor does the fact that a primary proposition may be put into a form in which it becomes secondary at all conflict with the views here maintained. For in the case thus supposed, it is not of the things connected together in the primary proposition that any direct account is taken, but only of the proposition itself considered as *true* or as *false*.

Great leading types of secondary propositions are not provided by Boole.

4. In the expression both of primary and of secondary propositions, the same symbols, subject, as it will appear, to the same laws, will be employed in this work. The difference between the two cases is a difference not of form but of interpretation. In both cases the actual relation which it is the object of the proposition to express will be denoted by the sign =. In the expression of primary propositions, the members thus connected will usually represent the "terms" of a proposition, or, as they are more particularly designated, its subject and predicate.

The equational algebra of logic for secondary propositions is the same as that for primary propositions.

Proposition II.

5. *To deduce a general method, founded upon the enumeration of possible varieties, for the expression of any class or collection of things, which may constitute a "term" of a Primary Proposition.*

How to express classes with complex descriptions (i.e., requiring more than a single letter) as algebraic terms in Boole's algebra of logic.

First, If the class or collection of things to be expressed is defined only by names or qualities common to all the individuals of which it consists, its expression will consist of a single term, in which the symbols expressive of those names or qualities will be combined without any connecting sign, as if by the algebraic process of multiplication. Thus, if x represent opaque substances, y polished substances, z stones, we shall have,

This paragraph is about classes expressed by constituents, like $x(1-y)(1-z)$. (See pages 74,75.)

xyz = opaque polished stones;

$xy(1-z)$ = opaque polished substances which are not stones;

$x(1-y)(1-z)$ = opaque substances which are not polished, and are not stones;

and so on for any other combination. Let it be observed, that each of these expressions satisfies the same law of duality, as the individual symbols which it contains. Thus,

The first hint that not all algebraic terms need be idempotent in Boole's algebra of logic.

$$xyz \times xyz = xyz\,;$$
$$xy(1-z) \times xy(1-z) = xy(1-z)\,;$$

and so on. Any such term as the above we shall designate as a "class term," because it expresses a class of things by means of the common properties or names of the individual members of such class.

"Class terms" are viewed as "constituents" on page 75.

Secondly, If we speak of a collection of things, different portions of which are defined by different properties, names, or attributes, the expressions for those different portions must be separately formed, and then connected by the sign +. But if the collection of which we desire to speak has been formed by excluding from some wider collection a defined portion of its members, the sign − must be prefixed to the symbolical expression of the excluded portion. Respecting the use of these symbols some further observations may be added.

This paragraph is about when to use '+', namely with terms designating pairwise disjoint classes; and when to use '−', namely one can use $s-t$ when t expresses a class that is a subclass of the class expressed by s.

6. Speaking generally, the symbol + is the equivalent of the conjunctions "and," "or," and the symbol −, the equivalent of the preposition "except." Of the conjunctions "and" and "or," the former is usually employed when the collection to be described forms the subject, the latter when it forms the predicate, of a proposition. "The scholar *and* the man of the world desire happiness," may be taken as an illustration of one of these cases. "Things possessing utility are *either* productive of pleasure *or* preventive of pain," may exemplify the other. Now whenever an expression involving these particles presents itself in a primary proposition, it becomes very important to know whether the groups or classes separated in thought by them are intended to be quite distinct from each other and mutually exclusive, or not. Does the expression, "Scholars and men of the world," include or exclude those who are both? Does the expression,

Interesting: "and" and "or" play distinct roles in modern propositional logic .

"Either productive of pleasure or preventive of pain," include or exclude things which possess both these qualities? I apprehend that in strictness of meaning the conjunctions "and," "or," do possess the power of separation or exclusion here referred to; that the formula, "All x's are either y's or z's," rigorously interpreted, means, "All x's are either y's, but not z's," or, "z's but not y's." But it must at the same time be admitted, that the "jus et norma loquendi" seems rather to favour an opposite interpretation. The expression, "Either y's or z's," would generally be understood to include things that are y's and z's at the same time, together with things which come under the one, but not the other. Remembering, however, that the symbol + does possess the separating power which has been the subject of discussion, we must resolve any disjunctive expression which may come before us into elements really separated in thought, and then connect their respective expressions by the symbol +.

A convenient assumption to justify his use of partial operations as 'natural'.

The symbol '+' does not have any "separating power"; in order for a term $s + t$ to be interpretable, s and t must refer to disjoint classes.

This "must resolve" demand only applies to putting the premisses in a proper form so that the corresponding equations are interpretable.

And thus, according to the meaning implied, the expression, "Things which are either x's or y's," will have two different symbolical equivalents. If we mean, "Things which are x's, but not y's, or y's, but not x's," the expression will be

$$x(1 - y) + y(1 - x)\,;$$

the symbol x standing for x's, y for y's. If, however, we mean, "Things which are either x's, or, if not x's, then y's," the expression will be

$$x + y(1 - x).$$

This expression supposes the admissibility of things which are both x's and y's at the same time. It might more fully be expressed in the form

$$xy + x(1 - y) + y(1 - x)\,;$$

but this expression, on addition of the two first terms, only reproduces the former one.

Expressing symmetric difference $x \triangle y$ in Boole's algebra.

Expressing $x \cup y$, the union of x and y, in Boole's algebra.

Expressing $x \cup y$ as a sum of constituents in Boole's algebra.

Let it be observed that the expressions above given satisfy the fundamental law of duality (III. 16). Thus we have

$$\begin{aligned}\big\{x(1 - y) + y(1 - x)\big\}^2 &= x(1 - y) + y(1 - x),\\ \big\{x + (1 - x)\big\}^2 &= x + y(1 - x).\end{aligned}$$

It will be seen hereafter, that this is but a particular manifestation

Second hint that not all terms need be idempotent in Boole's algebra.

By 'a particular manifestation of a general law' he probably meant Proposition IV on page 78: A sum of distinct constituents is idempotent.

of a general law of expressions representing "classes or collections of things."

7. The results of these investigations may be embodied in the following rule of expression.

Rule.—*Express simple names or qualities by the symbols x, y, z, &c., their contraries by $1-x$, $1-y$, $1-z$, &c.; classes of things defined by common names or qualities, by connecting the corresponding symbols as in multiplication; collections of things, consisting of portions different from each other, by connecting the expressions of those portions by the sign $+$. In particular, let the expression, "Either x's or y's," be expressed by $x(1-y)+y(1-x)$, when the classes denoted by x and y are exclusive, by $x+y(1-x)$ when they are not exclusive. Similarly let the expression, "Either x's, or y's, or z's," be expressed by $x(1-y)(1-z)+y(1-x)(1-z)+z(1-x)(1-y)$, when the classes denoted by x, y, and z, are designed to be mutually exclusive, by $x+y(1-x)+z(1-x)(1-y)$, when they are not meant to be exclusive, and so on.*

This Rule gives Boole's description of how to express classes, with simple or complex descriptions, by idempotent algebraic terms, all of whose subterms are idempotent; that is, by totally defined algebraic terms.

8. On this rule of expression is founded the converse rule of interpretation. Both these will be exemplified with, perhaps, sufficient fulness in the following instances. Omitting for brevity the universal subject "things," or "beings," let us assume

$$x = \text{hard}, y = \text{elastic}, z = \text{metals};$$

and we shall have the following results:

"Non-elastic metals," will be expressed by $z(1-y)$;

"Elastic substances with non-elastic metals," by $y+z(1-y)$;

"Hard substances, except metals," by $x-z$;

"Metallic substances, except those which are neither hard nor elastic," by $z-z(1-x)(1-y)$, or by $z\{1-(1-x)(1-y)\}$, *vide* (6), Chap. II.

In the last example, what we had really to express was "Metals, except not hard, not elastic, metals." Conjunctions used between *adjectives* are usually superfluous, and, therefore, must not be expressed symbolically.

Thus, "Metals hard and elastic," is equivalent to "Hard elastic metals," and expressed by xyz.

Take next the expression, "Hard substances, except those which are metallic and non-elastic, and those which are elastic and non-metallic." Here the word *those* means hard substances, so that the expression really means, *Hard substances except hard substances, metallic, non-elastic, and hard substances non-metallic, elastic*; the word *except* extending to both the classes which follow it. The complete expression is

$$x - \{xz(1-y) + xy(1-z)\};$$

or

$$x - xz(1-y) - xy(1-z).$$

9. The preceding Proposition, with the different illustrations which have been given of it, is a necessary preliminary to the following one, which will complete the design of the present chapter.

Proposition III.

To deduce from an examination of their possible varieties a general method for the expression of Primary or Concrete Propositions.

Boole ended up with just three master forms of primary propositions (see pages 64, 162).

A primary proposition, in the most general sense, consists of two terms, between which a relation is asserted to exist. These terms are not necessarily single-worded names, but may represent any collection of objects, such as we have been engaged in considering in the previous sections. The mode of expressing those terms is, therefore, comprehended in the general precepts above given, and it only remains to discover how the relations between the terms are to be expressed. This will evidently depend upon the nature of the relation, and more particularly upon the question whether, in that relation, the terms are understood to be universal or particular, i.e. whether we speak of the whole of that collection of objects to which a term refers, or indefinitely of the whole or of a part of it, the usual signification of the prefix, "some."

Boole extended the notion of 'term' in a proposition to include complex terms (just as he had done in MAL.)

Boole said one needs to consider whether the terms of a primary proposition refer to all or to some of a class.

Suppose that we wish to express a relation of identity between the two classes, "Fixed Stars" and "Suns," i.e. to express that "All fixed stars are suns," and "All suns are fixed stars." Here, if x stand for fixed stars, and y for suns, we shall have

$$x = y$$

for the equation required.

In the proposition, "All fixed stars are suns," the term "all fixed stars" would be called the *subject*, and " suns" the *predicate*. Suppose that we extend the meaning of the terms *subject* and *predicate* in the following manner. By subject let us mean the first term of any affirmative proposition, i.e. the term which precedes the copula *is* or *are*; and by predicate let us agree to mean the second term, i.e. the one which follows the copula; and let us admit the assumption that either of these may be universal or particular, so that, in either case, the whole class may be implied, or only a part of it. Then we shall have the following Rule for cases such as the one in the last example:—

10. RULE.—*When both Subject and Predicate of a Proposition are universal, form the separate expressions for them, and connect them by the sign =.*

$s = t$ expresses 'All S is all T' when s and t are algebraic terms expressing the classes S and T. This Rule applies to definitions.

This case will usually present itself in the expression of the definitions of science, or of subjects treated after the manner of pure science. Mr. Senior's definition of wealth affords a good example of this kind, viz.:

"Wealth consists of things transferable, limited in supply, and either productive of pleasure or preventive of pain."

Before proceeding to express this definition symbolically, it must be remarked that the conjunction *and* is superfluous. Wealth is really defined by its possession of three properties or qualities, not by its composition out of three classes or collections of objects. Omitting then the conjunction *and*, let us make

$$\begin{aligned} w &= \text{wealth.} \\ t &= \text{things transferable.} \\ s &= \text{limited in supply.} \\ p &= \text{productive of pleasure.} \\ r &= \text{preventive of pain.} \end{aligned}$$

Now it is plain from the nature of the subject, that the expression, "Either productive of pleasure or preventive of pain," in the above definition, is meant to be equivalent to "Either productive of pleasure; or, if not productive of pleasure, preventive of pain." Thus the class of things which the above expression, taken alone, would define, would consist of all things productive of pleasure, together with all things not productive of pleasure, but preventive of pain, and its symbolical expression would be

$$p + (1 - p)r.$$

If then we attach to this expression placed in brackets to denote that both its terms are referred to, the symbols s and t limiting its application to things "transferable" and "limited in supply," we obtain the following symbolical equivalent for the original definition, viz.:

$$w = st\{p + r(1 - p)\}. \quad (1)$$

If the expression, "Either productive of pleasure or preventive of pain," were intended to point out merely those things which are productive of pleasure without being preventive of pain, $p(1-r)$, or preventive of pain, without being productive of pleasure, $r(1 - p)$ (exclusion being made of those things which are both productive of pleasure and preventive of pain), the expression in symbols of the definition would be

$$w = st\{p(1 - r) + r(1 - p)\}. \quad (2)$$

All this agrees with what has before been more generally stated.

The reader may be curious to inquire what effect would be produced if we literally translated the expression, "Things productive of pleasure or preventive of pain," by $p + r$, making the symbolical equation of the definition to be

By 'literally translated' he meant: express 'or' by '+'.

$$w = st(p + r). \quad (3)$$

The answer is, that this expression would be equivalent to (2), with the additional implication that the classes of things denoted by stp and str are quite distinct, so that of things transferable and limited in supply there exist none in the universe which are at the same time both productive of pleasure and preventive of pain. How the full import of any equation may be determined will be explained hereafter. What has been said may show that before attempting to translate our data into the rigorous language of symbols, it is above all things necessary to ascertain the *intended* import of the words we are using. But this necessity cannot be regarded as an evil by those who value correctness of

Proof: Use the fact that (3) implies, by squaring both sides, $stpr = 0$. Then $(stp)(str) = 0$

For the "full import" of an equation, see page 82.

thought, and regard the right employment of language as both its instrument and its safeguard.

11. Let us consider next the case in which the predicate of the proposition is particular, e.g. "All men are mortal."

In this case it is clear that our meaning is, "All men are some mortal beings," and we must seek the expression of the predicate, "some mortal beings." Represent then by v, a class indefinite in every respect but this, viz., that some of its members are mortal beings, and let x stand for "mortal beings," then will vx represent "some mortal beings." Hence if y represent men, the equation sought will be

$$y = vx.$$

Translating "All y is x" as $y = vx$. Better choice: $y = yx$ (as in MAL).

From such considerations we derive the following Rule, for expressing an affirmative universal proposition whose predicate is particular:

Boole never mentioned that when there are several such premisses, they should have distinct v's. Else one could translate the premisses "All x is z" and "All y is z" as $x = vz$, $y = vz$, and conclude $x = y$, which does not follow from the original premisses.

Rule.—*Express as before the subject and the predicate, attach to the latter the indefinite symbol v, and equate the expressions.*

It is obvious that v is a symbol of the same kind as x, y, &c., and that it is subject to the general law,

$$v^2 = v, \quad \text{or} \quad v(1-v) = 0.$$

Boole started using the same v on page 118 for multiple premisses which are expressed by the form $X = vY$—it never led to a false conclusion because Boole always immediately eliminated the v, converting $X = vY$ into $X = XY$, or equivalently, into $X(1-Y) = 0$.

Thus, to express the proposition, "The planets are either primary or secondary," we should, according to the rule, proceed thus:

Let x represent planets (the subject);

$$y = \text{primary bodies;}$$
$$z = \text{secondary bodies;}$$

then, assuming the conjunction "or" to separate absolutely the class of "primary" from that of "secondary" bodies, so far as they enter into our consideration in the proposition given, we find for the equation of the proposition

$$x = v\{y(1-z) + z(1-y)\}. \quad (4)$$

It may be worth while to notice, that in this case the *literal* translation of the premises into the form

$$x = v(y+z) \quad (5)$$

would be exactly equivalent, v being an indefinite class symbol. The form (4) is, however, the better, as the expression

$$y\,(1-z)+z\,(1-y)$$

consists of terms representing classes quite distinct from each other, and satisfies the fundamental law of duality.

If we take the proposition, "The heavenly bodies are either suns, or planets, or comets," representing these classes of things by w, x, y, z, respectively, its expression, on the supposition that none of the heavenly bodies belong at once to two of the divisions above mentioned, will be

$$w \;=\; v\big\{x\,(1-y)\,(1-z)+y\,(1-x)\,(1-z)+z\,(1-x)\,(1-y)\,\big\}$$

If, however, it were meant to be implied that the heavenly bodies were either suns, or, if not suns, planets, or, if neither, suns nor planets, fixed stars, a meaning which does not exclude the supposition of some of them belonging at once to two or to all three of the divisions of suns, planets, and fixed stars,—the expression required would be

$$w \;=\; v\big\{x+y\,(1-x)+z\,(1-x)\,(1-y)\,\big\}. \tag{6}$$

The above examples belong to the class of descriptions, not definitions. Indeed the predicates of propositions are usually particular. When this is not the case, either the predicate is a singular term, or we employ, instead of the copula "is" or "are," some form of connexion, which implies that the predicate is to be taken universally.

12. Consider next the case of universal negative propositions, e.g. "No men are perfect beings."

Now it is manifest that in this case we do not speak of a class termed "no men," and assert of this class that all its members are "perfect beings." But we virtually make an assertion about "*all men*" to the effect that they are "*not perfect beings*." Thus the true meaning of the proposition is this:

"All men (subject) are (copula) not perfect (predicate);" whence, if y represent "men," and x "perfect beings," we shall have

$$y \;=\; v\,(1-x)\,,$$

and similarly in any other case. Thus we have the following Rule:

RULE.— *To express any proposition of the form "No x's are y's," convert it into the form "All x's are not y's," and then proceed as in the previous case.*

Better choice: $xy = 0$, as in MAL.

13. Consider, lastly, the case in which the subject of the proposition is particular, e.g. "Some men are not wise." Here, as has been remarked, the negative *not* may properly be referred, certainly, at least, for the ends of Logic, to the predicate *wise*; for we do not mean to say that it is not true that "Some men are wise," but we intend to predicate of "some men" a want of wisdom. The requisite form of the given proposition is, therefore, "Some men are not-wise." Putting, then, y for "men," x for "wise," i. e. "wise beings," and introducing v as the symbol of a class indefinite in all respects but this, that it contains some individuals of the class to whose expression it is prefixed, we have

$$vy = v\,(1 - x)\,.$$

Better choice: $v = vy(1 - x)$.

14. We may comprise all that we have determined in the following general Rule:

GENERAL RULE FOR THE SYMBOLICAL EXPRESSION OF PRIMARY PROPOSITIONS.

(Use the better choices given above for expressing propositions.)

1st. *If the proposition is affirmative, form the expression of the subject and that of the predicate. Should either of them be particular, attach to it the indefinite symbol v, and then equate the resulting expressions.*

2ndly. *If the proposition is negative, express first its true meaning by attaching the negative particle to the predicate, then proceed as above.*

One or two additional examples may suffice for illustration.

Ex.—"No men are placed in exalted stations, and free from envious regards."

Let y represent "men," x, "placed in exalted stations," z, "free from envious regards."

Now the expression of the class described as "placed in exalted station," and "free from envious regards," is xz. Hence the contrary class, i.e. they to whom this description does not apply, will be represented by $1 - xz$, and to this class all men are referred. Hence we have

$$y = v\,(1 - xz)\,.$$

If the proposition thus expressed had been placed in the equiva- lent form, "Men in exalted stations are not free from envious regards," its expression would have been

$$yx = v\,(1 - z)\,.$$

It will hereafter appear that this expression is really equivalent to the previous one, on the particular hypothesis involved, viz., that v is an indefinite class symbol.

Ex.—"No men are heroes but those who unite self-denial to courage."

Let $x =$ "men," $y =$ "heroes," $z =$ "those who practise self-denial," w, "those who possess courage."

The assertion really is, that "men who do not possess courage and practise self-denial are not heroes."

Hence we have

$$x\,(1 - zw) = v\,(1 - y)$$

for the equation required.

15. In closing this Chapter it may be interesting to compare together the great leading types of propositions symbolically expressed. If we agree to represent by X and Y the symbolical expressions of the "terms," or things related, those types will be

$$X = vY,$$
$$X = Y,$$
$$vX = vY.$$

In the first, the predicate only is particular; in the second, both terms are universal; in the third, both are particular. Some minor forms are really included under these. Thus, if $Y = 0$, the second form becomes

$$X = 0\,;$$

and if $Y = 1$ it becomes

$$X = 1\,;$$

Better choice: Any primary proposition can be put in one of the three forms:
$X = XY$,
$X = Y$,
$v = vXY$
where X, Y are totally defined algebraic terms. Use distinct v's for distinct premiss propositions.

both which forms admit of interpretation. It is further to be noticed, that the expressions X and Y, if founded upon a sufficiently careful analysis of the meaning of the "terms" of the proposition, will satisfy the fundamental law of duality which requires that we have

Actually want X, Y to be totally defined terms.

$$X^2 = X \quad \text{or} \quad X(1-X) = 0,$$
$$Y^2 = Y \quad \text{or} \quad Y(1-Y) = 0.$$

Chapter V

Of the Fundamental Principles of Symbolical Reasoning, and of the Expansion or Development of Expressions involving Logical Symbols.

Boole's Principles of Symbolical Reasoning are correct for total algebras, but they are not, in general, correct for partial algebras. However they are correct for his algebra of logic. (See the ArXiV preprint by SB and HPS.)

1. The previous chapters of this work have been devoted to the investigation of the fundamental laws of the operations of the mind in reasoning; of their development in the laws of the symbols of Logic; and of the principles of expression, by which that species of propositions called primary may be represented in the language of symbols. These inquiries have been in the strictest sense preliminary. They form an indispensable introduction to one of the chief objects of this treatise—the construction of a system or method of Logic upon the basis of an exact summary of the fundamental laws of thought. There are certain considerations touching the nature of this end, and the means of its attainment, to which I deem it necessary here to direct attention.

Below, Boole finally called the logical operation '+' in his algebra of logic by the numerical algebra name, 'addition'.

He stated that his laws involving $x + y$ were determined by considering the cases where x and y were disjoint classes, and that $x + y$ *seems* uninterpretable otherwise.

It would have helped the reader if he had clearly said early on that '$x + y$ *is* uninterpretable otherwise.'

Likewise it would have been good to have clearly said early on that $x - y$ is only defined if the class expressed by y is a subclass of the class expressed by x. (See page 93.)

2. I would remark in the first place that the generality of a method in Logic must very much depend upon the generality of its elementary processes and laws. We have, for instance, in the previous sections of this work investigated, among other things, the laws of that logical process of *addition* which is symbolized by the sign +. Now those laws have been determined from the study of instances, in all of which it has been a necessary condition, that the classes or things added together in thought should be mutually exclusive. The expression $x + y$ seems indeed uninterpretable, unless it be assumed that the things represented by x and the things represented by y are entirely separate; that they embrace no individuals in common. And conditions analogous to this have been involved in those acts of conception from the study of which the laws of the other symbolical operations have been ascertained. The question then arises, whether it is necessary to restrict the application of these symbolical laws and processes by the same conditions of interpretability under which the knowledge of them was obtained. If such restriction is necessary, it is manifest that no such thing as a general method in Logic is possible. On the other hand, if such restriction is unnecessary, in what light are we to contemplate processes which appear to be uninterpretable in that sphere of thought which they are designed to aid? These questions do not belong to the science of Logic alone. They are equally pertinent to every developed form of human reasoning which is based upon the employment of a symbolical language.

In contradiction to this 'manifest' claim, in Chapter X Boole presented a general method in Logic that avoided uninterpretable terms, a method that tended to require considerably more effort to use.

3. I would observe in the second place, that this apparent failure of correspondency between process and interpretation does not manifest itself in the *ordinary* applications of human reason. For no operations are there performed of which the meaning and the application are not seen; and to most minds it does not suffice that merely formal reasoning should connect their premises and their conclusions; but every step of the connecting train, every mediate result which is established in the course of demonstration, must be intelligible also. And without doubt, this is both an actual condition and an important safeguard, in the reasonings and discourses of common life.

Ordinary reasoning does not have uninterpretable steps.

There are perhaps many who would be disposed to extend the same principle to the general use of symbolical language as an instrument of reasoning. It might be argued, that as the laws or axioms which govern the use of symbols are established upon an investigation of those cases only in which interpretation is possible, we have no right to extend their application to other cases in which interpretation is impossible or doubtful, even though (as should be admitted) such application is employed in the intermediate steps of demonstration only. Were this objection conclusive, it must be acknowledged that slight advantage would accrue from the use of a symbolical method in Logic. Perhaps that advantage would be confined to the mechanical gain of employing short and convenient symbols in the place of more cumbrous ones. But the objection itself is fallacious. Whatever our *à priori* anticipations might be, it is an unquestionable fact that the validity of a conclusion arrived at

Indeed if one wants to use uninterpretable steps in an equational logic for partial algebras, one needs to properly show that such is permissible. Boole did not do this.

(Of course for total algebras there are no uninterpretable terms.)

by any symbolical process of reasoning, does not depend upon our ability to interpret the formal results which have presented themselves in the different stages of the investigation. There exist, in fact, certain general principles relating to the use of symbolical methods, which, as pertaining to the particular subject of Logic, I shall first state, and I shall then offer some remarks upon the nature and upon the grounds of their claim to acceptance.

Boole's assertion of this "unquestionable fact" is simply incorrect in general for partial algebras, but fortunately holds for his algebra of logic.

4. The conditions of valid reasoning, by the aid of symbols, are—

1st, That a fixed interpretation be assigned to the symbols employed in the expression of the data; and that the laws of the combination of those symbols be correctly determined from that interpretation.

2nd, That the formal processes of solution or demonstration be conducted throughout in obedience to all the laws determined as above, without regard to the question of the interpretability of the particular results obtained.

This 2nd condition can lead to problems when working with partial algebras; fortunately it does not in Boole's algebra of logic.

3rd, That the final result be interpretable in form, and that it be actually interpreted in accordance with that system of interpretation which has been employed in the expression of the data. Concerning these principles, the following observations may be made.

5. The necessity of a fixed interpretation of the symbols has already been sufficiently dwelt upon (II. 3). The necessity that the fixed result should be in such a form as to admit of that interpretation being applied, is founded on the obvious principle, that the use of symbols is a means towards an end, that end being the knowledge of some intelligible fact or truth. And that this end may be attained, the final result which expresses the symbolical conclusion must be in an interpretable form. It is, however, in connexion with the second of the above general principles or conditions (V. 4), that the greatest difficulty is likely to be felt, and upon this point a few additional words are necessary.

Indeed many readers of LT were uncomfortable with Boole's use of uninterpretable terms. It was not till 1976 that a proper justification of this aspect of Boole's work appeared (by Hailperin).

I would then remark, that the principle in question may be considered as resting upon a general law of the mind, the knowledge of which is not given to us *à priori*, i.e. antecedently to experience, but is derived, like the knowledge of the other laws of the mind, from the clear manifestation of the general principle in the particular instance. A single example of reasoning, in which symbols are employed in obedience to laws founded upon their interpretation, but without any sustained reference to that interpretation, the chain of demonstration conducting us through intermediate steps which are not interpretable, to a final result which is interpretable, seems not only to establish the validity of the particular application, but to make known to us the general law manifested therein. No accumulation of instances can properly add weight to such evidence. It may furnish us with clearer conceptions of that common element of truth upon which the application of the principle depends, and so prepare the way for its reception. It may, where the immediate force of the evidence is not felt, serve as a verification, *à posteriori*, of the practical validity of the principle in question. But this does not affect the position affirmed, viz., that the general principle must be seen in the particular instance,—seen to be general in application as well as true in the special example. The employment of the uninterpretable symbol $\sqrt{-1}$, in the intermediate processes of trigonometry, furnishes an illustration of what has been said. I apprehend that there is no mode of explaining that application which does not covertly assume the very principle in question. But that principle, though not, as I conceive, warranted by formal reasoning based upon other grounds, seems to deserve a place among those axiomatic truths which constitute, in some sense, the foundation of the possibility of general knowledge, and which may properly be regarded as expressions of the mind's own laws and constitution.

This 'single example' principle is just wrong. It was likely a product of Boole's desire to justify his work with uninterpretables in his algebra of logic.

Evidently Boole thought that the use of $\sqrt{-1}$ to derive trigonometric formulas could only be justified by his Principles of Symbolical Reasoning. It was widely known that complex numbers had a geometrical representation, and in the mid 1830s Hamilton had given the modern version of the complex numbers as ordered pairs of real numbers. It seems Boole was missing the modern concepts of *isomorphism* and *embedded in*, so that he regarded the geometric representation as just another interpretation of the laws of numbers, not realizing that this justified the use of $\sqrt{-1}$.

6. The following is the mode in which the principle above stated will be applied in the present work. It has been seen, that any system of propositions may be expressed by equations involving symbols x, y, z, which, whenever interpretation is possible, are subject to laws identical in form with the laws of a system of quantitative symbols, susceptible only of the values 0 and 1 (II. 15). But as the formal processes of reasoning depend only upon the laws of the symbols, and not upon the nature of their interpretation, we are permitted to treat the above symbols,

Change "It has been seen" to 'It has been asserted' (as regards the use of quantitative symbols).

x, y, z, as if they were quantitative symbols of the kind above described. *We may in fact lay aside the logical interpretation of the symbols in the given equation; convert them into quantitative symbols, susceptible only of the values* 0 *and* 1*; perform upon them as such all the requisite processes of solution; and finally restore to them their logical interpretation.* And this is the mode of procedure which will actually be adopted, though it will be deemed unnecessary to restate in every instance the nature of the transformation employed. The processes to which the symbols x, y, z, regarded as quantitative and of the species above described, are subject, are not limited by those conditions of thought to which they would, if performed upon purely logical symbols, be subject, and a freedom of operation is given to us in the use of them, without which, the inquiry after a general method in Logic would be a hopeless quest.

This is based on Boole's **R01**, stated on page 37.

Now the above system of processes would conduct us to no intelligible result, unless the final equations resulting therefrom were in a form which should render their interpretation, after restoring to the symbols their logical significance, possible. There exists, however, a general method of reducing equations to such a form, and the remainder of this chapter will be devoted to its consideration. I shall say little concerning the way in which the method renders interpretation possible,—this point being reserved for the next chapter,—but shall chiefly confine myself here to the mere process employed, which may be characterized as a process of "development." As introductory to the nature of this process, it may be proper first to make a few observations.

7. Suppose that we are considering any class of things with reference to this question, viz., the relation in which its members stand as to the possession or the want of a certain property x. As every individual in the proposed class either possesses or does not possess the property in question, we may divide the class into two portions, the former consisting of those individuals which possess, the latter of those which do not possess, the property. This possibility of dividing in thought the whole class into two constituent portions, is antecedent to all knowledge of the constitution of the class derived from any other source; of which knowledge the effect can only be to inform us, more or less precisely, to what further conditions the portions of the class which possess and which do not possess the given property are subject. Suppose, then, such knowledge is to the following effect, viz., that the members of that portion which possess the property x, possess also a certain property u, and that these conditions united are a sufficient definition of them. We may then represent that portion of the original class by the expression ux (II. 6). If, further, we obtain information that the members of the original class which do not possess the property x, are subject to a condition v, and are thus defined, it is clear, that those members will be represented by the expression $v(1-x)$. Hence the class in its totality will be represented by

$$ux + v(1-x);$$

which may be considered as a general developed form for the expression of any class of objects considered with reference to the possession or the want of a given property x.

The general form thus established upon purely logical grounds may also be deduced from distinct considerations of formal law, applicable to the symbols x, y, z, equally in their logical and in their quantitative interpretation already referred to (V. 6).

8. *Definition.*—Any algebraic expression involving a symbol x is termed a function of x, and may be represented under the abbreviated general form $f(x)$. Any expression involving two symbols, x and y, is similarly termed a function of x and y, and may be represented under the general form $f(x, y)$, and so on for any other case.

Thus the form $f(x)$ would indifferently represent any of the following functions, viz., x, $1-x$, $\frac{1+x}{1-x}$, &c.; and $f(x, y)$ would equally represent any of the forms $x+y$, $x-2y$, $\frac{x+y}{x-2y}$, &c.

On the same principles of notation, if in any function $f(x)$ we change x into 1, the result will be expressed by the form $f(1)$; if in the same function we change x into 0, the result will be expressed by the form $f(0)$. Thus, if $f(x)$ represent the

Boole used three levels of algebraic expressions:

- algebraic terms $t(\vec{x})$
- polynomials $p(\vec{x})$
- rational $p(\vec{x})/q(\vec{x})$, (quotients of polynomials).

It would have been better not to mention proper rational functions till later, since they are only used as a mnemonic device. See p. 82.

function $\frac{a+x}{a-2x}$, $f(1)$ will represent $\frac{a+1}{a-2}$, and $f(0)$ will represent $\frac{a}{a}$.

9. *Definition.*—Any function $f(x)$, in which x is a logical symbol, or a symbol of quantity susceptible only of the values 0 and 1, is said to be developed, when it is reduced to the form $ax + b(1-x)$, a and b being so determined as to make the result equivalent to the function from which it was derived.

Boole used the words

- 'expand(ed), expansion'

interchangeably with the words

- 'develop(ed), development'.

This definition assumes, that it is possible to represent any function $f(x)$ in the form supposed. The assumption is vindicated in the following Proposition.

Proposition I.

10 . *To develop any function $f(x)$ in which x is symbol.*

By the principle which has been asserted in this chapter, it is lawful to treat x as a quantitative symbol, susceptible only of the values 0 and 1.

This principle is on page 70, and is based on **R01**.

Assume then,

$$f(x) = ax + b(1-x),$$

and making $x = 1$, we have

$$f(1) = a.$$

Again, in the same equation making $x = 0$, we have

$$f(0) = b.$$

Hence the values of a and b are determined, and substituting them in the first equation, we have

$$\boxed{f(x) = f(1)\,x + f(0)\,(1-x)}\,; \qquad (1)$$

Expanding f about x.

as the development sought.[1] The second member of the equation

---

[1] To some it may be interesting to remark, that the development of $f(x)$ obtained in this chapter, strictly holds, in the logical system, the place of the expansion of $f(x)$ in ascending powers of x in the system of ordinary algebra. Thus it may be obtained by introducing into the expression of Taylor's well-known theorem, viz.:

$$f(x) = f(0) + f'(0)\,x + f''(0)\,\frac{x^2}{1\cdot 2} + f'''(0)\,\frac{x^3}{1\cdot 2\cdot 3}, \text{ \&c.} \qquad (1)$$

the condition $x(1-x) = 0$, whence we find $x^2 = x$, $x^3 = x$, &c., and

This power series approach to the expansion theorem was used in MAL.

adequately represents the function $f\,(x)$, whatever the form of that function may be. For x regarded as a quantitative symbol admits only of the values 0 and 1, and for each of these values the development

This **R01**-proof is all the proof one needs for the polynomial case. For $f(x)$ proper rational, see pages 87-92.

$$f\,(1)\,x \;+\; f\,(0)\,(1-x)\,,$$

assumes the same value as the function $f\,(x)$.

As an illustration, let it be required to develop the function $\dfrac{1+x}{1+2x}$. Here, when $x = 1$, we find $f\,(1) = \dfrac{2}{3}$, and when $x = 0$, we find $f\,(0) = \dfrac{1}{1}$, or 1. Hence the expression required is

$$\frac{1+x}{1+2x} \;=\; \frac{2}{3}x + 1 - x\,;$$

$= 1 - \dfrac{1}{3}x$

and this equation is satisfied for each of the values of which the symbol x is susceptible.

Proposition II.

To expand or develop a function involving any number of logical symbols.

Boole expanded $f(x,y)$ by first expanding it as a function of x, giving (2), and then expanding the result as a function of y to obtain (5).

Let us begin with the case in which there are two symbols, x and y, and let us represent the function to be developed by $f\,(x,y)$.

First, considering $f\,(x,y)$ as a function of x alone, and expanding it by the general theorem (1), we have

$$f\,(x,y) \;=\; f\,(1\,,y)\,x + f\,(0\,,y)\,(1-x)\;; \tag{2}$$

A partial expansion, expanding $f(x,y)$ about x.

$$f\,(x) \;=\; f\,(0) + \{f'\,(0) + \frac{f''\,(0)}{1\cdot 2} + \frac{f'''\,(0)}{1\cdot 2\cdot 3} + \;\&\text{c.}\}x. \tag{2}$$

But making in (1), $x = 1$, we get

$$f(1) \;=\; f(0) + f'(0) + \frac{f''(0)}{1\cdot 2} + \frac{f'''(0)}{1\cdot 2\cdot 3} + \;\&\text{c.}$$

whence

$$f'(0) + \frac{f''(0)}{1\cdot 2} + \;\&\text{c.} \;=\; f(1) - f(0),$$

and (2) becomes, on substitution,

$$\begin{aligned} f(x) \;&=\; f(0) + \{f(1) - f(0)\}x, \\ &=\; f(1)x + f(0)(1-x), \end{aligned}$$

the form in question. This demonstration in supposing $f\,(x)$ to be developable in a series of ascending powers of x is less general than the one in the text.

wherein $f(1, y)$ represents what the proposed function becomes, when in it for x; we write 1, and $f(0, y)$ what the said function becomes, when in it for x we write 0.

Now, taking the coefficient $f(1, y)$, and regarding it as a function of y, and expanding it accordingly, we have

$$f(1, y) = f(1, 1)\, y + f(1, 0)\,(1 - y)\,, \tag{3}$$

wherein $f(1, 1)$ represents what $f(1, y)$ becomes when y is made equal to 1, and $f(1, 0)$ what $f(1, y)$ becomes when y is made equal to 0.

In like manner, the coefficient $f(0, y)$ gives by expansion,

$$f(0, y) = f(0, 1)\, y + f(0, 0)\,(1 - y)\,. \tag{4}$$

Substitute in (2) for $f(1, y)$, $f(0, y)$, their values given in (3) and (4), and we have

$$f(x, y) = f(1, 1)\, xy + f(1, 0)\, x(1 - y) + f(0, 1)\,(1 - x)\, y + f(0, 0)\,(1 - x)\,(1 - y)\,, \tag{5}$$

for the expansion required. Here $f(1, 1)$ represents what $f(x, y)$ becomes when we make therein $x = 1$, $y = 1$; $f(1, 0)$ represents what $f(x, y)$ becomes when we make therein $x = 1$, $y = 0$, and so on for the rest.

Thus, if $f(x, y)$ represent the function $\dfrac{1-x}{1-y}$, we find

$$f(1, 1) = \frac{0}{0}, \qquad f(1, 0) = \frac{0}{1}, \qquad f(0, 1) = \frac{1}{0}, \qquad f(0, 0) = 1$$

whence the expansion of the given function is

$$\frac{0}{0} xy + 0x(1 - y) + \frac{1}{0}(1 - x)\, y + (1 - x)(1 - y)\,.$$

It will in the next chapter be seen that the forms $\frac{0}{0}$ and $\frac{1}{0}$, the former of which is known to mathematicians as the symbol of indeterminate quantity, admit, in such expressions as the above, of a very important logical interpretation.

Suppose, in the next place, that we have three symbols in the function to be expanded, which we may represent under the general form $f(x, y, z)$. Proceeding as before, we get

The *complete* expansion of $f(x, y)$, that is, the expansion about x, y. (This terminology is used on pages 101, 104.) Equation (5) is called a Theorem on page 76.

Formally expanding proper rational functions is only used as a mnemonic device in Boole's algorithm to solve a polynomial equation for a variable. This algorithm is used frequently in his chapters on probability.

$$
\begin{aligned}
f(x,y,z) &= f(1,1,1)\,xyz + f(1,1,0)\,xy(1-z) + f(1,0,1)\,x(1-y)\,z \\
&+ f(1,0,0)\,x(1-y)(1-z) + f(0,1,1)(1-x)\,yz \\
&+ f(0,1,0)(1-x)\,y(1-z) + f(0,0,1)(1-x)(1-y)\,z \\
&+ f(0,0,0)(1-x)(1-y)(1-z)
\end{aligned}
$$

Expanding f about x, y, z.

in which $f(1,1,1)$ represents what the function $f(x,y,z)$ becomes when we make therein $x = 1$, $y = 1$, $z = 1$, and so on for the rest.

11. It is now easy to see the general law which determines the expansion of any proposed function, and to reduce the method of effecting the expansion to a rule. But before proceeding to the expression of such a rule, it will be convenient to premise the following observations:—

Boole did not have a notation for a general expansion—here is one. Let $C_1(x) = x$ and $C_0(x) = 1 - x$. For σ a list of 0s and 1s of the same length as a list $\vec{x}$ of m variables, let $C_\sigma(\vec{x}) := \prod_i C_{\sigma(i)}(x_i)$. Then $f(\vec{x}) = \prod_\sigma f(\sigma)C_\sigma(\vec{x})$, where the product is over all σ.

Each form of expansion that we have obtained consists of certain terms, into which the symbols x, y, &c. enter, multiplied by coefficients, into which those symbols do not enter. Thus the expansion of $f(x)$ consists of two terms, x and $1 - x$, multiplied by the coefficients $f(1)$ and $f(0)$ respectively. And the expansion of $f(x,y)$ consists of the four terms xy, $x(1-y)$, $(1-x)\,y$, and $(1-x)$, $(1-y)$, multiplied by the coefficients $f(1,1)$, $f(1,0)$, $f(0,1)$, $f(0,0)$, respectively. The terms x, $1 - x$, in the former case, and the terms xy, $x(1-y)$, &c., in the latter, we shall call the *constituents* of the expansion. It is evident that they are in form independent of the form of the function to be expanded. Of the constituent xy, x and y are termed the *factors*.

CONSTITUENTS:
The constituents for one variable x and for two variables x, y are given.

NOTE: Margin notes will use the notions of x-constituents and x, y-constituents, and in general of $\vec{x}$-constituents $C_\sigma(\vec{x})$ for $\vec{x}$ a list of variables. $\mathfrak{C}(\vec{x})$ denotes the set of $\vec{x}$-constituents.

The general rule of development will therefore consist of two parts, the first of which will relate to the formation of the *constituents* of the expansion, the second to the determination of their respective coefficients. It is as follows:

1st. *To expand any function of the symbols x, y, z.* —Form a series of constituents in the following manner: Let the first constituent be the product of the symbols; change in this product any symbol z into $1 - z$, for the second constituent. Then in both these change any other symbol y into $1 - y$, for two more constituents. Then in the four constituents thus obtained change any other symbol x into $1 - x$, for four new constituents, and so on until the number of possible changes is exhausted.

An algorithm to form all 8 constituents of x, y, z. The "and so on until ..." indicates that Boole meant to consider an arbitrary number of variables.

2ndly. *To find the coefficient of any constituent.* —If that

How to find the coefficient of a constituent in the expansion of a function.

constituent involves x as a factor, change in the original function x into 1; but if it involves $1 - x$ as a factor, change in the original function x into 0. Apply the same rule with reference to the symbols y, z, &c.: the final calculated value of the function thus transformed will be the coefficient sought.

The sum of the constituents, multiplied each by its respective coefficient, will be the expansion required.

EXPANSION THEOREM: How to find the expansion of f.

12. It is worthy of observation, that a function may be developed with reference to symbols which it does not explicitly contain. Thus if, proceeding according to the rule, we seek to develop the function $1-x$, with reference to the symbols x and y, we have,

When	x = 1	and	y = 1	the given function	=	0 .
	x = 1	”	y = 0	” ”	=	0 .
	x = 0	”	y = 1	” ”	=	1.
	x = 0	”	y = 0	” ”	=	1.

Whence the development is

$$1 - x \ = \ 0xy + 0x\,(1-y) + (1-x)\,y + (1-x)\,(1-y) \ ;$$

and this is a true development. The addition of the terms $(1-x)\,y$ and $(1-x)\,(1-y)$ produces the function $1-x$.

The symbol 1 thus developed according to the rule, with respect to the symbol x, gives

$$x + 1 - x.$$

Developed with respect to x and y, it gives

$$xy + x\,(1-y) + (1-x)\,y + (1-x)\,(1-y)\,.$$

Similarly developed with respect to any set of symbols, it produces a series consisting of all possible constituents of those symbols.

13. A few additional remarks concerning the nature of the general expansions may with propriety be added. Let us take, for illustration, the general theorem (5), which presents the type of development for functions of two logical symbols.

In the first place, that theorem is perfectly true and intelligible when x and y are quantitative symbols of the species considered in this chapter, whatever algebraic form may be assigned to the function $f\,(x,y)$, and it may therefore be intelligibly employed

“algebraic form” means rational function.

in any stage of the process of analysis intermediate between the change of interpretation of the symbols from the logical to the quantitative system above referred to, and the final restoration of the logical interpretation.

This is based on **R01**.

Secondly. The theorem is perfectly true and intelligible when x and y are logical symbols, provided that the form of the function $f(x, y)$ is such as to represent a *class or collection of things,* in which case the second member is always logically interpretable. For instance, if $f(x, y)$ represent the function $1 - x + xy$, we obtain on applying the theorem

If $f(x, y)$ is idempotent, meaning $f(x, y)^2 = f(x, y)$ holds in $\mathbb{Z}_{01}$, then its expansion is always logically interpretable, that is, it is totally defined.

$$\begin{aligned} 1 - x + xy &= xy + 0x(1-y) + (1-x)y + (1-x)(1-y), \\ &= xy + (1-x)y + (1-x)(1-y), \end{aligned}$$

and this result is intelligible and true.

Thus we may regard the theorem as true and intelligible for quantitative symbols of the species above described, *always*; for logical symbols, *always when interpretable*. Whensoever therefore it is employed in this work it must be understood that the symbols x, y are quantitative and of the particular species referred to, if the expansion obtained is not interpretable.

The "it must be understood" comment is unnecessary. It is perhaps convenient to think of working in Boole's algebra of 0 and 1 as an aid to remembering the permissible derivation steps.

Given an equation $f = 0$, completely expanding f is a step towards an interpretable form of $f = 0$.

But though the expansion is not always immediately interpretable, it always conducts us at once to results which are interpretable. Thus the expression $x - y$ gives on development the form

$$x(1-y) - y(1-x),$$

which is not generally interpretable. We cannot take, in thought, from the class of things which are x's and not y's, the class of things which are y's and not x's, because the latter class is not contained in the former. But if the form $x - y$ presented itself as the first member of an equation, of which the second member was 0, we should have on development

$$x(1-y) - y(1-x) = 0.$$

Given an equation $s = t$, in modern terminology s is the left side of the equation, t the right side. Boole called s the first member of the equation, t the second member.

Now it will be shown in the next chapter that the above equation, x and y being regarded as quantitative and of the species described, is resolvable at once into the two equations

$$x(1-y) = 0, \quad y(1-x) = 0,$$

and these equations are directly interpretable in Logic when logical interpretations are assigned to the symbols x and y. And it may be remarked, that though *functions* do not necessarily become interpretable upon development, yet *equations* are always reducible by this process to interpretable forms.

Note: Polynomial functions become interpretable on development iff they are idempotent. (See pages 93, 122.)

14. The following Proposition establishes some important properties of constituents. In its enunciation the symbol t is employed to represent indifferently any constituent of an expansion. Thus if the expansion is that of a function of two symbols x and y, t represents any of the four forms xy, $x(1-y)$, $(1-x)y$, and $(1-x)(1-y)$. Where it is necessary to represent the constituents of an expansion by single symbols, and yet to distinguish them from each other, the distinction will be marked by suffixes. Thus t_1 might be employed to represent xy, t_2 to represent $x(1-y)$, and so on.

Proposition III.

Prop. III. Three main properties of constituents:

Recall: $\mathfrak{C}(\vec{x})$ is the set of $\vec{x}$-constituents.

For $s, t \in \mathfrak{C}(\vec{x})$ one has:

- $t^2 = t$
- $st = 0 \quad \text{if} \quad s \neq t$
- $1 = \sum \mathfrak{C}(\vec{x})$.

Any single constituent t of an expansion satisfies the law of duality whose expression is

$$t(1-t) = 0.$$

The product of any two distinct constituents of an expansion is equal to 0, and the sum of all the constituents is equal to 1.

1st. Consider the particular constituent xy. We have

$$xy \times xy = x^2y^2.$$

But $x^2 = x$, $y^2 = y$, by the fundamental law of class symbols; hence

$$xy \times xy = xy.$$

Or representing xy by t,

$$t \times t = t,$$

or

$$t(1-t) = 0.$$

Similarly the constituent $x(1-y)$ satisfies the same law. For we have

$$x^2 = x, \qquad (1-y)^2 = 1-y,$$

$$\therefore \ \left\{x(1-y)\right\}^2 = x(1-y), \qquad \text{or} \qquad t(1-t) = 0.$$

Now every factor of every constituent is either of the form x or of the form $1-x$. Hence the square of each factor is equal to that factor, and therefore the square of the product of the factors, i.e. of the constituent, is equal to the constituent; wherefore t representing any constituent, we have

$$t^2 = t, \quad \text{or} \quad t(1-t) = 0.$$

2ndly. The product of any two constituents is 0. This is evident from the general law of the symbols expressed by the equation $x(1-x) = 0$; for whatever constituents in the same expansion we take, there will be at least one factor x in the one, to which will correspond a factor $1-x$ in the other.

3rdly. The sum of all the constituents of an expansion is unity. This is evident from addition of the two constituents x and $1-x$, or of the four constituents, xy, $x(1-y)$, $(1-x)y$, $(1-x)(1-y)$. But it is also, and more generally, proved by expanding 1 in terms of any set of symbols (V. 12). The constituents in this case are formed as usual, and all the coefficients are unity.

15. With the above Proposition we may connect the following.

Proposition IV.

If V represent the sum of any series of constituents, the separate coefficients of which are 1, then is the condition satisfied,

$$V(1-V) = 0.$$

A sum of distinct constituents is idempotent.
Better: it is totally defined.

Let t_1, $t_2 \ldots t_n$ be the constituents in question, then

$$V = t_1 + t_2 \cdots + t_n.$$

Squaring both sides, and observing that $t_1^2 = t_1$, $t_1 t_2 = 0$, &c., we have

$$V^2 = t_1 + t_2 \cdots + t_n;$$

whence

$$V = V^2.$$

Therefore

$$V(1-V) = 0.$$

Chapter VI

Of the General Interpretation of Logical Equations, and the Resulting Analysis of Propositions. Also, of the Condition of Interpretability of Logical Functions.

Three cases of interpreting an equation as proposition(s) are considered:
Form I: The case $V = 0$ for V a polynomial;
Form II: The case $V = 1$ for V a polynomial;
Form III: The case $w = V$ where V results from formally solving a polynomial equation $f(\vec{x}, w) = 0$ for one of its variables w. This is done by a formal expansion of f about w, giving $Ew + E'(1 - w) = 0$, where E is $f(\vec{x}, 1)$ and E' is $f(\vec{x}, 0)$, and then solving for w by formal division, giving $w = E'/(E' - E)$. This rational expression is V.

1. It has been observed that the complete expansion of any function by the general rule demonstrated in the last chapter, involves two distinct sets of elements, viz., the constituents of the expansion, and their coefficients. I propose in the present chapter to inquire, first, into the interpretation of constituents, and afterwards into the mode in which that interpretation is modified by the coefficients with which they are connected.

By "any equation or function" is meant 'any rational function or equation involving rational functions'.

The terms "logical equation," "logical function," &c., will be employed generally to denote any equation or function involving the symbols x, y, &c., which may present itself either in the expression of a system of premises, or in the train of symbolical results which intervenes between the premises and the conclusion. If that function or equation is in a form not immediately interpretable in Logic, the symbols x, y, &c., must be regarded as quantitative symbols of the species described in previous chapters (II. 15), (V. 6), as satisfying the law,

Replace "must be regarded" by 'can be regarded'. This is just an application of **R01** (II.15 and V.6).

$$x\,(1 - x) \;=\; 0\,.$$

By the problem, then, of the interpretation of any such logical function or equation, is meant the reduction of it to a form in which, when logical values are assigned to the symbols x, y, &c., it shall become interpretable, together with the resulting interpretation. These conventional definitions are in accordance with the general principles for the conducting of the method of this treatise, laid down in the previous chapter.

Sounds like the reduction of a function to a totally defined term, and of an equation to an equation with totally defined terms. The former is not always possible, the latter is.

Proposition I.

2. *The constituents of the expansion of any function of the logical symbols x, y, &c., are interpretable, and represent the several exclusive divisions of the universe of discourse, formed by the predication and denial in every possible way of the qualities denoted by the symbols x, y, &c.*

Constituents partition the universe, as in a Venn diagram.

For greater distinctness of conception, let it be supposed that the function expanded involves two symbols x and y, with reference to which the expansion has been effected. We have then the following constituents, viz.:

$$xy, \; x(1-y), \; (1-x)y, \; (1-x)(1-y).$$

Of these it is evident, that the first xy represents that class of objects which at the same time possess both the elementary qualities expressed by x and y, and that the second $x(1-y)$ represents the class possessing the property x, but not the property y. In like manner the third constituent represents the class of objects which possess the property represented by y, but not that represented by x; and the fourth constituent $(1-x)(1-y)$, represents that class of objects, the members of which possess neither of the qualities in question.

Thus the constituents in the case just considered represent all the four classes of objects which can be described by affirmation and denial of the properties expressed by x and y. Those classes are distinct from each other. No member of one is a member of another, for each class possesses some property or quality contrary to a property or quality possessed by any other class. Again, these classes together make up the universe, for there is no object which may not be described by the presence or the absence of a proposed quality, and thus each individual thing in the universe may be referred to some one or other of the four classes made by the possible combination of the two given classes x and y, and their contraries.

The remarks which have here been made with reference to the constituents of $f(x, y)$ are perfectly general in character. The constituents of any expansion represent classes—those classes

General properties of constituents; see page 78.

are mutually distinct, through the possession of contrary qualities, and they together make up the universe of discourse.

3. These properties of constituents have their expression in the theorems demonstrated in the conclusion of the last chapter, and might thence have been deduced. From the fact that every constituent satisfies the fundamental law of the individual symbols, it might have been conjectured that each constituent would represent a class. From the fact that the product of any two constituents of an expansion vanishes, it might have been concluded that the classes they represent are mutually exclusive. Lastly, from the fact that the sum of the constituents of an expansion is unity, it might have been inferred, that the classes which they represent, together make up the universe.

4. Upon the laws of constituents and the mode of their interpretation above determined, are founded the analysis and the interpretation of logical equations. That all such equations admit of interpretation by the theorem of development has already been stated. I propose here to investigate the forms of possible solution which thus present themselves in the conclusion of a train of reasoning, and to show how those forms arise. Although, properly speaking, they are but manifestations of a single fundamental type or principle of expression, it will conduce to clearness of apprehension if the minor varieties which they exhibit are presented separately to the mind. The forms, which are three in number, are as follows:

This means the forms of equations that one derives, and how to interpret them.

FORM I.

5. The form we shall first consider arises when any logical equation $V = 0$ is developed, and the result, after resolution into its component equations, is to be interpreted. The function is supposed to involve the logical symbols x, y, &c., in combinations which are not fractional. Fractional combinations indeed only arise in the class of problems which will be considered when we come to speak of the third of the forms of solution above referred to.

In Form I, V is a polynomial. Only Form III ($w = V$) deals with proper rational V.

Proposition II.

To interpret the logical equation $V = 0$.

For simplicity let us suppose that V involves but two symbols,

x and y, and let us represent the development of the given equation by

$$axy + bx(1-y) + c(1-x)y + d(1-x)(1-y) \;=\; 0\,; \qquad (1)$$

a, b, c, and d being definite numerical constants.

Now, suppose that any coefficient, as a, does not vanish. Then multiplying each side of the equation by the constituent xy, to which that coefficient is attached, we have

$$axy \;=\; 0\,,$$

whence, as a does not vanish,

$$xy \;=\; 0\,,$$

and this result is quite independent of the nature of the other coefficients of the expansion. Its interpretation, on assigning to x and y their logical significance, is "No individuals belonging at once to the class represented by x, and the class represented by y, exist."

But if the coefficient a *does* vanish, the term axy does not appear in the development (1), and, therefore, the equation $xy = 0$ cannot thence be deduced.

In like manner, if the coefficient b does not vanish, we have

$$x(1-y) \;=\; 0\,,$$

which admits of the interpretation, "There are no individuals which at the same time belong to the class x, and do not belong to the class y."

Either of the above interpretations may, however, as will subsequently be shown, be exhibited in a different form.

The sum of the distinct interpretations thus obtained from the several terms of the expansion whose coefficients do not vanish, will constitute the complete interpretation of the equation $V = 0$. The analysis is essentially independent of the number of logical symbols involved in the function V, and the object of the proposition will, therefore, in all instances, be attained by the following Rule: —

RULE.—*Develop the function V, and equate to 0 every constituent whose coefficient does not vanish. The interpretation of these results collectively will constitute the interpretation of the given equation.*

By "the development" of a polynomial equation $V = 0$ he meant putting the expansion of V equal to 0.

This is the first time Boole used a variant of the word "multiply" within his algebra of logic.

An application of the torsion-free property.

By the "sum of the distinct interpretations" he meant the collection of the distinct interpretations.

Given an equation $V = 0$, V a polynomial, let $\mathfrak{C}(V)$ be the collection of constituents t with a nonzero coefficient in the expansion of V. Then $V = 0$ is equivalent to the collection of constituent equations

$$\{t = 0 : t \in \mathfrak{C}(V)\}.$$

NOTE: It is easy to show that this collection of constituent equations, and thus $V = 0$, is equivalent to the single equation $\sum \mathfrak{C}(V) \;=\; 0$.
$\sum \mathfrak{C}(V)$ is not only idempotent, it is totally defined.

6. Let us take as an example the definition of "clean beasts," laid down in the Jewish law, viz., "Clean beasts are those which both divide the hoof and chew the cud," and let us assume

x = clean beasts;
y = beasts dividing the hoof;
z = beasts chewing the cud;

Then the given proposition will be represented by the equation

$$x = yz$$

which we shall reduce to the form

$$x - yz = 0,$$

and seek that form of interpretation to which the present method leads. Fully developing the first member, we have

$$0xyz + xy(1-z) + x(1-y)z + x(1-y)(1-z) - (1-x)yz + 0(1-x)y(1-z) + 0(1-x)(1-y)z + 0(1-x)(1-y)(1-z).$$

Whence the terms, whose coefficients do not vanish, give

$$xy(1-z) = 0, \; xz(1-y) = 0, \; x(1-y)(1-z) = 0, \; (1-x)yz = 0.$$

These equations express a denial of the existence of certain classes of objects, viz.:

1st. Of beasts which are clean, and divide the hoof, but do not chew the cud.

2nd. Of beasts which are clean, and chew the cud, but do not divide the hoof.

3rd. Of beasts which are clean, and neither divide the hoof nor chew the cud.

4th. Of beasts which divide the hoof, and chew the cud, and are not clean.

Now all these several denials are really involved in the original proposition. And conversely, if these denials be granted, the original proposition will follow as a necessary consequence. They are, in fact, the separate elements of that proposition. Every primary proposition can thus be resolved into a series of denials of the existence of certain defined classes of things, and may, from that system of denials, be itself reconstructed. It might here be asked, how it is possible to make an assertive proposition

By a "series of denials" is meant the interpretation of a collection of constituent equations $t = 0$ as propositions.

out of a series of denials or negations? From what source is the positive element derived? I answer, that the mind assumes the existence of a universe not *à priori* as a fact independent of experience, but either *à posteriori* as a deduction from experience, or *hypothetically* as a foundation of the possibility of assertive reasoning. Thus from the Proposition, "There are no men who are not fallible," which is a negation or denial of the existence of "infallible men," it may be inferred either hypothetically, "All men (if men exist) are fallible," or absolutely, (experience having assured us of the existence of the race), "All men are fallible."

The form in which conclusions are exhibited by the method of this Proposition may be termed the form of "Single or Conjoint Denial."

This means one has *one* or *several* equations $t = 0$ to interpret as propositions.

FORM II.

7. As the previous form was derived from the development and interpretation of an equation whose second member is 0, the present form, which is supplementary to it, will be derived from the development and interpretation of an equation whose second member is 1. It is, however, readily suggested by the analysis of the previous Proposition.

The case $V = 1$, where V is polynomial as in Form I.

Thus in the example last discussed we deduced from the equation

$$x - yz = 0$$

the conjoint denial of the existence of the classes represented by the constituents

$$xy\,(1-z)\,, \quad xz\,(1-y)\,, \quad x\,(1-y)\,(1-z)\,, \quad (1-x)\,yz,$$

whose coefficients were not equal to 0. It follows hence that the remaining constituents represent classes which make up the universe. Hence we shall have

$$xyz + (1-x)\,y\,(1-z) + (1-x)\,(1-y)\,z + (1-x)\,(1-y)\,(1-z) = 1.$$

This is equivalent to the affirmation that all existing things belong to some one or other of the following classes, viz.:

1st. Clean beasts both dividing the hoof and chewing the cud.

2nd. Unclean beasts dividing the hoof, but not chewing the cud.

3rd. Unclean beasts chewing the cud, but not dividing the hoof.

4th. Things which are neither clean beasts, nor chewers of the cud, nor dividers of the hoof.

This form of conclusion may be termed the form of "Single or Disjunctive Affirmation," —single when but one constituent appears in the final equation; disjunctive when, as above, more constituents than one are there found.

This newly minted terminology is never used again.

Any equation, $V = 0$, wherein V satisfies the law of duality, may also be made to yield this form of interpretation by reducing it to the form $1 - V = 1$, and developing the first member. The case, however, is really included in the next general form. Both the previous forms are of slight importance compared with the following one.

FORM III.

8. In the two preceding cases the functions to be developed were equated to 0 and to 1 respectively. In the present case I shall suppose the corresponding function equated to any logical symbol w. We are then to endeavour to interpret the equation $V = w$, V being a function of the logical symbols x, y, z, &c. In the first place, however, I deem it necessary to show how the equation $V = w$, or, as it will usually present itself, $w = V$, arises.

This is not a clear rollout of this case. Form III arises when solving a polynomial equation $f(\vec{x}, w) = 0$ for a variable w. Expanding $f(\vec{x}, w)$ about w leads to the equation

$$\big(f(\vec{x},0) - f(\vec{x},1)\big)w = f(\vec{x},0).$$

Formal division leads to

$$w = \frac{f(\vec{x},0)}{f(\vec{x},0) - f(\vec{x},1)},$$

and the fraction on the right side is V. This is the only situation where Boole used proper rational functions. More details are given in the margin on the next page.
The discussion of Form III runs from this page to page 93.

Let us resume the definition of "clean beasts," employed in the previous examples, viz., "Clean beasts are those which both divide the hoof and chew the cud," and suppose it required to determine the relation in which "beasts chewing the cud" stand to "clean beasts" and "beasts dividing the hoof." The equation expressing the given proposition is

$$x = yz,$$

and our object will be accomplished if we can determine z as an interpretable function of x and y.

Now treating x, y, z as symbols of quantity subject to a peculiar law, we may deduce from the above equation, by solution,

$$z = \frac{x}{y}.$$

But this equation is not at present in an interpretable form. If we can reduce it to such a form it will furnish the relation required.

On developing the second member of the above equation, we have

On *formally* developing...

$$z = xy + \frac{1}{0}x(1-y) + 0(1-x)y + \frac{0}{0}(1-x)(1-y),$$

and it will be shown hereafter (Prop. 3) that this admits of the following interpretation:

"Beasts which chew the cud consist of all clean beasts (which also divide the hoof), together with an indefinite remainder (some, none, or all) of unclean beasts which do not divide the hoof."

SOLUTION THEOREM
The following contains what will be called Boole's Solution Theorem.

9. Now the above is a particular example of a problem of the utmost generality in Logic, and which may thus be stated:—"Given any logical equation connecting the symbols x, y, z, w, required an interpretable expression for the relation of the class represented by w to the classes represented by the other symbols x, y, z, &c."

Goal: Solve a polynomial equation $f(\vec{x}, w) = 0$ for w.

The solution of this problem consists in all cases in determining, from the equation *given*, the expression of the above symbol w, in terms of the other symbols, and rendering that expression interpretable by development. Now the equation given is always of the first degree with respect to each of the symbols involved. The required expression for w can therefore always be found. In fact, if we develop the given equation, whatever its form may be with respect to w, we obtain an equation of the form

Step 1: Expand $f(\vec{x}, w)$ about w. In item (1) we have $E = f(\vec{x}, 1)$ and $E' = f(\vec{x}, 0)$.

$$Ew + E'(1-w) = 0, \tag{1}$$

Step 2: Put the terms involving w on the right side.

E and E' being functions of the remaining symbols. From the above we have

$$E' = (E' - E)\,w.$$

Step 3: Formally solve for w by division in (2).

Therefore

$$w = \frac{E'}{E' - E} \tag{2}$$

Step 4: Formally expand the right side of (2).

and expanding the second member by the rule of development, it will only remain to interpret the result in logic by the next proposition.

Step 5: Use the expansion to find (a) an equation $g(\vec{x}) = 0$ in the variables $\vec{x}$ which gives necessary and sufficient conditions on $\vec{x}$ in order for a solution w of $f(\vec{x}, w) = 0$ to exist, and (b) given that $g(\vec{x}) = 0$ is satisfied, the expansion leads to the general solution for w as a function of $\vec{x}$ (and possibly a parameter v).

If the fraction $\dfrac{E'}{E'-E}$ has common factors in its numerator and denominator, we are not permitted to reject them, unless they are mere numerical constants. For the symbols x, y, &c., regarded as quantitative, may admit of such values 0 and 1 as to cause the common factors to become equal to 0, in which case the algebraic rule of reduction fails. This is the case contemplated in our remarks on the failure of the algebraic axiom of division (II. 14). To *express* the solution in the form (2), and without attempting to perform any unauthorized reductions, to interpret the result by the theorem of development, is a course strictly in accordance with the general principles of this treatise.

If the relation of the class expressed by $1 - w$ to the other classes, x, y, &c. is required, we deduce from (1), in like manner as above,

$$1 - w \ = \ \frac{E}{E - E'},$$

to the interpretation of which also the method of the following Proposition is applicable:

Proposition III.

10. *To determine the interpretation of any logical equation of the form $w = V$, in which w is a class symbol, and V a function of other class symbols quite unlimited in its form.*

Let the second member of the above equation be fully expanded. Each coefficient of the result will belong to some one of the four classes, which, with their respective interpretations, we proceed to discuss.

1st. Let the coefficient be 1. As this is the symbol of the universe, and as the product of any two class symbols represents those individuals which are found in both classes, any constituent which has unity for its coefficient must be interpreted without limitation, i.e. the whole of the class which it represents is implied.

2nd. Let the coefficient be 0. As in Logic, equally with Arithmetic, this is the symbol of Nothing, no part of the class

Example: One can reduce $2x/2x$ to x/x, but not to $2/2$. Cancelling the x would be an "unauthorized reduction".

(2) is just a formal solution that needs to be 'understood'.

There is nothing in the general principles in this treatise that allows the introduction of a new operation, much less to carry out a formal expansion of an expression involving it.

The formal expansion of a quotient is used to determine the solution. "Quite unlimited in its form" meant a rational function. Boole said the coefficients of the constituents will be of four kinds: 1, 0, $\frac{0}{0}$, and otherwise (which turns out to be $\frac{m}{n}$ for $0 \neq m \neq n$.) Constituents with coefficient 0 are deleted from the expansion, all coefficients $\frac{0}{0}$ are treated as arbitrary class-parameters, and all constituents with coefficient $\frac{m}{n}$ with $0 \neq m \neq n$ are removed from the expansion and set equal to 0. Boole tended to use the canonical coefficient $\frac{1}{0}$ for any $\frac{m}{n}$ with $0 \neq m \neq n$ (see page 156).

represented by the constituent to which it is prefixed must be taken.

3rd. Let the coefficient be of the form $\frac{0}{0}$. Now, as in Arithmetic, the symbol $\frac{0}{0}$ represents an *indefinite number*, except when otherwise determined by some special circumstance, analogy would suggest that in the system of this work the same symbol should represent an *indefinite class*. That this is its true meaning will be made clear from the following example:

Let us take the Proposition, "Men not mortal do not exist;" represent this Proposition by symbols; and seek, in obedience to the laws to which those symbols have been proved to be subject, a reverse definition of "mortal beings," in terms of "men."

Now if we represent "men" by y, and "mortal beings" by x, the Proposition, "Men who are not mortals do not exist," will be expressed by the equation

$$y\,(1-x) \;=\; 0\,,$$

from which we are to seek the value of x. Now the above equation gives

$$y - yx \;=\; 0\,, \quad \text{or} \quad yx \;=\; y.$$

Were this an ordinary algebraic equation, we should, in the next place, divide both sides of it by y. But it has been remarked in Chap. **11.** that the operation of division cannot be *performed* with the symbols with which we are now engaged. Our resource, then, is to *express* the operation, and develop the result by the method of the preceding chapter. We have, then, first,

Since division is not one of the fundamental operations, let's just try formal division and formal expansion to see if they lead to the solution.

$$x \;=\; \frac{y}{y},$$

and, expanding the second member as directed,

$$x \;=\; y + \frac{0}{0}\,(1-y)\,.$$

This implies that mortals (x) consist of all men (y), together with such a remainder of beings which are not men ($1-y$), as be indicated by the coefficient $\frac{0}{0}$. Now let us inquire what remainder of "not men" is implied by the premiss. It might happen that the remainder included all the beings who are not men, or it might include only some of them, and not others, or it might include none, and any one of these assumptions would be in perfect accordance with our premiss. In other words, whether those beings which are not men are *all*, or *some*, or *none*, of them *mortal*, the truth of the premiss which virtually asserts that all men are mortal, will be equally unaffected, and therefore the expression $\frac{0}{0}$ here indicates that *all*, *some*, or *none* of the class to whose expression it is affixed must be taken.

Although the above determination of the significance of the symbol $\frac{0}{0}$ is founded only upon the examination of a particular case, yet the principle involved in the demonstration is general, and there are no circumstances under which the symbol can present itself to which the same mode of analysis is inapplicable. We may properly term $\frac{0}{0}$ an *indefinite class symbol*, and may, if convenience should require, replace it by an uncompounded symbol v, subject to the fundamental law, $v(1-v) = 0$.

This v is not the v of 'some', introduced on page 61.

4th. It may happen that the coefficient of a constituent in an expansion does not belong to any of the previous cases. To ascertain its true interpretation when this happens, it will be necessary to premise the following theorem:

The 4th case is when the coefficient is of the form m/n, $0 \neq m \neq n$. (Boole never stated this.)

11. Theorem.—*If a function V, intended to represent any class or collection of objects, w, be expanded, and if the numerical coefficient, a, of any constituent in its development, do not satisfy the law,*

$$a\,(1-a) \;=\; 0\,,$$

then the constituent in question must be made equal to 0.

This theorem, which is valid for V a polynomial, is claimed by Boole to be valid for the formal expansion of a rational function V (obtained formally in the solution $w = V$ of a polynomial equation $f(\vec{x}, w) = 0$). To obtain the desired conclusions, Boole conveniently decided to treat $0/0$ and $1/0$ as numerical coefficients, with $0/0$ assumed to be idempotent (see bottom of next page), and $1/0$ considered, via dubious limit argument, to be non-idempotent.

To prove the theorem generally, let us represent the expansion given, under the form

$$w \;=\; a_1t_1 + a_2t_2 + a_3t_3 + \&c., \tag{1}$$

in which t_1, t_2, t_3, &c. represent the constituents, and a_1, a_2, a_3, &c. the coefficients; let us also suppose that a_1 and a_2 do not satisfy the law

$$a_1\,(1-a_1) \;=\; 0\,, \qquad a_2\,(1-a_2) \;=\; 0\,;$$

but that the other coefficients are subject to the law in question, so that we have

$$a_3{}^2 \;=\; a_3, \;\&\text{c.}$$

Now multiply each side of the equation (1) by itself. The result will be

$$w \;=\; a_1{}^2 t_1 + a_2{}^2 t_2 + \;\&\text{c.} \tag{2}$$

This is evident from the fact that it must represent the development of the equation

$$w \;=\; V^2,$$

but it may also be proved by actually squaring (1), and observing that we have

$$t_1{}^2 \;=\; t_1\,, \qquad t_2{}^2 \;=\; t_2, \qquad t_1 t_2 \;=\; 0\,, \qquad \&\text{c.}$$

by the properties of constituents. Now subtracting (2) from (1), we have

$$\left(a_1 - a_1{}^2\right) t_1 + \left(a_2 - a_2{}^2\right) t_2 \;\;=\;\; 0\,.$$

Or,

$$a_1\,(1 - a_1)\, t_1 + a_2\,(1 - a_2)\, t_2 \;\;=\;\; 0\,.$$

Multiply the last equation by t_1; then since $t_1 t_2 = 0$, we have

$$a_1\,(1 - a_1)\, t_1 \;=\; 0\,, \qquad \text{whence } t_1 \;=\; 0\,.$$

In like manner multiplying the same equation by t_2, we have

$$a_2\,(1 - a_2)\, t_2 \;=\; 0\,, \qquad \text{whence } t_2 \;=\; 0\,.$$

Thus it may be shown generally that any constituent whose coefficient is not subject to the same fundamental law as the symbols themselves must be separately equated to 0. The usual form under which such coefficients occur is $\frac{1}{0}$. This is the algebraic symbol of infinity. Now the nearer any number approaches to infinity (allowing such an expression), the more does it depart from the condition of satisfying the fundamental law above referred to.

The symbol $\frac{0}{0}$, whose interpretation was previously discussed, does not necessarily disobey the law we are here considering, for it admits of the numerical values 0 and 1 indifferently. Its actual interpretation, however, as an indefinite class symbol, cannot, I conceive, except upon the ground of analogy, be deduced

The torsion-free property of the algebra of numbers, namely if $ap(\vec{x}) = 0$ and a is a non-zero number then $p(\vec{x}) = 0$, for any polynomial $p(\vec{x})$. By **R01** it is valid for Boole's algebra of logic.

If $f(\vec{x}, w)$ is idempotent then the only 4th case coefficient is $1/0$. Otherwise we only know that the form is m/n, $0 \neq m \neq n$ (see page 156). Boole conveniently decided that such coefficients were not idempotent.

It was also convenient to assume that $0/0$ was idempotent. As far as taking on the values 0 and 1 indifferently, in analysis $0/0$ can take on any value considered as a limit.

from its arithmetical properties, but must be established experimentally.

12. We may now collect the results to which we have been led, into the following summary:

1st. The symbol 1, as the coefficient of a term in a development, indicates that the whole of the class which that constituent represents, is to be taken.

2nd. The coefficient 0 indicates that none of the class are to be taken.

3rd. The symbol $\frac{0}{0}$ indicates that a perfectly *indefinite* portion of the class, i.e. *some*, *none*, or *all* of its members are to be taken.

4th. Any other symbol as a coefficient indicates that the constituent to which it is prefixed must be equated to 0.

It follows hence that if the solution of a problem, obtained by development, be of the form

$$w = A + 0B + \frac{0}{0}C + \frac{1}{0}D,$$

Changing all coefficients m/n, $0 \neq m \neq n$, to $1/0$, as Boole preferred to do (see page 156), gives this form of the expansion of V in $w = V$ with V rational.

that solution may be resolved into the two following equations, viz.,

$$w = A + vC, \tag{3}$$
$$D = 0, \tag{4}$$

v being an indefinite class symbol. The interpretation of (3) shows what elements enter, or may enter, into the composition of w, the class of things whose definition is required; and the interpretation of (4) shows what relations exist among the elements of the original problem, in perfect independence of w.

The indefinite class symbol here is not the same as that previously used for "some". This v can be 0 (see the 3rd item above).

Such are the canons of interpretation. It may be added, that they are universal in their application, and that their use is always unembarrassed by exception or failure.

13. *Corollary.*—If V be an independently interpretable logical function, it will satisfy the symbolical law, $V(1 - V) = 0$.

This is likely meant to apply only to polynomial V.

By an independently interpretable logical function, I mean one which is interpretable, without presupposing any relation among the things represented by the symbols which it involves. Thus $x(1 - y)$ is independently interpretable, but $x - y$ is not so.

Being "independently interpretable" sounds like a totally defined algebraic term.

The latter function presupposes, as a condition of its interpretation, that the class represented by y is wholly contained in the class represented by x; the former function does not imply any such requirement.

This confirms that $x - y$ is not defined unless $y \subseteq x$.

Now if V be independently interpretable, and if w represent the collection of individuals which it contains, the equation $w = V$ will hold true without entailing as a consequence the vanishing of any of the constituents in the development of V; since such vanishing of constituents would imply relations among the classes of things denoted by the symbols in V. Hence the development of V will be of the form

$$a_1 t_1 + a_2 t_2 + \&\text{c.}$$

the coefficients a_1, a_2, &c. all satisfying the condition

$$a_1(1 - a_1) = 0, \quad a_2(1 - a_2) = 0, \quad \&c.$$

Hence by the reasoning of Prop. 4, Chap. v. the function V will be subject to the law

$$V(1 - V) = 0.$$

This result, though evident *à priori* from the fact that V is supposed to represent a class or collection of things, is thus seen to follow also from the properties of the constituents of which it is composed. The condition $V(1-V) = 0$ may be termed "the condition of interpretability of logical functions."

NOTE: V is idempotent iff V is equivalent to a totally defined algebraic term.

14. The general form of solutions, or logical conclusions developed in the last Proposition, may be designated as a "Relation between terms." I use, as before, the word "terms" to denote the parts of a proposition, whether simple or complex, which are connected by the copula "is" or "are." The classes of things represented by the individual symbols may be called the elements of the proposition.

NOTE: Terms s and t are *equivalent* if the equation $s = t$ is valid in Boole's algebra of logic, that is, $\mathbb{Z} \models_{01} s = t$.

15. Ex. 1.—Resuming the definition of "clean beasts," (VI.6), required a description of "unclean beasts."

Here, as before, x standing for " clean beasts," y for "beasts dividing the hoof," z for "beasts chewing the cud," we have

$$x = yz; \tag{5}$$

whence

$$1 - x = 1 - yz;$$

and developing the second member,

$$1 - x \ = \ y(1-z) \ + \ z(1-y) \ + \ (1-y)(1-z)\,;$$

which is interpretable into the following Proposition: *Unclean beasts are all which divide the hoof without chewing the cud, all which chew the cud without dividing the hoof, and all which neither divide the hoof nor chew the cud.*

Ex. 2.—The same definition being given, required a description of beasts which do not divide the hoof.

From the equation $x = yz$ we have

$$y \ = \ \frac{x}{z}\,;$$

therefore,

$$1 - y \ = \ \frac{z - x}{z}\,;$$

and developing the second member,

$$1 - y \ = \ 0\,xz \ + \ \frac{-1}{0}x(1-z) \ + \ (1-x)z \ + \ \frac{0}{0}(1-x)(1-z).$$

Here, according to the Rule, the term whose coefficients is $\frac{-1}{0}$, must be separately equated to 0, whence we have

$$1 - y \ = \ (1-x)z \ + \ v(1-x)(1-z),$$
$$x(1-z) \ = \ 0\,;$$

whereof the first equation gives by interpretation the Proposition: *Beasts which do not divide the hoof consist of all unclean beasts which chew the cud, and an indefinite remainder (some, none, or all) of unclean beasts which do not chew the cud.*

The second equation gives the Proposition: *There are no clean beasts which do not chew the cud.* This is one of the independent relations above referred to. We sought the direct relation of "Beasts not dividing the hoof," to "Clean beasts and beasts which chew the cud." It happens, however, that independently of any relation to beasts not dividing the hoof, there exists, in virtue of the premiss, a separate relation between clean beasts and beasts which chew the cud. This relation is also necessarily given by the process.

Ex. 3.—Let us take the following definition, viz.: "Responsible beings are all rational beings who are either free to act, or

This looks like Boole is employing rational arithmetic, that is, from $y = \frac{x}{z}$ one has

$$1 - y = 1 - \frac{x}{z} = \frac{z - x}{z}.$$

However this can be obtained without using rational arithmetic. From $x = yz$ one has $z - x = z(1 - y)$, thus formal division gives $1 - y = \frac{z - x}{z}$.

The "independent relations" are the constituent equations $t = 0$ for which the coefficient of t is not one of 0, 1, and 0/0.

have voluntarily sacrificed their freedom," and apply to it the preceding analysis.

Let	x	stand for	responsible beings.
	y	"	rational beings.
	z	"	those who are free to act,
	w	"	those who have voluntarily sacrificed their freedom of action.

In the expression of this definition I shall assume, that the two alternatives which it presents, viz.: "Rational beings free to act," and "Rational beings whose freedom of action has been voluntarily sacrificed," are mutually exclusive, so that no individuals are found at once in both these divisions. This will permit us to interpret the proposition literally into the language of symbols, as follows:

$$x = yz + yw. \tag{6}$$

Let us first determine hence the relation of "rational beings" to responsible beings, beings free to act, and beings whose freedom of action has been voluntarily abjured. Perhaps this object will be better stated by saying, that we desire to express the relation among the elements of the premiss in such a form as will enable us to determine how far rationality may be inferred from responsibility, freedom of action, a voluntary sacrifice of freedom, and their contraries.

From (6) we have

$$y = \frac{x}{z+w},$$

and developing the second member, but rejecting terms whose coefficients are 0,

$$y = \frac{1}{2}xzw + xz(1-w) + x(1-z)w + \frac{1}{0}x(1-z)(1-w) + \frac{0}{0}(1-x)(1-z)(1-w),$$

This is the first time Boole writes an explicit rational coefficient $\neq \pm 1, 0$.

whence, equating to 0 the terms whose coefficients are $\frac{1}{2}$ and $\frac{1}{0}$, we have

$$y = xz(1-w) + xw(1-z) + v(1-x)(1-z)(1-w); \tag{7}$$

$$xzw = 0; \tag{8}$$

$$x(1-z)(1-w) \;=\; 0\,; \tag{9}$$

whence by interpretation—

Direct Conclusion.—*Rational beings are all responsible beings who are either free to act, not having voluntarily sacrificed their freedom, or not free to act, having voluntarily sacrificed their freedom, together with an indefinite remainder (some, none, or all) of beings not responsible, not free, and not having voluntarily sacrificed their freedom.*

First Independent Relation.—*No responsible beings are at the same time free to act, and in the condition of having voluntarily sacrificed their freedom.*

Second.—*No responsible beings are not free to act, and at the same time in the condition of not having sacrificed their freedom.*

The independent relations above determined may, however, be put in another and more convenient form. Thus (8) gives

$$xw \;=\; \frac{0}{z} \;=\; 0z \;+\; \frac{0}{0}(1-z), \text{ on development;}$$

or,

$$xw \;=\; v(1-z)\,; \tag{10}$$

and in like manner (9) gives

$$x(1-w) \;=\; \frac{0}{1-z} \;=\; \frac{0}{0}z + 0(1-z)\,;$$

or,

$$x(1-w) \;=\; vz\,; \tag{11}$$

and (10) and (11) interpreted give the following Propositions:

1st. *Responsible beings who have voluntarily sacrificed their freedom are not free.*

2nd. *Responsible beings who have not voluntarily sacrificed their freedom are free.*

These, however, are merely different forms of the relations before determined.

16. In examining, these results, the reader must bear in mind, that the sole province of a method of inference or analysis, is to determine those relations which are necessitated by the *connexion* of the terms in the original proposition. Accordingly, in estimating the completeness with which this object is effected, we have nothing whatever to do with those other relations which may be suggested to our minds by the *meaning* of the terms employed, as distinct from their expressed connexion. Thus it seems obvious to remark, that "They who have voluntarily sacrificed their freedom are not free," this being a relation implied in the very meaning of the terms. And hence it might appear, that the first of the two independent relations assigned by the method is on the one hand needlessly limited, and on the other hand superfluous. However, if regard be had merely to the connexion of the terms in the original premiss, it will be seen that the relation in question is not liable to either of these charges. The solution, as expressed in the direct conclusion and the independent relations, conjointly, is perfectly complete, without being in any way superfluous.

If we wish to take into account the implicit relation above referred to, viz., "They who have voluntarily sacrificed their freedom are not free," we can do so by making this a distinct proposition, the proper expression of which would be

$$w = v(1 - z).$$

This equation we should have to employ together with that expressive of the original premiss. The mode in which such an examination must be conducted will appear when we enter upon the theory of systems of propositions in a future chapter. The sole difference of result to which the analysis leads is, that the first of the independent relations deduced above is superseded.

17. Ex. 4. — Assuming the same definition as in Example 2, let it be required to obtain a description of irrational persons.

We have

$$\begin{aligned} 1 - y &= 1 - \frac{x}{z + w} \\ &= \frac{z + w - x}{z + w} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2}xzw + 0xz(1-w) + 0x(1-z)w - \frac{1}{0}x(1-z)(1-w) \\ &+ (1-x)zw + (1-x)z(1-w) + (1-x)(1-z)w + \frac{0}{0}(1-x)(1-z)(1-w) \\ &= (1-x)zw + (1-x)z(1-w) + (1-x)(1-z)w + v(1-x)(1-z)(1-w) \\ &= (1-x)z + (1-x)(1-z)w + v(1-x)(1-z)(1-w), \end{aligned}$$

with $xzw = 0, \quad x(1-z)(1-w) = 0.$

The independent relations here given are the same as we before arrived at, as they evidently ought to be, since whatever relations prevail independently of the existence of a given class of objects y, prevail independently also of the existence of the contrary class $1 - y$.

The direct solution afforded by the first equation is:—*Irrational persons consist of all irresponsible beings who are either free to act, or have voluntarily sacrificed their liberty, and are not free to act; together with an indefinite remainder of irresponsible beings who have not sacrificed their liberty, and are not free to act.*

18. The propositions analyzed in this chapter have been of that species called definitions. I have discussed none of which the second or predicate term is particular, and of which the general type is $Y = vX$, Y and X being functions of the logical symbols x, y, z, &c., and v an indefinite class symbol. The analysis of such propositions is greatly facilitated (though the step is not an essential one) by the elimination of the symbol v, and this process depends upon the method of the next chapter. I postpone also the consideration of another important problem necessary to complete the theory of single propositions, but of which the analysis really depends upon the method of the reduction of systems of propositions to be developed in a future page of this work.

In all examples Boole immediately eliminated the indefinite class symbols in equations of the form $Y = vX$; but not in equations of the form $vX = vY$ (see page 124), which only occur in the examples in Chapter XV.

Chapter VII

On Elimination.

1. In the examples discussed in the last chapter, all the elements of the original premiss re-appeared in the conclusion, only in a different order, and with a different connexion. But it more usually happens in common reasoning, and especially when we have more than one premiss, that some of the elements are required not to appear in the conclusion. Such elements, or, as they are commonly called, "middle terms," may be considered as introduced into the original propositions only for the sake of that connexion which they assist to establish among the other elements, which are alone designed to enter into the expression of the conclusion.

2. Respecting such intermediate elements, or middle terms, some erroneous notions prevail. It is a general opinion, to which, however, the examples contained in the last chapter furnish a contradiction, that inference consists peculiarly in the elimination of such terms, and that the elementary type of this process is exhibited in the elimination of one middle term from two premises, so as to produce a single resulting conclusion into which that term does not enter. Hence it is commonly held, that *syllogism* is the basis, or else the common type, of all inference, which may thus, however complex its form and structure, be resolved into a series of syllogisms. The propriety of this view will be considered in a subsequent chapter. At present I wish to direct attention to an important, but hitherto unnoticed, point of difference between the system of Logic, as expressed by symbols, and that of common algebra, with reference to the subject of elimination. In the algebraic system we are able to eliminate one symbol from two equations, two symbols from three equations, and generally $n-1$ symbols from n equations. There thus exists a definite connexion between the number of independent equations given,

See Chap XV, pp. 238–240, for a discussion of the merits of syllogisms.

and the number of symbols of quantity which it is possible to eliminate from them. But it is otherwise with the system of Logic. No fixed connexion there prevails between the number of equations given representing propositions or premises, and the number of typical symbols of which the elimination can be effected. From a single equation an indefinite number of such symbols may be eliminated. On the other hand, from an indefinite number of equations, a single class symbol only may be eliminated. We may affirm, that in this peculiar system, the problem of elimination is resolvable under all circumstances alike. This is a consequence of that remarkable law of duality to which the symbols of Logic are subject. To the equations furnished by the premises given, there is added another equation or system of equations drawn from the fundamental laws of thought itself, and supplying the necessary means for the solution of the problem in question. Of the many consequences which flow from the law of duality, this is perhaps the most deserving of attention.

This repeats what was said on page 8.

3. As in Algebra it often happens, that the elimination of symbols from a given system of equations conducts to a mere identity in the form $0 = 0$, no independent relations connecting the symbols which remain; so in the system of Logic, a like result, admitting of a similar interpretation, may present itself. Such a circumstance does not detract from the generality of the principle before stated. The object of the method upon which we are about to enter is to eliminate any number of symbols from any number of logical equations, and to exhibit in the result the actual relations which remain. Now it may be, that no such residual relations exist. In such a case the truth of the method is shown by its leading us to a merely identical proposition.

4. The notation adopted in the following Propositions is similar to that of the last chapter. By $f(x)$ is meant any expression involving the logical symbol x, with or without other logical symbols. By $f(1)$ is meant what $f(x)$ becomes when x is therein changed into 1; by $f(0)$ what the same function becomes when x is changed into 0.

The variables of $f(x)$ *include* x; this is not the modern convention.

PROPOSITION I.

5. *If $f(x) = 0$ be any logical equation involving the class symbol x, with or without other class symbols, then will the equation*

$$f(1)f(0) \ = \ 0$$

be true, independently of the interpretation of x; and it will be the complete result of the elimination of x from the above equation.

In other words, the elimination of x from any given equation, $f(x) = 0$, will be effected by successively changing in that equation x into 1, and x into 0, and multiplying the two resulting equations together.

Similarly the complete result of the elimination of any class symbols, x, y, etc., from any equation of the form $V = 0$, will be obtained by completely expanding the first member of that equation in constituents of the given symbols, and multiplying together all the coefficients of those constituents, and equating the product to 0.

ELIMINATION THEOREM

Eliminating one variable x from $f(x) = 0$. In Schröder's *Algebra der Logik* this is called Boole's Main Theorem.

The "complete result" of eliminating $\vec{x}$ from $f(\vec{x}, \vec{y}) = 0$ is $\prod_\sigma f(\sigma, \vec{y}) = 0$, that is, setting the product of the coefficients of the $\vec{x}$-constituents in the expansion of $f(\vec{x}, \vec{y})$ about $\vec{x}$ equal to 0.

Boole's phrase "complete result" can be made precise as follows: there is $\vec{x}$ such that $f(\vec{x}, \vec{y}) = 0$ iff $\prod_\sigma f(\sigma, \vec{y}) = 0$.

This is the first of Boole's three proofs.

Developing the first member of the equation $f(x) = 0$, we have (V. 10),

$$f(1)x \ + \ f(0)(1-x) \ = \ 0\,;$$

or,

$$\big[f(1) - f(0)\big]x \ + \ f(0) \ = \ 0\,. \qquad (1)$$

$$\therefore \ x \ = \ \frac{f(0)}{f(0) - f(1)}\,;$$

and

$$1 - x \ = \ -\frac{f(1)}{f(0) - f(1)}\,.$$

Substitute these expressions for x and $1-x$ in the fundamental equation

$$x(1-x) \ = \ 0\,,$$

and there results

$$-\frac{f(0)f(1)}{\big[f(0) - f(1)\big]^2} \ = \ 0\,;$$

or,

$$f(1)f(0) \ = \ 0\,, \qquad (2)$$

the form required.

This is the rare occasion when Boole brings the algebra of rational functions into his algebra of logic. Such is not justified by anything stated so far.

6. It is seen in this process, that the elimination is really effected between the given equation $f(x) = 0$ and the universally true equation $x(1-x) = 0$, expressing the fundamental law of logical symbols, *qua* logical. There exists, therefore, no need of more than one premiss or equation, in order to render possible the elimination of a term, the necessary law of thought virtually supplying the other premiss or equation. And though the demonstration of this conclusion may be exhibited in other forms, yet the same element furnished by the mind itself will still be virtually present. Thus we might proceed as follows:

(Second proof.)

Multiply (1) by x, and we have

$$f(1)x = 0, \qquad (3)$$

and let us seek by the forms of ordinary algebra to eliminate x from this equation and (1).

Now if we have two algebraic equations of the form

$$ax + b = 0,$$
$$a'x + b' = 0;$$

it is well known that the result of the elimination of x is

$$ab' - a'b = 0 \qquad (4)$$

But comparing the above pair of equations with (1) and (3) respectively, we find

$$a = f(1) - f(0), \qquad b = f(0);$$
$$a' = f(1) \qquad b' = 0;$$

which, substituted in (4), give

$$f(1)f(0) = 0,$$

as before. In this form of the demonstration, the fundamental equation $x(1 - x) = 0$, makes its appearance in the derivation of (3) from (1).

This is the elimination theorem from the algebra of numbers that Boole used in MAL. (See page 109 for further comments on this theorem.)

This proof indeed shows $f(1)f(0) = 0$ follows from $f(x) = 0$. A briefer write up is as follows.

From $f(x) = 0$ one has, by the expansion theorem, $f(1)x = 0$ and $f(0)(1-x) = 0$. These two equations give

$f(1)f(0)x = 0$
$f(1)f(0)(1 - x) = 0.$

Adding these two gives $f(1)f(0) = 0$.

7. I shall add yet another form of the demonstration, partaking of a half logical character, and which may set the demonstration of this important theorem in a clearer light.

(Third proof.)

This "half logical" proof is, in general, false; it does not clarify anything beyond Boole's willingness to try formal applications of results determined in a different setting.

We have as before

$$f(1)x + f(0)(1 - x) = 0.$$

Multiply this equation first by x, and secondly by $1 - x$, we get

$$f(1)x = 0, \qquad f(0)(1 - x) = 0.$$

From these we have by solution and development,

$$f(1) = \frac{0}{x} = \frac{0}{0}(1-x), \text{ on development,}$$
$$f(0) = \frac{0}{1-x} = \frac{0}{0}x.$$

Boole's solution theorem does not apply here; $f(1)$ and $f(0)$ need not be idempotent, hence they need not represent classes.

The direct interpretation of these equations is—

1st. Whatever individuals are included in the class represented by $f(1)$, are not x's.

2nd. Whatever individuals are included in the class represented by $f(0)$, are x's.

Whence by common logic, there are no individuals at once in the class $f(1)$ and in the class $f(0)$, i.e. there are no individuals in the class $f(1)f(0)$. Hence,

$$f(1)f(0) = 0. \quad (5)$$

Or it would suffice to multiply together the developed equations, whence the result would immediately follow.

8. The theorem (5) furnishes us with the following Rule :

TO ELIMINATE ANY SYMBOL FROM A PROPOSED EQUATION.

RULE.—*The terms of the equation having been brought, by transposition if necessary, to the first side, give to the symbol successively the values 1 and 0 , and multiply the resulting equations together.*

The first part of the Proposition is now proved.

The first part of the discussion of Proposition I gives three 'proofs' of the result of the elimination of a single variable. Using this there follows in §9 a detailed demonstration of the elimination of two variables, one at a time. Boole considered this sufficient to validate his formulation of the general elimination theorem in Proposition I.

9. Consider in the next place the general equation

$$f(x, y) = 0;$$

the first member of which represents any function of x, y, and other symbols.

By what has been shown, the result of the elimination of y from this equation will be

$$f(x,1)f(x,0) = 0;$$

for such is the form to which we are conducted by successively changing in the given equation y into 1 , and y into 0 , and multiplying the results together.

Again, if in the result obtained we change successively x into 1 , and x into 0 , and multiply the results together, we have

$$f(1,1)f(1,0)f(0,1)f(0,0) = 0; \quad (6)$$

as the final result of elimination.

Eliminating two variables x, y from $f(x, y) = 0$.

But the four factors of the first member of this equation are the four coefficients of the complete expansion of $f(x, y)$, the first member of the original equation; whence the second part of the Proposition is manifest.

EXAMPLES.

10. Ex. 1. — Having given the Proposition, "All men are mortal," and its symbolical expression, in the equation,

$$y = vx,$$

in which y represents "men," and x "mortals," it is required to eliminate the indefinite class symbol v, and to interpret the result.

Here bringing the terms to the first side, we have

$$y - vx = 0.$$

When $v = 1$ this becomes

$$y - x = 0;$$

and when $v = 0$ it becomes

$$y = 0;$$

and these two equations multiplied together, give

$$y - yx = 0,$$

or

$$y(1 - x) = 0,$$

it being observed that $y^2 = y$.

The above equation is the required result of elimination, and its interpretation is, *Men who are not mortal do not exist*, — an obvious conclusion.

If from the equation last obtained we seek a description of beings who are not mortal, we have

$$x = \frac{y}{y},$$

$$\therefore\ 1 - x = \frac{0}{y}.$$

Whence, by expansion, $1 - x = \frac{0}{0}(1 - y)$, which interpreted gives, *They who are not mortal are not men*. This is an example of what in the common logic is called conversion by contraposition, or negative conversion.[1]

Ex. 2.—Taking the Proposition, "No men are perfect," as represented by the equation

$$y = v(1-x),$$

wherein y represents "men," and x "perfect beings," it is required to eliminate v, and find from the result a description both of *perfect beings* and of *imperfect beings.* We have

$$y - v(1-x) = 0\,.$$

Whence, by the rule of elimination,

$$\{y-(1-x)\} \times y = 0\,,$$

or

$$y - y(1-x) = 0\,,$$

or

$$yx = 0\,;$$

which is interpreted by the Proposition, *Perfect men do not exist.* From the above equation we have

$$x = \frac{0}{y} = \frac{0}{0}(1-y) \text{ by development;}$$

whence, by interpretation, *No perfect beings are men.* Similarly,

$$1-x = 1-\frac{0}{y} = \frac{y}{y} = y + \frac{0}{0}(1-y),$$

which, on interpretation, gives, *Imperfect beings are all men with an indefinite remainder of beings, which are not men.*

11. It will generally be the most convenient course, in the treatment of propositions, to eliminate first the indefinite class symbol v, wherever it occurs in the corresponding equations. This will only modify their form, without impairing their significance. Let us apply this process to one of the examples of Chap. IV. For the Proposition, "No men are placed in exalted stations and free from envious regards," we found the expression

This is only for propositions that translate into the form $X = vY$. See page 124.

$$y = v(1-xz),$$

and for the equivalent Proposition, "Men in exalted stations are not free from envious regards," the expression

$$yx = v(1-z)\,;$$

[1] Whately's Logic, Book II. chap. II. sec. 4.

and it was observed that these equations, v being an indefinite class symbol, were themselves equivalent. To prove this, it is only necessary to eliminate from each the symbol v. The first equation is

$$y - v(1 - xz) = 0,$$

whence, first making $v = 1$, and then $v = 0$, and multiplying the results, we have

$$(y - 1 + xz)y = 0,$$

or

$$yxz = 0.$$

Now the second of the given equations becomes on transposition

$$yx - v(1 - z) = 0;$$

whence

$$(x - 1 + z)yx = 0,$$

or

$$yxz = 0,$$

as before. The reader will easily interpret the result.

12. Ex. 3.—As a subject for the general method of this chapter, we will resume Mr. Senior's definition of wealth, viz.: "Wealth consists of things transferable, limited in supply, and either productive of pleasure or preventive of pain." We shall consider this definition, agreeably to a former remark, as including all things which possess at once both the qualities expressed in the last part of the definition, upon which assumption we have, as our representative equation,

$$w = st\{pr + p(1 - r) + r(1 - p)\},$$

or

$$w = st\{p + r(1 - p)\},$$

wherein

w	stands for	wealth.
s	"	things limited in supply.
t	"	things transferable.
p	"	things productive of pleasure.
r	"	things preventive of pain.

From the above equation we can eliminate any symbols that we do not desire to take into account, and express the result by solution and development, according to any proposed arrangement of subject and predicate.

Let us first consider what the expression for w, wealth, would be if the element r, referring to prevention of pain, were eliminated. Now bringing the terms of the equation to the first side, we get

$$w - st(p + r - rp) = 0\,.$$

Making $r = 1$, the first member becomes $w - st$, and making $r = 0$ it becomes $w - stp$; whence we have by the Rule,

$$(w - st)(w - stp) = 0 \qquad (7)$$

or

$$w - wstp - wst + stp = 0\,; \qquad (8)$$

whence

$$w = \frac{stp}{st + stp - 1}\,;$$

the development of the second member of which equation gives

$$w = stp + \frac{0}{0}st(1 - p). \qquad (9)$$

Whence we have the conclusion,—*Wealth consists of all things limited in supply, transferable, and productive of pleasure, and an indefinite remainder of things limited in supply, transferable, and not productive of pleasure.* This is sufficiently obvious.

First time Boole used the word 'multiplication' as referring to an operation in his algebra of logic.

Let it be remarked that it is not necessary to perform the multiplication indicated in (7), and reduce that equation to the form (8), in order to determine the expression of w in terms of the other symbols. The process of development may in all cases be made to supersede that of multiplication. Thus if we develop (7) in terms of w, we find

$$(1 - sf)(1 - stp)w + stp(1 - w) = 0\,,$$

whence

$$w = \frac{stp}{stp - (1 - st)(1 - stp)}\,;$$

and this equation developed will give, as before,

$$w = stp + \frac{0}{0}st(1 - p).$$

13. Suppose next that we seek a description of things limited in supply, as dependent upon their relation to wealth, transferableness, and tendency to produce pleasure, omitting all reference to the prevention of pain.

From equation (8), which is the result of the elimination of r from the original equation, we have

$$w - s\,(wt \;+\; wtp - tp) \;=\; 0\,;$$

whence

$$\begin{aligned} s \;&=\; \frac{w}{wt \;+\; wtp - tp} \\ &=\; wtp \;+\; wt\,(1-p) \;+\; \frac{1}{0}w\,(1-t)\,p \;+\; \frac{1}{0}w\,(1-t)\,(1-p) \\ &\quad + 0\,(1-w)\,tp \;+\; \frac{0}{0}\,(1-w)\,t\,(1-p) \;+\; \frac{0}{0}\,(1-w)\,(1-t)\,p \\ &\quad + \frac{0}{0}\,(1-w)\,(1-t)\,(1-p)\,. \end{aligned}$$

We will first give the direct interpretation of the above solution, term by term; afterwards we shall offer some general remarks which it suggests; and, finally, show how the expression of the conclusion may be somewhat abbreviated.

First, then, the direct interpretation is, Things limited in supply consist of *All wealth transferable and productive of pleasure—all wealth transferable, and not productive of pleasure,—an indefinite amount of what is not wealth, but is either transferable, and not productive of pleasure, or intransferable and productive of pleasure, or neither transferable nor productive of pleasure.*

To which the terms whose coefficients are $\frac{1}{0}$ permit us to add the following independent relations, viz.:

1st. *Wealth that is intransferable, and productive of pleasure, does not exist.*

2ndly. *Wealth that is intransferable, and not productive of pleasure, does not exist.*

14. Respecting this solution I suppose the following remarks are likely to be made.

First, it may be said, that in the expression above obtained for "things limited in supply," the term "All wealth transferable," &c., is in part redundant; since all wealth is (as implied in the original proposition, and directly asserted in the *independent relations*) necessarily transferable.

I answer, that although in ordinary speech we should not deem it necessary to add to "wealth" the epithet "transferable," if another part of our reasoning had led us to express the conclusion, that there is no wealth which is not transferable, yet it pertains to the perfection of this method that it in all cases fully defines the objects represented by each term of the conclusion, by stating the relation they bear to each quality or element of distinction that we have chosen to employ. This is necessary in order to keep the different parts of the solution really distinct and independent, and actually prevents redundancy. Suppose that the pair of terms we have been considering had not contained the word "transferable," and had unitedly been "All wealth," we could then logically resolve the single term "All wealth" into the two terms "All wealth transferable," and "All wealth intransferable." But the latter term is shown to disappear by the "independent relations." Hence it forms no part of the description required, and is therefore redundant. The remaining term agrees with the conclusion actually obtained.

Solutions in which there cannot, by logical divisions, be produced any superfluous or redundant terms, may be termed *pure solutions.* Such are all the solutions obtained by the method of development and elimination above explained. It is proper to notice, that if the common algebraic method of elimination were adopted in the cases in which that method is possible in the present system, we should not be able to depend upon the purity of the solutions obtained. Its want of generality would not be its only defect.

In MAL, Boole used only the "common algebraic method of elimination," described here on page 102. It failed to be adequate to the task of dealing with the Aristotelian syllogisms, leading to ad hoc tricks. In LT, Chapter VII, Boole gave the correct elimination theorem for his algebra of logic.

15. In the second place, it will be remarked, that the conclusion contains two terms, the aggregate significance of which would be more conveniently expressed by a single term. Instead of "All wealth productive of pleasure, and transferable," and "All wealth not productive of pleasure, and transferable," we might simply say, "All wealth transferable." This remark is quite just. But it must be noticed that whenever any such simplifications are possible, they are immediately suggested by the form of the equation we have to interpret; and if that equation be reduced to its simplest form, then the interpretation to which it conducts will be in its simplest form also. Thus in the original solution the terms wtp and $wt(1-p)$, which have unity for their coefficient, give, on addition, wt; the terms $w\,(1-t)\,p$ and $w\,(1-t)\,(1-p)$, which have $\frac{1}{0}$ for their coefficient give $w\,(1-t)$; and the terms $(1-w)\,(1-t)\,p$ and $(1-w)\,(1-t)\,(1-p)$, which have $\frac{0}{0}$ for their coefficient, give $(1-w)\,(1-t)$. Whence the complete solution is

$$s \;=\; wt \;+\; \frac{0}{0}\,(1-w)\,(1-t) \;+\; \frac{0}{0}\,(1-w)\,t\,(1-p)\,,$$

with the independent relation,

$$w\,(1-t) \;=\; 0\,, \text{ or } w \;=\; \frac{0}{0}t.$$

The interpretation would now stand thus:—

1st. *Things limited in supply consist of all wealth transferable, with an indefinite remainder of what is not wealth and not transferable, and of transferable articles which are not wealth, and are not productive of pleasure.*

2nd. *All wealth is transferable.*

This is the simplest form under which the general conclusion, with its attendant condition, can be put.

16. When it is required to eliminate two or more symbols from a proposed equation we can either employ (6) Prop. I., or eliminate them in succession, the order of the process being indifferent. From the equation

$$w \;=\; st\,(p+r-pr)\,,$$

we have eliminated r, and found the result,

$$w - wst - wstp \;+\; stp \;=\; 0\,.$$

Suppose that it had been required to eliminate both r and t, then taking the above as the first step of the process, it remains to eliminate from the last equation t. Now when $t = 1$ the first member of that equation becomes

$$w - ws - wsp \;+\; sp,$$

and when $t = 0$ the same member becomes w. Whence we have

$$w\,(w - ws - wsp \;+\; sp) \;=\; 0\,,$$

or

$$w - ws \;=\; 0\,,$$

for the required result of elimination.

If from the last result we determine w, we have

$$w = \frac{0}{1-s} = \frac{0}{0}s,$$

whence "*All wealth is limited in supply.*" As p does not enter into the equation, it is evident that the above is true, irrespectively of any relation which the elements of the conclusion bear to the quality "productive of pleasure."

Resuming the original equation, let it be required to eliminate s and t. We have

$$w = st(p + r - pr).$$

Instead, however, of separately eliminating s and t according to the Rule, it will suffice to treat st as a single symbol, seeing that it satisfies the fundamental law of the symbols by the equation

$$st(1 - st) = 0\,.$$

Placing, therefore, the given equation under the form

$$w - st(p + r - pr) = 0\,;$$

and making st successively equal to 1 and to 0 , and taking the product of the results, we have

$$(w - p - r + pr)w = 0\,,$$

or

$$w - wp - wr + wpr = 0\,,$$

for the result sought.

As a particular illustration, let it be required to deduce an expression for "things productive of pleasure" (p), in terms of "wealth" (w), and "things preventive of pain" (r).

We have, on solving the equation,

$$\begin{aligned} p &= \frac{w(1-r)}{w(1-r)} \\ &= \frac{0}{0}wr + w(1-r) + \frac{0}{0}(1-w)r + \frac{0}{0}(1-w)(1-r) \\ &= w(1-r) + \frac{0}{0}wr + \frac{0}{0}(1-w). \end{aligned}$$

Whence the following conclusion:—*Things productive of pleasure* *are, all wealth not preventive of pain, an indefinite amount of wealth that is preventive of pain, and an indefinite amount of what is not wealth.*

From the same equation we get

$$1 - p \;=\; 1 - \frac{w(1-r)}{w(1-r)} \;=\; \frac{0}{w(1-r)},$$

which developed, gives

$$w(1-p) \;=\; \frac{0}{0}wr \;+\; \frac{0}{0}(1-w)\cdot r \;+\; \frac{0}{0}(1-w)\cdot(1-r)$$
$$=\; \frac{0}{0}wr \;+\; \frac{0}{0}(1-w).$$

Whence, *Things not productive of pleasure are either wealth, preventive of pain, or what is not wealth.*

Equally easy would be the discussion of any similar case.

17. In the last example of elimination, we have eliminated the compound symbol *st* from the given equation, by treating it as a single symbol. The same method is applicable to any combination of symbols which satisfies the fundamental law of individual symbols. Thus the expression $p+r-pr$ will, on being multiplied by itself, reproduce itself, so that if we represent $p+r-pr$ by a single symbol as y, we shall have the fundamental law obeyed, the equation

One can eliminate idempotent terms.

$$y \;=\; y^2, \quad \text{or} \quad y(1-y) \;=\; 0\,,$$

being satisfied. For the rule of elimination for symbols is founded upon the supposition that each individual symbol is subject to that law; and hence the elimination of any function or combination of such symbols from an equation, may be effected by a single operation, whenever that law is satisfied by the function.

Though the forms of interpretation adopted in this and the previous chapter show, perhaps better than any others, the direct significance of the symbols 1 and $\frac{0}{0}$, modes of expression more agreeable to those of common discourse may, with equal truth and propriety, be employed. Thus the equation (9) may be interpreted in the following manner: *Wealth is either limited in supply, transferable, and productive of pleasure, or limited in supply,* *transferable, and not productive of pleasure.* And reversely, *Whatever is limited in supply, transferable, and productive of pleasure, is wealth.* Reverse interpretations, similar to the above, are always furnished when the final development introduces terms having unity as a coefficient.

18. NOTE.—The fundamental equation $f(1)f(0) = 0$, expressing the result of the elimination of the symbol x from any equation $f(x) = 0$, admits of a remarkable interpretation.

It is to be remembered, that by the equation $f(x) = 0$ is implied some proposition in which the individuals represented by the class x, suppose "men," are referred to, together, it may be, with other individuals; and it is our object to ascertain whether there is implied in the proposition any relation among the other individuals, independently of those found in the class *men*. Now the equation $f(1) = 0$ expresses what the original proposition would become if *men* made up the universe, and the equation $f(0) = 0$ expresses what that original proposition would become if men ceased to exist, wherefore the equation $f(1)f(0) = 0$ expresses what in virtue of the original proposition would be equally true on either assumption, i. e. equally true whether "men" were "all things" or "nothing." Wherefore the theorem expresses that *what is equally true, whether a given class of objects embraces the whole universe or disappears from existence, is independent of that class altogether, and vice versâ.* Herein we see another example of the interpretation of formal results, immediately deduced from the mathematical laws of thought, into general axioms of philosophy.

Was Boole claiming that $f(1) = 0$ and $f(0) = 0$ hold iff $f(x) = 0$ for all x? This is obvious from the expansion theorem: $f(x) = f(1)x + f(0)(1 - x)$.

Chapter VIII

On the Reduction of Systems of Propositions.

1. In the preceding chapters we have determined sufficiently for the most essential purposes the theory of single primary propositions, or, to speak more accurately, of primary propositions expressed by a single equation. And we have established upon that theory an adequate method. We have shown how any element involved in the given system of equations may be eliminated, and the relation which connects the remaining elements deduced in any proposed form, whether of denial, of affirmation, or of the more usual relation of subject and predicate. It remains that we proceed to the consideration of systems of propositions, and institute with respect to them a similar series of investigations. We are to inquire whether it is possible from the equations by which a system of propositions is expressed to eliminate, *ad libitum*, any number of the symbols involved; to deduce by interpretation of the result the whole of the relations implied among the remaining symbols; and to determine in particular the expression of any single element, or of any interpretable combination of elements, in terms of the other elements, so as to present the conclusion in any admissible form that may be required. These questions will be answered by showing that it is possible to reduce any system of equations, or any of the equations involved in a system, to an equivalent single equation, to which the methods of the previous chapters may be immediately applied. It will be seen also, that in this reduction is involved an important extension of the theory of single propositions, which in the previous discussion of the subject we were compelled to forego. This circumstance is not peculiar in its nature. There are many special departments of science which cannot be completely surveyed from within, but require to be studied also from an external point of view, and to be regarded in connexion with

In the margin on page 83 it was noted that any equation $V = 0$ is equivalent to $\sum \mathfrak{C}(V) = 0$, the left side being a sum of constituents.

This easily generalizes to a system of equations $V_i = 0$, namely the system is equivalent to the single equation $\sum \bigcup_i \mathfrak{C}(V_i) = 0$.

However, finding the $\mathfrak{C}(V_i)$ may not be practical if there is a large number of variables. This chapter looks at possibly faster ways by reducing a system to a single equation.

other and kindred subjects, in order that their full proportions be understood.

This chapter will exhibit two distinct modes of reducing systems of equations to equivalent single equations. The first of these rests upon the employment of arbitrary constant multipliers. It is a method sufficiently simple in theory, but it has the inconvenience of rendering the subsequent processes of elimination and development, when they occur, somewhat tedious. It was, however, the method of reduction first discovered, and partly on this account, and partly on account of its simplicity, it has been thought proper to retain it. The second method does not require the introduction of arbitrary constants, and is in nearly all respects preferable to the preceding one. It will, therefore, generally be adopted in the subsequent investigations of this work.

2. We proceed to the consideration of the first method.

First method of REDUCTION: A system of equations $V_0 = 0, \ldots, V_n = 0$ is equivalent to the single equation $V_0 + \sum\{c_i V_i : 1 \leq i \leq n\} = 0$, where the c_i are arbitrary constants.

Proposition I.

Any system of logical equations may be reduced to a single equivalent equation, by multiplying each equation after the first by a distinct arbitrary constant quantity, and adding all the results, including the first equation, together.

By Prop. 2, Chap, VI., the interpretation of any single equation, $f(x, y..) = 0$ is obtained by equating to 0 those constituents of the development of the first member, whose coefficients do not vanish. And hence, if there be given two equations, $f(x, y..) = 0$, and $F(x, y..) = 0$, their united import will be contained in the system of results formed by equating to 0 all those constituents which thus present themselves in both, or in either, of the given equations developed according to the Rule of Chap. VI. Thus let it be supposed, that we have the two equations

$$xy - 2x = 0, \tag{1}$$

$$x - y = 0; \tag{2}$$

The development of the first gives

$$-xy - 2x(1 - y) = 0;$$

whence, $$xy = 0, \qquad x(1 - y) = 0. \tag{3}$$

The development of the second equation gives

$$x(1-y) - y(1-x) = 0;$$

whence, $$x(1-y) = 0, \quad y(1-x) = 0. \tag{4}$$

The constituents whose coefficients do not vanish in both developments are xy, $x(1-y)$, and $(1-x)y$, and these would together give the system

$$xy = 0, \quad x(1-y) = 0, \quad (1-x)y = 0; \tag{5}$$

which is equivalent to the two systems given by the developments separately, seeing that in those systems the equation $x(1-y) = 0$ is repeated. Confining ourselves to the case of binary systems of equations, it remains then to determine a single equation, which on development shall yield the same constituents with coefficients which do not vanish, as the given equations produce.

Now if we represent by

$$V_1 = 0, \quad V_2 = 0,$$

the given equations, V_1 and V_2 being functions of the logical symbols x, y, z, &c.; then the single equation

$$V_1 + cV_2 = 0, \tag{6}$$

c being an arbitrary constant quantity, will accomplish the required object. For let At represent any term in the full development V_1, wherein t is a constituent and A its numerical coefficient, and let Bt represent the corresponding term in the full development of V_2, then will the corresponding term in the development of (6) be

$$(A + cB)t.$$

The coefficient of t vanishes if A and B both vanish, but not otherwise. For if we assume that A and B do not both vanish, and at the same time make

$$A + cB = 0, \tag{7}$$

the following cases alone can present themselves.

1st. That A vanishes and B does not vanish. In this case the above equation becomes

$$cB = 0,$$

and requires that $c = 0$. But this contradicts the hypothesis that c is an *arbitrary* constant.

2nd. That B vanishes and A does not vanish. This assumption reduces (7) to

$$A = 0,$$

by which the assumption is itself violated.

3rd. That neither A nor B vanishes. The equation (7) then gives

$$c = \frac{-A}{B}$$

which is a definite value, and, therefore, conflicts with the hypothesis that c is arbitrary.

Hence the coefficient $A + cB$ vanishes when A and B both vanish, but not otherwise. Therefore, the same constituents will appear in the development of (6), with coefficients which do not vanish, as in the equations $V_1 = 0, V_2 = 0$, singly or together. And the equation $V_1 + cV_2 = 0$, will be equivalent to the system $V_1 = 0, V_2 = 0$.

By similar reasoning it appears, that the general system of equations

$$V_1 = 0, \quad V_2 = 0, \quad V_3 = 0, \quad \&\text{c.};$$

may be replaced by the single equation

$$V_1 + cV_2 + c'V_3 + \&\text{c.} = 0,$$

c, c', &c., being arbitrary constants. The equation thus formed may be treated in all respects as the ordinary logical equations of the previous chapters. The arbitrary constants c_1, c_2, &c., are not *logical* symbols. They do not satisfy the law,

The constants c_1, c_2 should be c, c'.

$$c_1(1 - c_1) = 0, \quad c_2(1 - c_2) = 0.$$

But their introduction is justified by that general principle which has been stated in (II. 15) and (V. 6), and exemplified in nearly all our subsequent investigations, viz., that equations involving the symbols of Logic may be treated in all respects as if those symbols were symbols of quantity, subject to the special law $x(1 - x) = 0$, until in the final stage of solution they assume a form interpretable in that system of thought with which Logic is conversant.

The general principle is **R01**.

3. The following example will serve to illustrate the above method.

A popular example in the subsequent literature.

Ex. 1.—Suppose that an analysis of the properties of a particular class of substances has led to the following general conclusions, viz.:

1st. That wherever the properties A and B are combined, either the property C, or the property D, is present also; but they are not jointly present.

2nd. That wherever the properties B and C are combined, the properties A and D are either both present with them, or both absent.

3rd. That wherever the properties A and B are both absent, the properties C and D are both absent also; and vice versa, where the properties C and D are both absent, A and B are both absent also.

Let it then be required from the above to determine what may be concluded in any particular instance from the presence of the property A with respect to the presence or absence of the properties B and C, paying no regard to the property D.

Represent	the property A by x;
"	the property B by y;
"	the property C by z;
"	the property D by w.

Then the symbolical expression of the premises will be

However, Boole's algebra of logic was not a popular tool for analyzing the above example.

$$xy = v\{w(1-z) + z(1-w)\};$$
$$yz = v\{xw + (1-x)(1-w)\};$$
$$(1-x)(1-y) = (1-z)(1-w).$$

From the first two of these equations, separately eliminating the indefinite class symbol v, we have

$$xy\{1 - w(1-z) - z(1-w)\} = 0;$$
$$yz\{1 - xw - (1-x)(1-w)\} = 0.$$

Now if we observe that by development

$$1 - w(1-z) - z(1-w) = wz + (1-w)(1-z),$$

and

$$1 - xw - (1-x)(1-w) = x(1-w) + w(1-x),$$

and in these expressions replace, for simplicity,

First time Boole used $\bar{x}$ for $1-x$ in LT.

$$1-x \text{ by } \bar{x}, \quad 1-y \text{ by } \bar{y}, \quad \&c.,$$

we shall have from the three last equations,

$$xy(wz + \bar{w}\bar{z}) = 0; \qquad (1)$$
$$yz(x\bar{w} + \bar{x}w) = 0; \qquad (2)$$
$$\bar{x}\bar{y} = \bar{w}\bar{z}; \qquad (3)$$

and from this system we must eliminate w.

Multiplying the second of the above equations by c, and the third by c', and adding the results to the first, we have

$$xy(wz + \bar{w}\bar{z}) + cyz(x\bar{w} + \bar{x}w) + c'(\bar{x}\bar{y} - \bar{w}\bar{z}) = 0.$$

When w is made equal to 1, and therefore $\bar{w}$ to 0, the first member of the above equation becomes

$$xyz + c\bar{x}yz + c'\bar{x}\bar{y}.$$

And when in the same member w is made 0 and $\bar{w} = 1$, it becomes

$$xy\bar{z} + cxyz + c'\bar{x}\bar{y} - c'\bar{z}.$$

Hence the result of the elimination of w may be expressed in the form

$$(xyz + c\bar{x}yz + c'\bar{x}\bar{y})(xy\bar{z} + cxyz + c'\bar{x}\bar{y} - c'\bar{z}) = 0; \qquad (4)$$

and from this equation x is to be determined.

Were we now to proceed as in former instances, we should multiply together the factors in the first member of the above equation ; but it may be well to show that such a course is not at all necessary. Let us develop the first member of (4) with reference to x, the symbol whose expression is sought, we find

$$yz(y\bar{z} + cyz - c'\bar{z})x + (cyz + c'\bar{y})(c'\bar{y} - c'z)(1-x) = 0;$$

or,

$$cyzx + (cyz + c'\bar{y})(c'\bar{y} - c'\bar{z})(1-x) = 0;$$

whence we find,

$$x = \frac{(cyz + c'\bar{y})(c'\bar{y} - c'\bar{z})}{(cyz + c'\bar{y})(c'\bar{y} - c'\bar{z}) - cyz};$$

and developing the second member with respect to y and z,

$$x = 0yz + \frac{0}{0}y\bar{z} + \frac{c'^2}{c'^2}\bar{y}z + \frac{0}{0}\bar{y}\bar{z};$$

or,

$$x = (1-y)\,z + \frac{0}{0}y\,(1-z) + \frac{0}{0}\,(1-y)\,(1-z)\ ;$$

or,

$$x = (1-y)\,z + \frac{0}{0}\,(1-z)\ ;$$

the interpretation of which is, *Wherever the property A is present, there either C is present and B absent, or C is absent.* And inversely, *Wherever the property C is present, and the property B absent, there the property A is present.*

These results may be much more readily obtained by the method next to be explained. It is, however, satisfactory to possess different modes, serving for mutual verification, of arriving at the same conclusion.

4. We proceed to the second method.

Proposition II.

If any equations, $V_1 = 0$, $V_2 = 0$, &c., are such that the developments of their first members consist only of constituents with positive coefficients, those equations may be combined together into a single equivalent equation by addition.

Second REDUCTION method (special case): If the expansions of the V_i have only nonnegative coefficients then the system of equations $\{V_i = 0 : 1 \leq i \leq n\}$ is equivalent to the single equation $\sum_i V_i = 0.$

For, as before, let At represent any term in the development of the function V_1, Bt the corresponding term in the development of V_2 and so on. Then will the corresponding term in the development of the equation

$$V_1 + V_2 + \&\text{c.} = 0\,, \tag{1}$$

formed by the addition of the several given equations, be

$$(A + B + \&\text{c.})\,t.$$

But as by hypothesis the coefficients A, B, &c. are none of them negative, the aggregate coefficient $A + B +$ &c. in the derived equation will only vanish when the separate coefficients A, B, &c. vanish together. Hence the same constituents will appear in the development of the equation (1) as in the several equations $V_1 = 0$, $V_2 = 0$, &c. of the original system taken collectively, and therefore the interpretation of the equation (1) will be equivalent

LT has $A + B$, &c., an error.

to the collective interpretations of the several equations from which it is derived.

Proposition III.

5. *If* $V_1 = 0, V_2 = 0$, *&c. represent any system of equations, the terms of which have by transposition been brought to like first side, then the combined interpretation of the system will be involved in the single equation,*

$$V_1^2 + V_2^2 + \&c. = 0,$$

formed by adding together the squares of the given equations.

Second REDUCTION method: The system
$\{V_i = 0 : 1 \le i \le n\}$
is equivalent to the single equation
$\sum_i V_i^2 = 0.$

NOTE: This reduction step often introduces uninterpretable algebraic terms into Boole's equational derivations. For example, reducing the two equations $x = y$, $x = z$ gives
$(x - y)^2 + (x - z)^2 = 0,$
which is not idempotent since the coefficient of the constituent $x(1 - y)(1 - z)$ in the expansion of the left side of the equation is 2.

For let any equation of the system, as $V_1 = 0$, produce on development an equation

$$a_1 t_1 + a_2 t_2 + \&c. = 0$$

in which t_1, t_2, &c. are constituents, and a_1, a_2, &c. their corresponding coefficients. Then the equation $V_1^2 = 0$ will produce on development an equation

$${a_1}^2 t_1 + {a_2}^2 t_2 + \&c. = 0,$$

as may be proved either from the law of the development or by squaring the function $a_1 t_1 + a_2 t_2$, &c. in subjection to the conditions

$${t_1}^2 = t_1, \qquad {t_2}^2 = t_2,, \qquad t_1 t_2 = 0$$

assigned in Prop. 3, Chap. v. Hence the constituents which appear in the expansion of the equation $V_1^2 = 0$, are the same with those which appear in the expansion of the equation $V_1 = 0$, and they have positive coefficients. And the same remark applies to the equations $V_2 = 0$, &c. Whence, by the last Proposition, the equation

$$V_1^2 + V_2^2 + \&c. = 0$$

will be equivalent in interpretation to the system of equations

$$V_1 = 0, \qquad V_2 = 0, \qquad \&c.$$

Corollary.—Any equation, $V = 0$, of which the first member already satisfies the condition

$$V^2 = V, \qquad \text{or} \qquad V(1 - V) = 0,$$

This Corollary says that

$$V^2 = V$$

implies V is equal to a sum of distinct constituents. (The proof is given in the next margin note.)
The converse of the Corollary is valid, and is stated in Chap. V. Prop. 4.

does not need (as it would remain unaffected by) the process of squaring. Such equations are, indeed, immediately developable into a series of constituents, with coefficients equal to 1, Chap. v. Prop. 4.

Let the full expansion of V be $\sum_i a_i t_i$. From $V^2 = V$ one has $(a_i^2 - a_i)t_i = 0$ for each i. But $V^2 = V$ does not imply $t_i = 0$ (Boole omitted this step). Thus $V^2 = V$ implies $a_i^2 = a_i$, so a_i is 0 or 1, for all i.
Note: Boole's proof using Chap. V. Prop. 4. is not correct.

Proposition IV.

6. *Whenever the equations of a system have by the above process of squaring, or by any other process, been reduced to a form such that all the constituents exhibited in their development have positive coefficients, any derived equations obtained by elimination will possess the same character, and may be combined with the other equations by addition.*

Suppose the expansion of V has all coefficients non-negative. If elimination of some variables in $V = 0$ gives $W = 0$, then the expansion of W also has all coefficients non-negative.

Suppose that we have to eliminate a symbol x from any equation $V = 0$, which is such that none of the constituents, in the full development of its first member, have negative coefficients. That expansion may be written in the form

$$V_1 x + V_0(1-x) = 0$$

V_1 and V_0 being each of the form

$$a_1 t_1 + a_2 t_2 \ldots + a_n t_n,$$

in which $t_1 t_2 \ldots t_n$ are constituents of the other symbols, and $a_1 a_2 \ldots a_n$ in each case positive or vanishing quantities. The result of elimination is

$$V_1 V_2 = 0;$$

and as the coefficients in V_1 and V_2 are none of them negative, there can be no negative coefficients in the product $V_1 V_2$. Hence the equation $V_1 V_2 = 0$ may be added to any other equation, the coefficients of whose constituents are positive, and the resulting equation will combine the full significance of those from which it was obtained.

Proposition V.

7. *To deduce from the previous Propositions a practical rule or method for the reduction of systems of equations expressing propositions in Logic.*

We have by the previous investigations established the following points, viz.:

1st. That any equations which are of the form $V = 0$, V satisfying the fundamental law of duality $V(1-V) = 0$, may be combined together by simple addition.

2ndly. That any other equations of the form $V = 0$ may be reduced, by the process of squaring, to a form in which the same principle of combination by mere addition is applicable.

It remains then only to determine what equations in the actual expression of propositions belong to the former, and what to the latter, class.

Now the general types of propositions have been set forth in the conclusion of Chap. IV. The division of propositions which they represent is as follows:

1st. Propositions, of which the subject is universal, and the predicate particular.

The symbolical type (IV. 15) is

$$X = vY,$$

X and Y satisfying the law of duality. Eliminating v, we have

$$X(1-Y) = 0, \tag{1}$$

and this will be found also to satisfy the same law. No further reduction by the process of squaring is needed.

2nd. Propositions of which both terms are universal, and of which the symbolical type is

$$X = Y,$$

X and Y separately satisfying the law of duality. Writing the equation in the form $X - Y = 0$, and squaring, we have

$$X - 2XY + Y = 0,$$

or,

$$X(1-Y) + Y(1-X) = 0. \tag{2}$$

The first member of this equation satisfies the law of duality, as is evident from its very form.

We may arrive at the same equation in a different manner. The equation

$$X = Y$$

is equivalent to the two equations

$$X = vY, \; Y = vX,$$

(for to affirm that X's are identical with Y's is to affirm both that All X's are Y's, and that All Y's are X's). Now these equations give, on elimination of v,

$$X(1-Y) = 0, \quad Y(1-X) = 0,$$

which added, produce (2).

3rd. Propositions of which both terms are particular. The form of such propositions is

$$vX = vY,$$

but v is not quite arbitrary, and therefore must not be eliminated. For v is the representative of *some*, which, though it may include in its meaning *all*, does not include *none*. We must therefore transpose the second member to the first side, and square the resulting equation according to the rule.

First time Boole mentioned: Do not eliminate the v in the form $vX = vY$.

The result will obviously be

$$vX(1-Y) + vY(1-X) = 0.$$

The above conclusions it may be convenient to embody in a Rule, which will serve for constant future direction.

8. Rule.— *The equations being so expressed as that the terms X and Y in the following typical forms obey the law of duality, change the equations*

$$X = vY \quad \text{into} \quad X(1-Y) = 0,$$
$$X = Y \quad \text{into} \quad X(1-Y) + Y(1-X) = 0.$$
$$vX = vY \quad \text{into} \quad vX(1-Y) + vY(1-X) = 0.$$

Any equation which is given in the form $X = 0$ will not need transformation, and any equation which presents itself in the form $X = 1$ may be replaced by $1 - X = 0$, as appears from the second of the above transformations.

Given X, Y idempotent algebraic terms, how to put primary propositions in the form $V = 0$ with V idempotent.

NOTE: If X, Y are totally defined, this gives V totally defined.

When the equations of the system have thus been reduced, any of them, as well as any equations derived from them by the process of elimination, may be combined by addition.

Since all coefficients of the constituents in the above reduced forms are non-negative.

9. Note.—It has been seen in Chapter IV. that in literally translating the terms of a proposition, without attending to its real meaning, into the language of symbols, we may produce equations in which the terms X and Y do not obey the law of duality. The equation $w = st(p + r)$, given in (3) Prop. 3 of the chapter referred to, is of this kind. Such equations, however, as it has been seen, have a meaning. Should it, for curiosity, or for any other motive, be determined to employ them, it will be best to reduce them by the Rule (VI. 5).

10. Ex. 2.—Let us take the following Propositions of Elementary Geometry:

1st. Similar figures consist of all whose corresponding angles are equal, and whose corresponding sides are proportional.

2nd. Triangles whose corresponding angles are equal have their corresponding sides proportional, and *vice versâ*.

To represent these premises, let us make

s = similar.
t = triangles.
q = having corresponding angles equal.
r = having corresponding sides proportional.

Then the premises are expressed by the following equations:

$$s = qr, \quad (1)$$
$$tq = tr. \quad (2)$$

Reducing by the Rule, or, which amounts to the same thing, bringing the terms of these equations to the first side, squaring each equation, and then adding, we have

$$s + qr - 2qrs + tq + tr - 2tqr = 0. \quad (3)$$

Let it be required to deduce a description of dissimilar figures formed out of the elements expressed by the terms, *triangles*, having corresponding angles equal, having corresponding sides proportional.

We have from (3),

$$s = \frac{tq + qr + rt - 2tqr}{2qr - 1},$$
$$\therefore\ 1 - s = \frac{qr - tq - rt + 2tqr - 1}{2qr - 1}. \quad (4)$$

And fully developing the second member, we find

$$1 - s = 0tqr + 2tq(1-r) + 2tr(1-q) + t(1-q)(1-r) + 0(1-t)qr + (1-t)q(1-r) + (1-t)r(1-q) + (1-t)(1-q)(1-r). \quad (5)$$

In the above development two of the terms have the coefficient 2, these must be equated to 0 by the Rule, then those terms whose coefficients are 0 being rejected, we have

$$\begin{aligned} 1-s &= t(1-q)(1-r) + (1-t)q(1-r) + (1-t)r(1-q) \\ &\qquad + (1-t)(1-q)(1-r)\,; \quad (6) \\ tq(1-r) &= 0\,; \quad (7) \\ tr(1-q) &= 0\,; \quad (8) \end{aligned}$$

the direct interpretation of which is

1st. *Dissimilar figures consist of all triangles which have not their corresponding angles equal and sides proportional, and of all figures not being triangles which have either their angles equal, and sides not proportional, or their corresponding sides proportional, and angles not equal, or neither their corresponding angles equal nor corresponding sides proportional.*

2nd. *There are no triangles whose corresponding angles are equal. and sides not proportional.*

3rd. *There are no triangles whose corresponding sides are proportional and angles not equal.*

11. Such are the immediate interpretations of the final equation. It is seen, in accordance with the general theory, that in deducing a description of a particular class of objects, viz., dissimilar figures, in terms of certain other elements of the original premises, we obtain also the independent relations which exist among those elements in virtue of the same premises. And that this is not superfluous information, even as respects the immediate object of inquiry, may easily be shown. For example, the independent relations may always be made use of to reduce, if it be thought desirable, to a briefer form, the expression of that relation which is directly sought. Thus if we write (7) in the form

$$0 = tq(1-r),$$

and add it to (6), we get, since

$$t(1-q)(1-r) + tq(1-r) = t(1-r),$$

$$\begin{aligned} 1-s = t(1-r) + (1-t)q(1-r) + (1-t)r(1-q) \\ + (1-t)(1-q)(1-r), \end{aligned}$$

which, on interpretation, would give for the first term of the description of dissimilar figures, "Triangles whose corresponding sides are not proportional," instead of the fuller description originally obtained. A regard to convenience must always determine the propriety of such reduction.

12. A reduction which is always advantageous (VII. 15) consists in collecting the terms of the immediate description sought, as of the second member of (5) or (6), into as few groups as possible. Thus the third and fourth terms of the second member of (6) produce by addition the single term $(1-t)(1-q)$. If this reduction be combined with the last, we have

$$1 - s = t(1-r) + (1-t)q(1-r) + (1-t)(1-q),$$

the interpretation of which is

Dissimilar figures consist of all triangles whose corresponding sides are not proportional, and all figures not being triangles which have either their corresponding angles unequal, or their corresponding angles equal, but sides not proportional.

The fulness of the general solution is therefore not a superfluity. While it gives us all the information that we seek, it provides us also with the means of expressing that information in the mode that is most advantageous.

13. Another observation, illustrative of a principle which has already been stated, remains to be made. Two of the terms in the full development of $1 - s$ in (5) have 2 for their coefficients, instead of $\frac{1}{0}$. It will hereafter be shown that this circumstance indicates that the two premises were not independent. To verify this, let us resume the equations of the premises in their reduced forms, viz.,

$$\begin{aligned} s(1-qr) + qr(1-s) &= 0\,, \\ tq(1-r) + tr(1-q) &= 0\,. \end{aligned}$$

Now if the first members of these equations have any common constituents, they will appear on multiplying the equations together. If we do this we obtain

$$stq(1-r) + str(1-q) = 0\,.$$

Whence there will result

$$stq(1-r) = 0, \quad str(1-q) = 0,$$

these being equations which are deducible from either of the primitive ones. Their interpretations are—

Similar triangles which have their corresponding angles equal have their corresponding sides proportional.

Similar triangles which have their corresponding sides proportional have their corresponding angles equal.

And these conclusions are equally deducible from either premiss *singly.* In this respect, according to the definitions laid down, the premises are not independent.

14. Let us, in conclusion, resume the problem discussed in illustration of the first method of this chapter, and endeavour to ascertain, by the present method, what may be concluded from the presence of the property C, with reference to the properties A and B.

We found on eliminating the symbols v the following equations, viz.:

$$xy(wz + \bar{w}\bar{z}) = 0, \quad (1)$$

$$yz(x\bar{w} + \bar{x}w) = 0, \quad (2)$$

$$\bar{x}\bar{y} = \bar{w}\bar{z}. \quad (3)$$

From these we are to eliminate w and determine z. Now (1) and (2) already satisfy the condition $V(1-V) = 0$. The third equation gives, on bringing the terms to the first side, and squaring

$$\bar{x}\bar{y}(1-\bar{w}\bar{z}) + \bar{w}\bar{z}(1-\bar{x}\bar{y}) = 0. \quad (4)$$

Adding (1) (2) and (4) together, we have

$$xy(wz + \bar{w}\bar{z}) + yz(x\bar{w} + \bar{x}w) + \bar{x}\bar{y}(1-\bar{w}\bar{z}) + \bar{w}\bar{z}(1-\bar{x}\bar{y}) = 0.$$

Eliminating w, we get

$$(xyz + yz\bar{x} + \bar{x}\bar{y})\{xy\bar{z} + yzx + \bar{x}\bar{y}z + \bar{z}(1-\bar{x}\bar{y})\} = 0.$$

Now, on multiplying the terms in the second factor by those in the first successively, observing that

$$x\bar{x} = 0, \quad y\bar{y} = 0, \quad z\bar{z} = 0,$$

nearly all disappear, and we have only left

$$xyz + \bar{x}\bar{y}z = 0; \tag{5}$$

whence

$$\begin{aligned} z &= \frac{0}{xy + \bar{x}\bar{y}} \\ &= 0xy + \frac{0}{0}x\bar{y} + \frac{0}{0}\bar{x}y + 0\bar{x}\bar{y} \\ &= \frac{0}{0}x\bar{y} + \frac{0}{0}\bar{x}y, \end{aligned}$$

furnishing the interpretation. *Wherever the property C is found, either the property A or the property B will be found with it, but not both of them together.*

From the equation (5) we may readily deduce the result arrived at in the previous investigation by the method of arbitrary constant multipliers, as well as any other proposed forms of the relation between x, y, and z; e. g. *If the property B is absent, either A and C will be jointly present, or C will be absent.* And conversely, *If A and C are jointly present, B will be absent.* The converse part of this conclusion is founded on the presence of a term xz with unity for its coefficient in the developed value of $\bar{y}$.

Chapter IX

On Certain Methods of Abbreviation.

1. Though the three fundamental methods of development, elimination, and reduction, established and illustrated in the previous chapters, are sufficient for all the practical ends of Logic, yet there are certain cases in which they admit, and especially the method of elimination, of being simplified in an important degree; and to these I wish to direct attention in the present chapter. I shall first demonstrate some propositions in which the principles of the above methods of abbreviation are contained, and I shall afterwards apply them to particular examples.

This chapter is about observations and techniques that can be useful to shorten the derivation of results in Boole's algebra of logic.

Let us designate as class terms any terms which satisfy the fundamental law $V(1-V) = 0$. Such terms will individually be constituents; but, when occurring together, will not, as do the terms of a development, necessarily involve the same symbols in each. Thus $ax+bxy+cyz$ may be described as an expression consisting of three class terms, x, xy, and yz, multiplied by the coefficients a, b, c respectively. The principle applied in the two following Propositions, and which, in some instances, greatly abbreviates the process of elimination, is that of the *rejection of superfluous class terms*; those being regarded as superfluous which do not add to the constituents of the final result.

A "class term" was defined on page 55 as a constituent (for some list of variables).

Proposition I.

2. *From any equation, $V = 0$, in which V consists of a series of class terms having positive coefficients, we are permitted to reject any term which contains another term as a factor, and to change every positive coefficient to unity.*

By a "series of" is surely meant "a sum of", so $V = \sum_i n_i t_i$ where the $n_i > 0$, and each t_i is a constituent for some list of variables. Let V_j be the sum of t_i for $i \neq j$. Proposition I says that if t_i is a factor of t_j then $V = 0 \iff V_j = 0$.

For the significance of this series of positive terms, depends only upon the number and nature of the constituents of its final expansion, i.e. of its expansion with reference to all the symbols

This is the first mention of the phrase "positive term", which is never defined. It also occurs on pages 131, 132, 142, 154. It surely means a positive number times a constituent.

which it involves, and not at all upon the actual values of the coefficients (VI. 5). Now let x be any term of the series, and xy any other term having x as a factor. The expansion of x with reference to the symbols x and y will be

$$xy \ + \ x(1-y),$$

and the expansion of the sum of the terms x and xy will be

$$2xy \ + \ x(1-y).$$

But by what has been said, these expressions occurring in the first member of an equation, of which the second member is 0 , and of which all the coefficients of the first member are positive, are equivalent; since there must exist simply the two constituents xy and $x(1-y)$ in the final expansion, whence will simply arise the resulting equations

$$xy \ = \ 0, \quad x(1-y) \ = \ 0.$$

And, therefore, the aggregate of terms $x + xy$ may be replaced by the single term x.

The same reasoning applies to all the cases contemplated in the Proposition. Thus, if the term x is repeated, the aggregate $2x$ may be replaced by x, because under the circumstances the equation $x = 0$ must appear in the final reduction.

Proposition II.

3. *Whenever in the process of elimination we have to multiply together two factors, each consisting solely of positive terms, satisfying the fundamental law of logical symbols, it is permitted to reject from both factors any common term, or from either factor any term which is divisible by a term in the other factor; provided always, that the rejected term be added to the product of the resulting factors.*

In the enunciation of this Proposition, the word "divisible" is a term of convenience, used in the algebraic sense, in which xy and $x(1-y)$ are said to be divisible by x.

To render more clear the import of this Proposition, let it be supposed that the factors to be multiplied together are $x + y + z$ and $x + yw + t$. It is then asserted, that from these two factors we may reject the term x, and that from the second factor we may reject the term yw, provided that these terms be transferred

Defn: A positive term is of the form mt with $m > 0$ and t a constituent (for some set of variables).
Let SPT be the collection of sums of positive terms.

For $i = 1, 2$ let $V_i = m_i t_i + A_i$ where $m_i t_i$ is a positive term and $A_i \in SPT$.

If $t_1 | t_2$ then for any W,
$$V_1 V_2 W = 0$$
is equivalent to
$$\big(t_2 + (t_1 + A_1)A_2\big)W = 0.$$

If $t_1 = t_2 = t$ then
$$V_1 V_2 W = 0$$
is equivalent to
$$\big(t + A_1 A_2\big)W = 0.$$

This can be useful in the elimination procedure.

to the final product. Thus, the resulting factors being $y+z$ and t, if to their product $yt+zt$ we add the terms x and yw, we have

$$x \ + \ yw \ + \ yt \ + \ zt,$$

as an expression equivalent to the product of the given factors $x+y+z$ and $x+yw+t$; equivalent namely in the process of elimination.

End of Example

Let us consider, first, the case in which the two factors have a common term x, and let us represent the factors by the expressions $x+P$, $x+Q$, supposing P in the one case and Q in the other to be the sum of the positive terms additional to x.

Start of Proof.
x is used here to denote a positive term mt.

Now,

$$(x+P)(x+Q) \ = \ x \ + \ xP \ + \ xQ \ + \ PQ.$$

But the process of elimination consists in multiplying certain factors together, and equating the result to 0. Either then the second member of the above equation is to be equated to 0, or it is a factor of some expression which is to be equated to 0.

If the former alternative be taken, then, by the last Proposition, we are permitted to reject the terms xP and xQ, inasmuch as they are positive terms having another term x as a factor. The resulting expression is

$$x \ + \ PQ,$$

which is what we should obtain by rejecting x from both factors, and adding it to the product of the factors which remain.

Taking the second alternative, the only mode in which the second member of (1) can affect the final result of elimination must depend upon the number and nature of its constituents, both which elements are unaffected by the rejection of the terms xP and xQ. For that development of x includes all possible constituents of which x is a factor.

Consider finally the case in which one of the factors contains a term, as xy, divisible by a term, x, in the other factor.

x is a positive term mt and xy is a positive term ntt'.

Let $x+P$ and $xy \ + \ Q$ be the factors. Now

$$(x+P)(xy+Q) \ = \ xy \ + \ xQ \ + \ xyP \ + \ PQ.$$

But by the reasoning of the last Proposition, the term xyP may be rejected as containing another positive term xy as a factor, whence we have

$$xy + xQ + PQ$$
$$= xy + (x+P)Q.$$

But this expresses the rejection of the term xy from the second factor, and its transference to the final product. Wherefore the Proposition is manifest.

PROPOSITION III.

4. *If t be any symbol which is retained in the final result of the elimination of any other symbols from any system of equations, the result of such elimination may be expressed in the form*

$$Et + E(1-t) = 0,$$

in which E is formed by making in the proposed system $t = 1$, and eliminating the same other symbols; and E' by making in the proposed system $t = 0$, and eliminating the same other symbols.

Eliminating $\vec{x}$ from $f(\vec{x}, t, \vec{y}) = 0$ gives $\prod_\sigma f(\sigma, t, \vec{y}) = 0$, which, by expansion about t, is equivalent to $t \prod_\sigma f(\sigma, 1, \vec{y}) + (1-t) \prod_\sigma f(\sigma, 0, \vec{y}) = 0$.

Boole wrote below that, in general, the two products in the last equation are much less complex than the single one in the previous equation.

For let $\phi(t) = 0$ represent the final result of elimination. Expanding this equation, we have

$$\phi(1)\,t + \phi(0)\,(1-t) = 0.$$

Now by whatever process we deduce the function $\phi(t)$ from the proposed system of equations, by the same process should we deduce $\phi(1)$, if in those equations t were changed into 1; and by the same process should we deduce $\phi(0)$, if in the same equations t were changed into 0. Whence the truth of the proposition is manifest.

5. Of the three propositions last proved, it may be remarked, that though quite unessential to the strict development or application of the general theory, they yet accomplish important ends of a practical nature. By Prop. 1 we can simplify the results of addition; by Prop. 2 we can simplify those of multiplication; and by Prop. 3 we can break up any tedious process of elimination into two distinct processes, which will in general be of a much less complex character. This method will be very frequently adopted, when the final object of inquiry is the determination of the value of t, in terms of the other symbols which remain after the elimination is performed.

An argument of Aristotle.

6. Ex. 1.—Aristotle, in the Nicomachean Ethics, Book II. Cap. 3, having determined that actions are virtuous, not as possessing in themselves a certain character, but as implying a certain condition of mind in him who performs them, viz., that he perform them knowingly, and with deliberate preference, and for their own sakes, and upon fixed principles of conduct, proceeds in the two following chapters to consider the question, whether virtue is to be referred to the genus of Passions, or Faculties, or Habits, together with some other connected points. He grounds his investigation upon the following premises, from which, also, he deduces the general doctrine and definition of moral virtue, of which the remainder of the treatise forms an exposition.

PREMISES.

1. Virtue is either a passion (πάθος), or a faculty (δύναμις), or a habit (ἕξις).

2. Passions are not things according to which we are praised or blamed, or in which we exercise deliberate preference.

3. Faculties are not things according to which we are praised or blamed, and which are accompanied by deliberate preference.

4. Virtue is something according to which we are praised or blamed, and which is accompanied by deliberate preference.

5. Whatever art or science makes its work to be in a good state avoids extremes, and keeps the mean in view relative to human nature (τὸ μέσον ... πρὸς ἡμᾶς)

6. Virtue is more exact and excellent than any art or science.

This is an argument *à fortiori.* If science and true art shun defect and extravagance alike, much more does virtue pursue the undeviating line of moderation. If *they* cause their work to be in a good state, much more reason have to we to say that Virtue causeth her peculiar work to be "in a good state." Let the final premiss be thus interpreted. Let us also pretermit all reference to praise or blame, since the mention of these in the premises accompanies only the mention of deliberate preference, and this is an element which we purpose to retain. We may then assume as our representative symbols—

v = virtue.
p = passions.
f = faculties.
h = habits.
d = things accompanied by deliberate preference.

g = things causing their work to be in a good state.
m = things keeping the mean in view relative to human nature.

The letter "v" is not the only one used to express "some"; this example uses "q".

Using, then, q as an indefinite class symbol, our premises will be expressed by the following equations:

$$\begin{aligned}
v &= q\big\{p\,(1-f)\,(1-h) \;+\; f\,(1-p)\,(1-h) \;+\; h\,(1-p)\,(1-f)\big\}.\\
p &= q\,(1-d)\,.\\
f &= q\,(1-d)\,.\\
v &= qd.\\
g &= qm.\\
v &= qg.
\end{aligned}$$

And separately eliminating from these the symbols q,

$$\begin{aligned}
v\big\{1-p\,(1-f)\,(1-h)-f\,(1-p)\,(1-h)&\\
-h\,(1-p)\,(1-f)\big\} &= 0\,. &(1)\\
pd &= 0\,. &(2)\\
fd &= 0\,. &(3)\\
v\,(1-d) &= 0\,. &(4)\\
g\,(1-m) &= 0\,. &(5)\\
v\,(1-g) &= 0\,. &(6)
\end{aligned}$$

We shall first eliminate from (2), (3), and (4) the symbol d, and then determine v in relation to p, f, and h. Now the addition of (2), (3), and (4) gives

$$(p+f)\,d \;+\; v\,(1-d) \;=\; 0\,.$$

From which, eliminating d in the ordinary way, we find

$$(p+f)\,v \;=\; 0\,. \qquad (7)$$

Adding this to (1), and determining v, we find

$$v = \frac{0}{p+f+1-p\,(1-f)\,(1-h)-f\,(1-p)\,(1-h)-h\,(1-f)\,(1-p)}.$$

Whence by development,

$$v = \frac{0}{0}h\,(1-f)\,(1-p)\,.$$

The interpretation of this equation is: *Virtue is a habit, and not a faculty or a passion.*

Next, we will eliminate f, p, and g from the original system of equations, and then determine v in relation to h, d, and m. We will in this case eliminate p and f together. On addition of (1), (2), and (3), we get

$$v\big\{1 - p(1-f)(1-h) - f(1-p)(1-h) - h(1-p)(1-f)\big\} + pd + fd = 0\,.$$

Developing this with reference to p and f, we have

$$(v+2d)pf + (vh+d)p(1-f) + (vh+d)(1-p)f + v(1-h)(1-p)(1-f) = 0\,.$$

Whence the result of elimination will be

$$(v+2d)(vh+d)(vh+d)v(1-h) = 0\,.$$

Now $v + 2d = v + d + d$, which by Prop. I. is reducible to $v + d$. The product of this and the second factor is

$$(v+d)(vh+d),$$

which by Prop. II. reduces to $d + v(vh)$ or $vh + d$.

In like manner, this result, multiplied by the third factor, gives simply $vh + d$. Lastly, this multiplied by the fourth factor, $v(1-h)$, gives, as the final equation,

$$vd(1-h) = 0 \tag{8}$$

It remains to eliminate g from (5) and (6). The result is

$$v(1-m) = 0 \tag{9}$$

Finally, the equations (4), (8), and (9) give on addition

$$v(1-d) + vd(1-h) + v(1-m) = 0$$

from which we have

$$v = \frac{0}{1-d + d(1-h) + 1-m}.$$

And the development of this result gives

$$v = \frac{0}{0}hdm,$$

of which the interpretation is,—*Virtue is a habit accompanied by* *deliberate preference, and keeping in view the mean relative to human nature.*

Properly speaking, this is not a definition, but a description of virtue. It is *all*, however, that can be correctly inferred from the premises. Aristotle specially connects with it the necessity of prudence, to determine the safe and middle line of action; and there is no doubt that the ancient theories of virtue generally partook more of an intellectual character than those (the theory of utility excepted) which have most prevailed in modern days. Virtue was regarded as consisting in the right state and habit of the whole mind, rather than in the single supremacy of conscience or the moral facility. And to some extent those theories were undoubtedly right. For though unqualified obedience to the dictates of conscience is an essential element of virtuous conduct, yet the conformity of those dictates with those unchanging principles of rectitude (αἰώνια δίκαια)which are founded in, or which rather are themselves the foundation of the constitution of things, is another element. And generally this conformity, in any high degree at least, is inconsistent with a state of ignorance and mental hebetude. Reverting to the particular theory of Aristotle, it will probably appear to most that it is of too negative a character, and that the shunning of extremes does not afford a sufficient scope for the expenditure of the nobler energies of our being. Aristotle seems to have been imperfectly conscious of this defect of his system, when in the opening of his seventh book he spoke of an "heroic virtue"[1] rising above the measure of human nature.

7. I have already remarked (VIII. 1) that the theory of single equations or propositions comprehends questions which cannot be fully answered, except in connexion with the theory of systems of equations. This remark is exemplified when it is proposed to determine from a given single equation the relation, not of some single elementary class, but of some compound class, involving in its expression more than one element, in terms of the remaining elements. The following particular example, and the succeeding general problem, are of this nature.

[1] τὴν ὑπὲρ ἡμᾶς ἀρετὴν ἡρωϊκήν τινα και θειαν—Nic. Eth. Book vii.

Ex. 2.—Let us resume the symbolical expression of the definition of wealth employed in Chap, VII., viz.,

$$w = st\{p + r(1-p)\},$$

wherein, as before,

w = wealth,
s = things limited in supply,
t = things transferable,
p = things productive of pleasure,
r = things preventive of pain;

and suppose it required to determine hence the relation of things transferable and productive of pleasure, to the other elements of the definition, viz., wealth, things limited in supply, and things preventive of pain.

The expression for things transferable and productive of pleasure is tp. Let us represent this by a new symbol y. We have then the equations

$$w = st\{p + r(1-p)\}, \quad (1)$$
$$y = tp, \quad (2)$$

Boole's LT failed to number these two equations, starting the numbering with (3) below.

from which, if we eliminate t and p, we may determine y as a function of w, s, and r. The result interpreted will give the relation sought.

Bringing the terms of these equations to the first side, we have

$$w - stp - str(1-p) = 0.$$
$$y - tp = 0. \quad (3)$$

And adding the squares of these equations together,

$$w + stp + str(1-p) - 2wstp - 2wstr(1-p) + y + tp - 2ytp = 0. \quad (4)$$

Developing the first member with respect to t and p, in order to eliminate those symbols, we have

$$(w + s - 2ws + 1 - y)\,tp + (w + sr - 2wsr + y)\,t(1-p)$$
$$+ (w+y)(1-t)p + (w+y)(1-t)(1-p); \quad (5)$$

and the result of the elimination of t and p will be obtained by equating to 0 the product of the four coefficients of

$$tp, \quad t(1-p), \quad (1-t)p, \quad \text{and } (1-t)(1-p).$$

Or, by Prop. 3, the result of the elimination of t and p from the above equation will be of the form

$$Ey \ + \ E'(1-y)\,,$$

wherein E is the result obtained by changing in the given equation y into 1, and then eliminating t and p; and E' the result obtained by changing in the same equation y into 0, and then eliminating t and p. And the mode in each case of eliminating t and p is to multiply together the coefficients of the four constituents tp, $t(1-p)$, &c.

If we make $y = 1$, the coefficients become—

1st. $w(1-s) \ + \ s(1-w)$

2nd. $1 \ + \ w(1-sr) \ + \ s(1-w)r$, equivalent to 1 by Prop. I.

3rd and 4th. $1+w$, equivalent to 1 by Prop. I.

Hence the value of E will be

$$w(1-s) \ + \ s(1-w)\,.$$

Again, in (5) making $y = 0$, we have for the coefficients—

1st. $1 \ + \ w(1-s) \ + \ s(1-w)$, equivalent to 1.

2nd. $w(1-sr) \ + \ sr(1-w)$.

3rd and 4th. w.

The product of these coefficients gives

$$E' = w(1-sr)\,.$$

The equation from which y is to be determined, therefore, is

$$\big\{w(1-s) \ + \ s(1-w)\big\}y \ + \ w(1-sr)(1-y) \ = \ 0\,,$$

$$\therefore \ y = \frac{w(1-sr)}{w(1-sr) - w(1-s) - s(1-w)}\,;$$

and expanding the second member,

$$\begin{aligned} y \ &= \frac{0}{0}wsr \ + \ ws(1-r) \ + \ \frac{1}{0}w(1-s)r \ + \ \frac{1}{0}w(1-s)(1-r) \\ &\quad + 0(1-w)sr \ + \ 0(1-w)s(1-r) \ + \ \frac{0}{0}(1-w)(1-s)r \\ &\qquad + \ \frac{0}{0}(1-w)(1-s)(1-r)\,; \end{aligned}$$

whence reducing

$$y = ws(1-r) \ + \ \frac{0}{0}wsr \ + \ \frac{0}{0}(1-w)(1-s), \tag{6}$$

$$\text{with} \quad w(1-s) = 0 \,. \tag{7}$$

The interpretation of which is—

1st. *Things transferable and productive of pleasure consist of all wealth (limited in supply and) not preventive of pain, an indefinite amount of wealth (limited in supply and) preventive of pain, and an indefinite amount of what is not wealth and not limited in supply.*

2nd. *All wealth is limited in supply.*

I have in the above solution written in parentheses that part of the full description which is implied by the accompanying independent relation (7).

8. The following problem is of a more general nature, and will furnish an easy practical rule for problems such as the last.

General Problem.

Given any equation connecting the symbols x, y..w, z..

Required to determine the logical expression of any class expressed in any way by the symbols x, y.. in terms of the remaining symbols, w, z, &c.

Let us confine ourselves to the case in which there are but two symbols, x, y, and two symbols, w, z, a case sufficient to determine the general Rule.

Let $V = 0$ be the given equation, and let $\phi(x, y)$ represent the class whose expression is to be determined.

Assume $t = \phi(x, y)$, then, from the above two equations, x and y are to be eliminated.

Now the equation $V = 0$ may be expanded in the form

$$Axy \ + \ Bx(1-y) \ + \ C(1-x)y \ + \ D(1-x)(1-y) \ = \ 0 \,, \tag{1}$$

A, B, C, and D being functions of the symbols w and z.

Again, as $\phi(x, y)$ represents a class or collection of things, it must consist of a constituent, or series of constituents, whose coefficients are 1.

Wherefore if the *full* development of $\phi(x, y)$ be represented in the form

$$axy \;+\; bx(1-y) + c(1-x)y \;+\; d(1-x)(1-y),$$

the coefficients a, b, c, d must each be 1 or 0 .

Now reducing the equation $t = \phi(x, y)$ by transposition and squaring, to the form

$$t\bigl\{1-\phi(x,y)\bigr\} \;+\; \phi(x,y)(1-t) \;=\; 0\,;$$

and expanding with reference to x and y, we get

$$\begin{aligned}\bigl\{t(1-a) \;+\; a(1-t)\bigr\}xy \;+\; \{t(1-b) \;+\; b(1-t)\}x(1-y)\\ +\bigl\{t(1-c) \;+\; c(1-t)\bigr\}(1-x)y\\ +\bigl\{t(1-d) \;+\; d(1-t)\bigr\}(1-x)(1-y) \;=\; 0\,;\end{aligned}$$

whence, adding this to (1), we have

$$\begin{aligned}\{A \;+\; t(1-a) \;+\; a(1-t)\}xy\\ +\bigl\{B \;+\; t(1-b) \;+\; b(1-t)\bigr\}x(1-y) \;+\; \&c. \;=\; 0\,.\end{aligned}$$

Let the result of the elimination of x and y be of the form

$$Et \;+\; E'(1-t) \;=\; 0\,,$$

then E will, by what has been said, be the reduced product of what the coefficients of the above expansion become when $t = 1$, and E' the product of the same factors similarly reduced by the condition $t = 0$.

Hence E will be the reduced product

$$(A+1-a)(B+1-b)(C+1-c)(D+1-d).$$

Considering any factor of this expression, as $A + 1 - a$, we see that when $a = 1$ it becomes A, and when $a = 0$ it becomes $1 + A$, which reduces by Prop. I. to 1. Hence we may infer that E will be the product of the coefficients of those constituents in the development of V whose coefficients in the development of $\phi(x, y)$ are 1.

Moreover E' will be the reduced product

$$(A+a)(B+b)(C+c)(D+d).$$

Considering any one of these factors, as $A+a$, we see that this becomes A when $a = 0$, and reduces to 1 when $a = 1$; and so on for the others. Hence E' will be the product of the coefficients of those constituents in the development of y, whose coefficients in the development $\phi(x, y)$ are 0. Viewing these cases together, we may establish the following Rule:

9. *To deduce from a logical equation the relation of any class expressed by a given combination of the symbols x, y, &c, to the classes represented by any other symbols involved in the given equation.*

RULE.—*Expand the given equation with reference to the symbols x, y. Then form the equation*

$$Et \;+\; E'(1-t) = 0\,,$$

in which E is the product of the coefficients of all those constituents in the above development, whose coefficients in the expression of the given class are 1, and E' the product of the coefficients of those constituents of the development whose coefficients in the expression of the given class are 0. The value of t deduced from the above equation by solution and interpretation will be the expression required.

NOTE.—*Although in the demonstration of this Rule V is supposed to consist solely of positive terms, it may easily be shown that this condition is unnecessary, and the Rule general, and that no preparation of the given equation is really required.*

10. Ex. 3.—The same definition of wealth being given as in Example 2, required an expression for *things transferable, but not productive of pleasure*, $t(1-p)$, in terms of the other elements represented by w, s, and r.

The equation

$$w - stp - str(1-p) = 0\,,$$

gives, when squared,

$$w \;+\; stp \;+\; str(1-p) - 2wstp - 2wstr(1-p) \;=\; 0\,;$$

and developing the first member with respect to t and p,

$$(w + s - 2ws)tp \;+\; (w + sr - 2wsr)t(1-p) \;+\; w(1-t)p \\ + w(1-t)(1-p) \;=\; 0\,.$$

The coefficients of which it is best to exhibit as in the following equation;

$$\big\{w(1-s) \;+\; s(1-w)\big\}tp \;+\; w(1-sr) \;+\; sr(1-w)t(1-p) \;+\; w(1-t)p \\ + w(1-t)(1-p) \;=\; 0\,.$$

Let the function $t(1-p)$ to be determined, be represented by z; then the full development of z in respect of t and p is

$$z \;=\; 0tp \;+\; t(1-p) \;+\; 0(1-t)p \;+\; 0(1-t)(1-p).$$

Hence, by the last problem, we have

$$Ez \;+\; E'(1-z) \;=\; 0\,;$$

where
$$E = w(1-sr) \;+\; sr(1-w)\,;$$
$$E' \;=\; \bigl\{w(1-s) \;+\; s(1-w)\bigr\} \times w \times w = w(1-s)\,;$$
$$\therefore\; \bigl\{w(1-sr) \;+\; sr(1-w)\bigr\}z \;+\; w(1-s)(1-z) \;=\; 0\,.$$

Hence,

$$z \;=\; \frac{w(1-s)}{2wsr - ws - sr}$$
$$= \frac{0}{0}wsr \;+\; 0ws(1-r) \;+\; \frac{1}{0}w(1-s)r \;+\; \frac{1}{0}w(1-s)(1-r),$$
$$+\; 0(1-w)sr \;+\; \frac{0}{0}(1-w)s(1-r) \;+\; \frac{0}{0}(1-w)(1-s)r$$
$$+\; \frac{0}{0}(1-w)(1-s)(1-r).$$

Or,
$$z = \frac{0}{0}wsr \;+\; \frac{0}{0}(1-w)s(1-r) \;+\; \frac{0}{0}(1-w)(1-s),$$

with
$$w(1-s) \;=\; 0\,.$$

Hence, *Things transferable and not productive of pleasure are either wealth (limited in supply and preventive of pain); or things which are not wealth, but limited in supply and not preventive of pain; or things which are not wealth, and are unlimited in supply.*

The following results, deduced in a similar manner, will be easily verified:

Things limited in supply and productive of pleasure which are not wealth,—are intransferable.

Wealth that is not productive of pleasure is transferable, limited in supply, and preventive of pain.

Things limited in supply which are either wealth, or are productive of pleasure, but not both,—are either transferable and preventive of pain, or intransferable.

11. From the domain of natural history a large number of curious examples might be selected. I do not, however, conceive that such applications would possess any independent value. They would, for instance, throw no light upon the true principles of classification in the science of zoology. For the discovery of these, some basis of positive knowledge is requisite,—some acquaintance with organic structure, with teleological adaptation; and this is a species of knowledge which can only be derived from the use of external means of observation and analysis. Taking, however, any collection of propositions in natural history, a great number of logical problems present themselves, without regard to the system of classification adopted. Perhaps in forming such examples, it is better to avoid, as superfluous, the mention of that property of a class or species which is immediately suggested by its name, e.g. the ring-structure in the annelida, a class of animals including the earth-worm and the leech.

Ex. 4.—1. The annelida are soft-bodied, and either naked or enclosed in a tube.

2. The annelida consist of all invertebrate animals having red blood in a double system of circulating vessels.

Assume $a =$ annelida;
$s =$ soft-bodied animals;
$n =$ naked;
$t =$ enclosed in a tube;
$i =$ invertebrate;
$r =$ having red blood, &c.

Then the propositions given will be expressed by the equations

$$a = vs\{n(1-t) + t(1-n)\}; \quad (1)$$

$$a = ir; \quad (2)$$

to which we may add the implied condition,

$$nt = 0. \quad (3)$$

On eliminating v, and reducing the system to a single equation, we have

$$a\{1-sn(1-t)-st(1-n)\} + a(1-ir) + ir(1-a) + nt = 0. \quad (4)$$

Suppose that we wish to obtain the relation in which soft-bodied animals enclosed in tubes arc placed (by virtue of the premises) with respect to the following elements, viz., the possession of red blood, of an external covering, and of a vertebral column.

We must first eliminate a. The result is

$$ir\{1 - sn(1-t) - st(1-n)\} + nt = 0.$$

Then (IX. 9) developing with respect to s and t, and reducing the first coefficient by Prop. 1, we have

$$nst + ir(1-n)s(1-t) + (ir+n)(1-s)t + ir(1-s)(1-t) = 0. \quad (5)$$

Hence, if $st = w$, we find

$$nw + ir(1-n) \times (ir+n) \times ir(1-w) = 0;$$

or,

$$nw + ir(1-n)(1-w) = 0;$$
$$\therefore w = \frac{ir(1-n)}{ir(1-n)-n}$$
$$= 0irn + ir(1-n) + 0i(1-r)n + \frac{0}{0}i(1-r)(1-n)$$
$$+ 0(1-i)rn + \frac{0}{0}(1-i)r(1-n) + 0(1-i)(1-r)n$$
$$+ \frac{0}{0}(1-i)(1-r)(1-n);$$

or,

$$w = ir(1-n) + \frac{0}{0}i(1-r)(1-n) + \frac{0}{0}(1-i)(1-n).$$

Hence, *soft-bodied animals enclosed in tubes consist of all invertebrate animals having red blood and not naked, and an indefinite remainder of invertebrate animals not having red blood and not naked, and of vertebrate animals which are not naked.*

And in an exactly similar manner, the following reduced equations, the interpretation of which is left to the reader, have been deduced from the development (5).

$$s(1-t) = irn + \frac{0}{0}i(1-n) + \frac{0}{0}(1-i)$$
$$(1-s)t = \frac{0}{0}(1-i)r(1-n) + \frac{0}{0}(1-r)(1-n)$$
$$(1-s)(1-t) = \frac{0}{0}i(1-r) + \frac{0}{0}(1-i).$$

In none of the above examples has it been my object to exhibit in any special manner the power of the method. That, I conceive, can only be fully displayed in connexion with the mathematical theory of probabilities. I would, however, suggest to any who may be desirous of forming a correct opinion upon this point, that they examine by the rules of ordinary logic the following problem, *before* inspecting its solution; remembering at the same time, that whatever complexity it possesses might be multiplied indefinitely, with no other effect than to render its solution by the method of this work more operose, but not less certainly attainable.

Only probability studies can show the real power of his algebra of logic.

Ex. *5.* Let the observation of a class of natural productions be supposed to have led to the following general results.

A popular example from Boole's LT.

1st, That in whichsoever of these productions the properties A and C are missing, the property E is found, together with one of the properties B and D, but not with both.

2nd, That wherever the properties A and D are found while E is missing, the properties B and C will either both be found, or both be missing.

3rd, That wherever the property A is found in conjunction with either B or E, or both of them, there either the property C or the property D will be found, but not both of them. And conversely, wherever the property C or D is found singly, there the property A will be found in conjunction with either B or E, or both of them.

Let it then be required to ascertain, first, what in any particular instance may be concluded from the ascertained presence of the property A, with reference to the properties B, C, and D; also whether any relations exist independently among the properties B, C, and D. Secondly, what may be concluded in like manner respecting the property B, and the properties A, C, and D.

It will be observed, that in each of the three data, the information conveyed respecting the properties A, B, C, and D, is complicated with another element, E, about which we desire to say nothing in our conclusion. It will hence be requisite to eliminate the symbol representing the property E from the system of equations, by which the given propositions will be expressed.

Let us represent the property A by x, B by y, C by z, D by w, E by v. The data are

Note that q is now used for the indefinite class symbol.

$$\bar{x}\bar{z} = qv(y\bar{w} + w\bar{y})\,; \qquad (1)$$

$$\bar{v}xw = q(yz + \bar{y}\bar{z})\,; \qquad (2)$$

$$xy + xv\bar{y} = w\bar{z} + z\bar{w}\,; \qquad (3)$$

$\bar{x}$ standing for $1-x$, &c., and q being an indefinite class symbol. Eliminating q separately from the first and second equations, and adding the results to the third equation reduced by (5), Chap.VIII., we get

$$\bar{x}\bar{z}(1 - vy\bar{w} - vw\bar{y}) + \bar{v}xw(y\bar{z} + z\bar{y}) + (xy + xv\bar{y})(wz + \bar{w}\bar{z}) + (w\bar{z} + z\bar{w})(1 - xy - xv\bar{y}) = 0\,. \qquad (4)$$

From this equation v must be eliminated, and the value of x determined from the result. For effecting this object, it will be convenient to employ the method of Prop. 3 of the present chapter.

Let then the result of elimination be represented by the equation

$$Ex + E'(1-x) = 0\,.$$

To find E make $x = 1$ in the first member of (4), we find

$$\bar{v}w(y\bar{z} + z\bar{y}) + (y + v\bar{y})(wz + \bar{w}\bar{z}) + (w\bar{z} + z\bar{w})\bar{v}\bar{y}.$$

Eliminating v, we have

$$(wz + \bar{w}\bar{z})\big\{w(y\bar{z} + z\bar{y}) + y(wz + \bar{w}\bar{z}) + \bar{y}(w\bar{z} + z\bar{w})\big\}\,;$$

which, on actual multiplication, in accordance with the conditions $w\bar{w} = 0$, $z\bar{z} = 0$, &c., gives

$$E = wz + y\bar{w}\bar{z}$$

Next, to find E' make $x = 0$ in (4), we have

$$z(1 - vy\bar{w} - v\bar{y}w) + w\bar{z} + z\bar{w}.$$

whence, eliminating v, and reducing the result by Propositions 1 and 2, we find

$$E' = w\bar{z} + z\bar{w} + \bar{y}\bar{w}\bar{z}\,;$$

and, therefore, finally we have

$$(wz + y\bar{w}\bar{z})x + (w\bar{z} + z\bar{w} + \bar{y}\bar{w}\bar{z})\bar{x} = 0\,; \qquad (5)$$

from which

$$x = \frac{w\bar{z} + z\bar{w} + \bar{y}\bar{w}\bar{z}}{w\bar{z} + z\bar{w} + \bar{y}\bar{w}\bar{z} - wz - y\bar{w}\bar{z}}$$

wherefore, by development,

$$\begin{aligned} x = & \ 0yzw + yz\bar{w} + y\bar{z}w + 0y\bar{z}\bar{w} \\ & +0\bar{y}zw + \bar{y}z\bar{w} + \bar{y}\bar{z}w + \bar{y}\bar{z}\bar{x}\,; \end{aligned}$$

or, collecting the terms in vertical columns,

$$x = z\bar{w} + \bar{z}w + \bar{y}\bar{z}\bar{w}\,; \qquad (6)$$

the interpretation of which is—

In whatever substances the property A is found, there will also be found either the property C or the property D, but not both, or else the properties B, C, and D, will all be wanting. And conversely, *where either the property C or the property D is found singly, or the properties B, C, and D, are together missing, there the property A will be found.*

It also appears that there is no independent relation among the properties B, C, and D.

Secondly, we are to find y. Now developing (5) with respect to that symbol,

$$(xwz + x\bar{w}\bar{z} + \bar{x}w\bar{z} + \bar{x}z\bar{w})y + (xwz + \bar{x}w\bar{z} + \bar{x}z\bar{w} + \bar{x}\bar{z}\bar{w})\bar{y} = 0\,;$$

whence, proceeding as before,

$$y = \bar{x}\bar{w}\bar{z} + \frac{0}{0}(\bar{x}wz + xw\bar{z} + xz\bar{w}), \qquad (7)$$

$$xzw = 0\,; \qquad (8)$$

$$\bar{x}z\bar{z}w = 0\,; \qquad (9)$$

$$\bar{x}z\bar{w} = 0\,; \qquad (10)$$

From (10) reduced by solution to the form

$$\bar{x}z = \frac{0}{0}w\,;$$

we have the independent relation,—*If the property A is absent and C present, D is present.* Again, by addition and solution (8) and (9) give

$$xz + \bar{x}\bar{z} = \frac{0}{0}\bar{w}.$$

Whence we have for the general solution and the remaining independent relation:

1st. *If the property B be present in one of the productions, either the properties A, C, and D, are all absent, or some one alone of them is absent.* And conversely, *if they are all absent it may be concluded that the property A is present* (7).

2nd. *If A and C are both present or both absent, D will be absent, quite independently of the presence or absence of B* (8) and (9).

I have not attempted to verify these conclusions.

A curious admission.

Chapter X

Of the Conditions of a Perfect Method.

1. The subject of Primary Propositions has been discussed at length, and we are about to enter upon the consideration of Secondary Propositions. The interval of transition between these two great divisions of the science of Logic may afford a fit occasion for us to pause, and while reviewing some of the past steps of our progress, to inquire what it is that in a subject like that with which we have been occupied constitutes perfection of method. I do not here speak of that perfection only which consists in power, but of that also which is founded in the conception of what is fit and beautiful. It is probable that a careful analysis of this question would conduct us to some such conclusion as the following, viz., that a perfect method should not only be an efficient one, as respects the accomplishment of the objects for which it is designed, but should in all its parts and processes manifest a certain unity and harmony. This conception would be most fully realized if even the very forms of the method were suggestive of the fundamental principles, and if possible of the one fundamental principle, upon which they are founded. In applying these considerations to the science of Reasoning, it may be well to extend our view beyond the mere analytical processes, and to inquire what is best as respects not only the mode or form of deduction, but also the system of data or premises from which the deduction is to be made.

2. As respects mere power, there is no doubt that the first of the methods developed in Chapter VIII. is, within its proper sphere, a perfect one. The introduction of arbitrary constants makes us independent of the forms of the premises, as well as of any conditions among the equations by which they are represented. But it seems to introduce a foreign element, and while it is a more laborious, it is also a less elegant form of solution than the second method of reduction demonstrated in the same chapter. There are, however, conditions under which the latter method assumes a more perfect form than it otherwise bears. To make the one fundamental condition expressed by the equation

$$x(1-x) = 0,$$

the universal type of form, would give a unity of character to both processes and results, which would not else be attainable. Were brevity or convenience the only valuable quality of a method, no advantage would flow from the adoption of such a principle. For to impose upon every step of a solution the character above described, would involve in some instances no slight labour of preliminary reduction. But it is still interesting to know that this can be done, and it is even of some importance to be acquainted with the conditions under which such a form of solution would spontaneously present itself. Some of these points will be considered in the present chapter.

A stronger requirement is that all terms be totally defined, that is, every subterm of a term satisfies the idempotent law.

One can do the algebra of logic using only idempotent terms, but at the cost of much labor. (One can replace 'idempotent' in this statement with 'totally defined'.)

Proposition I.

3. *To reduce any equation among logical symbols to the form $V = 0$, in which V satisfies the law of duality,*

$$V(1-V) = 0.$$

V denotes a polynomial in this chapter.

It is shown in Chap. V. Prop. 4, that the above condition is satisfied whenever V is the sum of a series of constituents. And it is evident from Prop. 2, Chap. VI. that all equations are equivalent which, when reduced by transposition to the form $V = 0$, produce, by development of the first member, the same series of constituents with coefficients which do not vanish; the particular numerical values of those coefficients being immaterial.

One can convert any polynomial equation $W = 0$ into an equivalent equation $V = 0$ with V idempotent — let V be the sum of the constituents in the expansion of W that have nonzero coefficients. (This V is not only idempotent, it is totally defined.)

Hence the object of this Proposition may always be accomplished by bringing all the terms of an equation to the first side, fully expanding that member, and changing in the result all the coefficients which do not vanish into unity, except such as have already that value.

A convenient notation is to let $\widehat{W}$ denote this conversion of a polynomial W into an idempotent polynomial.

But as the development of functions containing many symbols conducts us to expressions inconvenient from their great length, it is desirable to show how, in the only cases which do practically offer themselves to our notice, this source of complexity may be avoided.

The great primary forms of equations have already been discussed in Chapter VIII. They are—

$$X = vY,$$
$$X = Y,$$
$$vX = vY.$$

Whenever the conditions $X(1-X)=0\,, Y(1-Y)=0$, are satisfied, we have seen that the two first of the above equations conduct us to the forms

$$X(1-Y) = 0\,, \qquad (1)$$
$$X(1-Y) + Y(1-X) = 0\,; \qquad (2)$$

and under the same circumstances it may be shown that the last of them gives

$$v\{X(1-Y) + Y(1-X)\} = 0\,; \qquad (3)$$

all which results obviously satisfy, in their first members, the condition

$$V(1-V) = 0\,.$$

Now as the above are the forms and conditions under which the equations of a logical system properly expressed do actually present themselves, it is always possible to reduce them by the above method into subjection to the law required. Though, however, the separate equations may thus satisfy the law, their equivalent sum (VIII. 4) may not do so, and it remains to show how upon it also the requisite condition may be imposed.

Let us then represent the equation formed by adding the several reduced equations of the system together, in the form

$$v + v' + v'', \&c. = 0\,, \qquad (4)$$

this equation being singly equivalent to the system from which it was obtained. We suppose v, v', v'', &c. to be class terms (IX. 1) satisfying the conditions

$$v(1-v)=0\,, \quad v'(1-v') = 0\,, \quad \&c.$$

Now the full interpretation of (4) would be found by developing

Since complete expansions of terms W can be very large, Boole looked for cases where one could convert $W = 0$ with a reasonable amount of work into an equation $V = 0$ with V idempotent.

NOTE: The method he gave below for such cases actually gives a totally defined V if the X, Y in the primary equations are totally defined.

Given equational premisses in the above "great primary forms", where X and Y are idempotent, convert each premiss into the form $V = 0$ with V idempotent, as per (1)–(3) above. (If the X and Y are totally defined, then the V will be totally defined.) Let the resulting equations be

$$v = 0\,,\ v' = 0\,, \ldots$$

By Prop. V, pages 122–123, one can reduce this system to the single equation

$$v + v' + \cdots = 0,$$

but the left side may not be idempotent.

the first member with respect to all the elementary symbols x, y, &c. which it contains, and equating to 0 all the constituents whose coefficients do not vanish; in other words, all the constituents which are found in either v, v', v'', &c. But those constituents consist of—1st, such as are found in v; 2nd, such as are not found in v, but are found in v'; 3rd, such as are neither found in v nor v', but are found in v'', and so on. Hence they will be such as are found in the expression

$$v + (1 - v)v' + (1 - v)(1 - v')v'' + \&c., \tag{5}$$

an expression in which no constituents are repeated, and which obviously satisfies the law $V(1 - V) = 0$.

Putting the term in (5) equal to 0 gives a single equation

$$v+(1-v)v'+(1-v)(1-v')v''+\cdots = 0$$

equivalent to (4), but with the advantage that the left side is idempotent.
NOTE: If the v, v', ... are totally defined, then so is (5).

Thus if we had the expression

$$(1 - t) + v + (1 - z) + tzw,$$

in which the terms $1 - t$, $1 - z$ are bracketed to indicate that they are to be taken as single class terms, we should, in accordance with (5), reduce it to an expression satisfying the condition $V(1 - V) = 0$, by multiplying all the terms after the first by t, then all after the second by $1 - v$; lastly, the term which remains after the third by z; the result being

$$1 - t + tv + t(1 - v)(1 - z) + t(1 - v)zw. \tag{6}$$

4. All logical equations then are reducible to the form $V = 0$, V satisfying the law of duality. But it would obviously be a higher degree of perfection if equations always presented themselves in such a form, without preparation of any kind, and not only exhibited this form in their original statement, but retained it unimpaired after those additions which are necessary in order to reduce systems of equations to single equivalent forms. That they do not spontaneously present this feature is not properly attributable to defect of method, but is a consequence of the fact that our premises are not always complete, and accurate, and independent. They are not complete when they involve material (as distinguished from formal) relations, which are not expressed. They are not accurate when they imply relations which are not intended. But setting aside these points, with which, in the present instance, we are less concerned, let it be considered in what sense they may fail of being independent.

A system of equations with all terms totally defined can be reduced to the form $V = 0$ with V totally defined, without encountering any uninterpretables enroute.

5. A system of propositions may be termed independent, when it is not possible to deduce from any portion of the system a conclusion deducible from any other portion of it. Supposing the equations representing those propositions all reduced to the form

$$V = 0,$$

then the above condition implies that no constituent which can be made to appear in the development of a particular function V of the system, can be made to appear in the development of any other function V' of the same system. When this condition is not satisfied, the equations of the system are not independent. This may happen in various cases. Let all the equations satisfy in their first members the law of duality, then if there appears a positive term x in the expansion of one equation, and a term xy in that of another, the equations are not independent, for the term x is further developable into $xy + x(1 - y)$, and the equation

$$xy = 0$$

is thus involved in both the equations of the system. Again, let a term xy appear in one equation, and a term xz in another. Both these may be developed so as to give the common constituent xyz. And other cases may easily be imagined in which premises which appear at first sight to be quite independent are not really so. Whenever equations of the form $V = 0$ are thus not truly independent, though individually they may satisfy the law of duality,

$$V(1 - V) = 0,$$

the equivalent equation obtained by adding them together will not satisfy that condition, unless sufficient reductions by the method of the present chapter have been performed. When, on the other hand, the equations of a system both satisfy the above law, and are independent of each other, their sum will also satisfy the same law. I have dwelt upon these points at greater length than would otherwise have been necessary, because it appears to me to be important to endeavour to form to ourselves, and to keep before us in all our investigations, the pattern of an ideal perfection,—the object and the guide of future efforts. In

In modern logic, a set of statements is independent if none can be deduced from the others. This is not Boole's definition. If one thinks of an equation as asserting that some region of a Venn diagram is empty, then a system of propositions is Boole-independent if the various regions that are asserted to be empty are pairwise disjoint.

For idempotent terms $v, v', \ldots$, their sum $v + v' + \cdots$ is idempotent iff the equations $v = 0$, $v' = 0$, …are independent in the sense of Boole.

the present class of inquiries the chief aim of improvement of method should be to facilitate, as far as is consistent with brevity, the transformation of equations, so as to make the fundamental condition above adverted to universal.

In connexion with this subject the following Propositions are deserving of attention.

Proposition II.

If the first member of any equation $V = 0$ satisfy the condition $V(1-V) = 0$, and if the expression of any symbol t of that equation be determined as a developed function of the other symbols, the coefficients of the expansion can only assume the forms 1, 0, $\frac{0}{0}$, $\frac{1}{0}$.

V is a polynomial.

For if the equation be expanded with reference to t, we obtain as the result,

$$Et \ + \ E'(1-t), \tag{1}$$

E and E' being what V becomes when t is successively changed therein into 1 and 0. Hence E and E' will themselves satisfy the conditions

$$E(1-E) \ = \ 0\,, \quad E'(1-E') \ = \ 0\,. \tag{2}$$

Now (1) gives

$$t \ = \ \frac{E'}{E'-E},$$

the second member of which is to be expanded as a function of the remaining symbols. It is evident that the only numerical values which E and E' can receive in the calculation of the coefficients will be 1 and 0. The following cases alone can therefore arise:

1st. $E' = 1$, $E = 1$, then $\dfrac{E'}{E'-E} \ = \ \dfrac{1}{0}$.

2nd. $E' = 1$, $E = 0$, then $\dfrac{E'}{E'-E} \ = \ 1$.

3rd. $E' = 0$, $E = 1$, then $\dfrac{E'}{E'-E} \ = \ 0$.

4th. $E' = 0$, $E = 0$, then $\dfrac{E'}{E'-E} \ = \ \dfrac{0}{0}$.

Whence the truth of the Proposition is manifest.

6. It may be remarked that the forms 1, 0, and $\frac{0}{0}$ appear in the solution of equations independently of any reference to the condition $V(1-V)=0$. But it is not so with the coefficient $\frac{1}{0}$. The terms to which this coefficient is attached when the above condition is satisfied may receive any other value except the three values 1, 0, and $\frac{0}{0}$, when that condition is not satisfied. It is permitted, and it would conduce to uniformity, to change any coefficient of a development not presenting itself in any of the four forms referred to in this Proposition into $\frac{1}{0}$, regarding this as the symbol proper to indicate that the coefficient to which it is attached should be equated to 0. This course I shall frequently adopt.

If a coefficient of a constituent is not one of 1, 0, and 0/0, then it is of the form m/n with $0 \neq m \neq n$.
Boole liked to change all such coefficients to 1/0, his canonical choice of coefficient for the constituents that must be set equal to 0.

Proposition III.

7. *The result of the elimination of any symbols x, y, &c. from an equation $V=0$, of which the first member identically satisfies the law of duality,*

$$V(1-V) \;=\; 0\,,$$

may be obtained by developing the given equation with reference to the other symbols, and equating to 0 the sum of those constituents whose coefficients in the expansion are equal to unity.

Elimination Shortcut(?):
If $V(\vec{x},\vec{y})$ is idempotent, the result of eliminating $\vec{x}$ from $V(\vec{x},\vec{y})=0$ is $W(\vec{y})=0$ where $W(\vec{y})$ is the sum of the $\vec{y}$-constituents $C_\tau(\vec{y})$ for which $V(\vec{x},\tau)=1$,
i.e., $W(\vec{y})$ is
$\sum\{C_\tau(\vec{y}) : V(\vec{x},\tau)=1\}$.

Suppose that the given equation $V=0$ involves but three symbols, x, y, and t, of which x and y are to be eliminated. Let the development of the equation, with respect to t, be

$$At + B(1-t) \;=\; 0\,, \tag{1}$$

A and B being free from the symbol t.

By Chap. IX. Prop. 3, the result of the elimination of x and y from the given equation will be of the form

$$Et + E'(1-t) \;=\; 0\,, \tag{2}$$

in which E is the result obtained by eliminating the symbols x and y from the equation $A=0$, E' the result obtained by eliminating from the equation $B=0$.

Now A and B must satisfy the condition

$$A(1-A)=0\,,\quad B(1-B)\ =\ 0$$

Hence A (confining ourselves for the present to this coefficient) will either be 0 or 1, or a constituent, or the sum of a part of the constituents which involve the symbols x and y. If $A=0$ it is evident that $E=0$; if A is a single constituent, or the sum of a part of the constituents involving x and y, E will be 0. For the *full* development of A, with respect to x and y, will contain terms with vanishing coefficients, and E is the product of *all* the coefficients. Hence when $A=1$, E is equal to A, but in other cases E is equal to 0. Similarly, when $B=1$, E' is equal to B, but in other cases E' vanishes. Hence the expression (2) will consist of that part, if any there be, of (1) in which the coefficients A, B are unity. And this reasoning is general. Suppose, for instance, that V involved the symbols x, y, z, t, and that it were required to eliminate x and y. Then if the development of V, with reference to z and t, were

$$zt + xz(1-t) + y(1-z)t + (1-z)(1-t),$$

the result sought would be

$$zt + (1-z)(1-t) = 0\,,$$

this being that portion of the development of which the coefficients are unity.

Hence, if from any system of equations we deduce a single equivalent equation $V=0$, V satisfying the condition

$$V(1-V)=0\,,$$

the ordinary processes of elimination may be entirely dispensed with, and the single process of development made to supply their place.

8. It may be that there is no practical advantage in the method thus pointed out, but it possesses a theoretical unity and completeness which render it deserving of regard, and I shall accordingly devote a future chapter (XIV.) to its illustration. The progress of applied mathematics has presented other and signal examples of the reduction of systems of problems or equations to the dominion of some central but pervading law.

9. It is seen from what precedes that there is one class of propositions to which all the special appliances of the above methods of preparation are unnecessary. It is that which is characterized by the following conditions:

First, That the propositions are of the ordinary kind, implied by the use of the copula *is* or *are*, the predicates being particular.

Secondly, That the terms of the proposition are intelligible without the supposition of any understood relation among the elements which enter into the expression of those terms.

Thirdly, That the propositions are independent.

We may, if such speculation is not altogether vain, permit ourselves to conjecture that these are the conditions which would be obeyed in the employment of language as an instrument of expression and of thought, by unerring beings, declaring simply what they mean, without suppression on the one hand, and without repetition on the other. Considered both in their relation to the idea of a perfect language, and in their relation to the processes of an exact method, these conditions are equally worthy of the attention of the student.

Chapter XI

Of Secondary Propositions, and of the Principles of their Symbolical Expression.

1. The doctrine has already been established in Chap. IV., that every logical proposition may be referred to one or the other of two great classes, viz., Primary Propositions and Secondary Propositions. The former of these classes has been discussed in the preceding chapters of this work, and we are now led to the consideration of Secondary Propositions, i.e. of Propositions concerning, or relating to, other propositions regarded as true or false. The investigation upon which we are entering will, in its general order and progress, resemble that which we have already conducted. The two inquiries differ as to the subjects of thought which they recognise, not as to the formal and scientific laws which they reveal, or the methods or processes which are founded upon those laws. Probability would in some measure favour the expectation of such a result. It consists with all that we know of the uniformity of Nature, and all that we believe of the immutable constancy of the Author of Nature, to suppose, that in the mind, which has been endowed with such high capabilities, not only for converse with surrounding scenes, but for the knowledge of itself, and for reflection upon the laws of its own constitution, there should exist a harmony and uniformity not less real than that which the study of the physical sciences makes known to us. Anticipations such as this are never to be made the primary rule of our inquiries, nor are they in any degree to divert us from those labours of patient research by which we ascertain what is the actual constitution of things within the particular province submitted to investigation. But when the grounds of resemblance have been properly and independently determined, it is not inconsistent, even with purely scientific ends, to make that resemblance a subject of meditation, to trace its extent, and to receive the intimations of truth, yet undiscovered, which it may seem to us to convey. The necessity of a final appeal to fact is not thus set aside, nor is the use of analogy extended beyond its proper sphere,—the suggestion of relations which independent inquiry must either verify or cause to be rejected.

2. *Secondary Propositions are those which concern or relate to Propositions considered as true or false.* The relations of *things* we express by primary propositions. But we are able to make Propositions themselves also the subject of thought, and to express our judgments concerning them. The expression of any such judgment constitutes a secondary proposition. There exists no proposition whatever of which a competent degree of knowledge would not enable us to make one or the other of these two assertions, viz., either that the proposition is true, or that it is false; and each of these assertions is a secondary proposition. "It is true that the sun shines;" "It is not true that the planets shine by their own light;" are examples of this kind. In the former example the Proposition "The sun shines," is asserted to be true. In the latter, the Proposition, "The planets shine by their own light," is asserted to be false. Secondary propositions also include all judgments by which we express a relation or dependence among propositions. To this class or division we may refer conditional propositions, as, "If the sun shine the day will be fair." Also most disjunctive propositions, as, "Either the sun will shine, or the enterprise will be postponed." In the former example we express the dependence of the truth of the Proposition, "The day will be fair," upon the truth of the Proposition, "The sun will shine." In the latter we express a relation between the two Propositions, "The sun will shine," "The enterprise will be postponed," implying that the truth of the one excludes the truth of the other. To the same class of secondary propositions we must also refer all those propositions which assert the simultaneous truth or falsehood of propositions, as, "It is not true both that 'the sun will shine' and that 'the journey will be postponed.' " The elements of distinction which we have noticed may even be blended together in the same secondary proposition. It may involve both the disjunctive element expressed by *either*, *or*, and the conditional element expressed by *if*; in addition to which, the connected propositions may themselves be of a compound

Boole's secondary propositions seem to include (are the same as?) the non-primary propositions in the closure of the primary propositions under Boolean combinations and the operators '—- is true' and '—-is false'.

character. *If* "the sun shine," *and* "leisure permit," then *either* "the enterprise shall be commenced," *or* "some preliminary step shall be taken." In this example a number of propositions are connected together, not arbitrarily and unmeaningly, but in such a manner as to express a *definite* connexion between them,—a connexion having reference to their respective truth or falsehood. This combination, therefore, according to our definition, forms a Secondary Proposition.

The theory of Secondary Propositions is deserving of attentive study, as well on account of its varied applications, as for that close and harmonious analogy, already referred to, which it sustains with the theory of Primary Propositions. Upon each of these points I desire to offer a few further observations.

3. I would in the first place remark, that it is in the form of secondary propositions, at least as often as in that of primary propositions, that the reasonings of ordinary life are exhibited. The discourses, too, of the moralist and the metaphysician are perhaps less often concerning things and their qualities, than concerning principles and hypotheses, concerning truths and the mutual connexion and relation of truths. The conclusions which our narrow experience suggests in relation to the great questions of morals and society yet unsolved, manifest, in more ways than one, the limitations of their human origin; and though the existence of universal principles is not to be questioned, the partial formulae which comprise our knowledge of their application are subject to conditions, and exceptions, and failure. Thus, in those departments of inquiry which, from the nature of their subject-matter, should be the most interesting of all, much of our actual knowledge is hypothetical. That there has been a strong tendency to the adoption of the same forms of thought in writers on speculative philosophy, will hereafter appear. Hence the introduction of a general method for the discussion of hypothetical and the other varieties of secondary propositions, will open to us a more interesting field of applications than we have before met with.

4. The discussion of the theory of Secondary Propositions is in the next place interesting, from the close and remarkable analogy which it bears with the theory of Primary Propositions. It will appear, that the formal laws to which the operations of the mind are subject, are identical in expression in both cases. The mathematical processes which are founded on those laws are, therefore, identical also. Thus the methods which have been investigated in the former portion of this work will continue to be available in the new applications to which we are about to proceed. But while the laws and processes of the method remain unchanged, the rule of interpretation must be adapted to new conditions. Instead of classes of things, we shall have to substitute propositions, and for the relations of classes and individuals, we shall have to consider the connexions of propositions or of events. Still, between the two systems, however differing in purport and interpretation, there will be seen to exist a pervading harmonious relation, an analogy which, while it serves to facilitate the conquest of every yet remaining difficulty, is of itself an interesting subject of study, and a conclusive proof of that unity of character which marks the constitution of the human faculties.

Proposition I.

5. *To investigate the nature of the connexion of Secondary Propositions with the idea of Time.*

It is necessary, in entering upon this inquiry, to state clearly the nature of the analogy which connects Secondary with Primary Propositions.

Primary Propositions express relations among things, viewed as component parts of a universe within the limits of which, whether coextensive with the limits of the actual universe or not, the matter of our discourse is confined. The relations expressed are essentially *substantive*. Some, or all, or none, of the members of a given class, are also members of another class. The subjects to which primary propositions refer—the relations among those subjects which they express—are all of the above character.

But in treating of secondary propositions, we find ourselves concerned with another class both of subjects and relations. For the subjects with which we have to do are themselves propositions, so that the question may be asked,—Can we regard these subjects also as *things*, and refer them, by analogy with the previous case, to a universe of their own? Again, the relations among these subject propositions are relations of coexistent truth or falsehood, not of substantive equivalence. We do not say, when expressing the connexion of two distinct propositions, that the one *is* the other, but use some such forms of speech as the following, according to the meaning which we desire to convey: "*Either* the proposition X is true, *or* the proposition Y is true;" "If the proposition X is true, the proposition Y is true;" "The propositions X and Y are jointly true;" and so on.

Now, in considering any such relations as the above, we are not called upon to inquire into the whole extent of their possible meaning (for this might involve us in metaphysical questions of causation, which are beyond the proper limits of science); but it suffices to ascertain some meaning which they undoubtedly possess, and which is adequate for the purposes of logical deduction. Let us take, as an instance for examination, the conditional proposition, "If the proposition X is true, the proposition Y is true." An undoubted meaning of this proposition is, that the *time* in which the proposition X is true, is *time* in which the proposition Y is true. This indeed is only a relation of coexistence, and may or may not exhaust the meaning of the proposition, but it is a relation really involved in the statement of the proposition, and further, it suffices for all the purposes of logical inference.

Boole associated to a proposition the time during which it was true, and said this was all one needed for the logic of secondary propositions.

The language of common life sanctions this view of the essential connexion of secondary propositions with the notion of time. Thus we limit the application of a primary proposition by the word "some," but that of a secondary proposition by the word "sometimes." To say, "Sometimes injustice triumphs," is equivalent to asserting that there are times in which the proposition "Injustice now triumphs," is a true proposition. There are indeed propositions, the truth of which is not thus limited to particular periods or conjunctures; propositions which are true throughout all time, and have received the appellation of "eternal truths." Plato and Aristotle, by the latter of whom, especially, it is employed to denote the contrast between the abstract verities of science, such as the propositions of geometry which are always

The role of 'some' in primary propositions is played by 'sometimes' in secondary propositions. After this remark, Boole does not mention the use of 'sometimes' in secondary propositions.

true, and those contingent or phænomenal relations of things which are sometimes true and sometimes false. But the forms of language in which both kinds of propositions are expressed manifest a common dependence upon the idea of time; in the one case as limited to some finite duration, in the other as stretched out to eternity.

6. It may indeed be said, that in ordinary reasoning we are often quite unconscious of this notion of time involved in the very language we are using. But the remark, however just, only serves to show that we commonly reason by the aid of words and the forms of a well-constructed language, without attending to the ulterior grounds upon which those very forms have been established. The course of the present investigation will afford an illustration of the very same principle. I shall avail myself of the notion of time in order to determine the laws of the expression of secondary propositions, as well as the laws of combination of the symbols by which they are expressed. But when those laws and those forms are once determined, this notion of time (essential, as I believe it to be, to the above end) may practically be dispensed with. We may then pass from the forms of common language to the closely analogous forms of the symbolical instrument of thought here developed, and use its processes, and interpret its results, without any conscious recognition of the idea of time whatever.

Proposition II.

7. *To establish a system of notation for the expression of Secondary Propositions, and to show that the symbols which it involves are subject to the same laws of combination as the corresponding symbols employed in the expression of Primary Propositions.*

Let us employ the capital letters X, Y, Z, to denote the elementary propositions concerning which we desire to make some assertion touching their truth or falsehood, or among which we seek to express some relation in the form of a secondary proposition. And let us employ the corresponding small letters x, y, z, considered as expressive of mental operations, in the following sense, viz.: Let x represent an act of the mind by which we fix our regard upon that portion of time for which the proposition X is true; and let this meaning be understood when it is asserted that *x denotes* the time for which the proposition X is true. Let us further employ the connecting signs +, -, =, &c., in the following sense, viz.: Let $x + y$ denote the aggregate of those portions of time for which the propositions X and Y are respectively true, those times being entirely separated from each other. Similarly let $x - y$ denote that remainder of time which is left when we take away from the portion of time for which X is true, that (by supposition) included portion for which Y is true. Also, let $x = y$ denote that the time for which the proposition X is true, is identical with the time for which the proposition Y is true. We shall term x the *representative symbol* of the proposition X, &c.

In the treatment of secondary propositions in MAL, x denoted the *cases* in which the proposition X was true. This was a standard method for converting hypothetical, disjunctive, etc. propositions into propositions about classes (see page 176).

From the above definitions it will follow, that we shall always have

$$x + y = y + x,$$

for either member will denote the same aggregate of time.

Let us further represent by xy the performance in succession of the two operations represented by y and x, i.e. the whole mental operation which consists of the following elements, viz., 1st, The mental selection of that portion of time for which the proposition Y is true. 2ndly, The mental selection, out of that portion of time, of such portion as it contains of the time in which the proposition X is true,—the result of these successive processes being the fixing of the mental regard upon the whole of that portion of time for which the propositions X and Y are both true.

xy denotes the composition $x \circ y$ of the two operators denoted by x and y, exactly as in MAL.

From this definition it will follow, that we shall always have

$$xy = yx. \tag{1}$$

For whether we select mentally, first that portion of time for which the proposition Y is true, then out of the result that contained portion for which X is true; or first, that portion of time for which the proposition X is true, then out of the result that contained portion of it for which the proposition Y is true; we shall arrive at the same final result, viz., that portion of time for which the propositions X and Y are both true.

By continuing this method of reasoning it may be established, that the laws of combination of the symbols x, y, z, &c., in the species of interpretation here assigned to them, are identical in expression with the laws of combination of the same symbols, in the interpretation assigned to them in the first part of this treatise. The reason of this final identity is apparent. For in both cases it is the same faculty, or the same combination of faculties, of which we study the operations; operations, the essential character of which is unaffected, whether we suppose them to be engaged upon that universe of things in which all existence is contained, or upon that whole of time in which all events are realized, and to some part, at least, of which all assertions, truths, and propositions, refer.

Thus, in addition to the laws above stated, we shall have by (4), Chap, II., the law whose expression is

$$x(y+z) \;=\; xy+xz\,; \tag{2}$$

and more particularly the fundamental law of duality (2) Chap, II., whose expression is

$$x^2 \;=\; x, \quad \text{or,} \quad x(1-x) \;=\; 0\,; \tag{3}$$

a law, which while it serves to distinguish the system of thought in Logic from the system of thought in the science of quantity, gives to the processes of the former a completeness and a generality which they could not otherwise possess.

8. Again, as this law (3) (as well as the other laws) is satisfied by the symbols 0 and 1, we are led, as before, to inquire whether those symbols do not admit of interpretation in the present system of thought. The same course of reasoning which we before pursued shows that they do, and warrants us in the two following positions, viz.:

1st, That in the expression of secondary propositions, 0 represents *nothing* in reference to the element of time.

2nd, That in the same system 1 represents the universe, or whole of time, to which the discourse is supposed in any manner to relate.

As in primary propositions the universe of discourse is sometimes limited to a small portion of the actual universe of things, and is sometimes co-extensive with that universe; so in secondary propositions, the universe of discourse may be limited to a single day or to the passing moment, or it may comprise the whole duration of time. It may, in the most literal sense, be "eternal." Indeed, unless there is some limitation expressed or implied in the nature of the discourse, the proper interpretation of the symbol 1 in secondary propositions is "eternity;" even as its proper interpretation in the primary system is the actually existent universe.

9. Instead of appropriating the symbols x, y, z, to the representation of the truths of propositions, we might with equal propriety apply them to represent the occurrence of events. In fact, the occurrence of an event both implies, and is implied by, the truth of a proposition, viz., of the proposition which asserts the occurrence of the event. The one signification of the symbol x necessarily involves the other. It will greatly conduce to convenience to be able to employ our symbols in either of these really equivalent interpretations which the circumstances of a problem may suggest to us as most desirable; and of this liberty I shall avail myself whenever occasion requires. In problems of pure Logic I shall consider the symbols x, y, &c. as representing elementary propositions, among which relation is expressed in the premises. In the mathematical theory of probabilities, which, as before intimated (I. 12), rests upon a basis of Logic, and which it is designed to treat in a subsequent portion of this work, I shall employ the same symbols to denote the simple events, whose implied or required frequency of occurrence it counts among its elements.

Proposition III.

10 . *To deduce general Rules for the expression of Secondary Propositions.*

In the various inquiries arising out of this Proposition, fulness of demonstration will be the less necessary, because of the exact analogy which they bear with similar inquiries already completed with reference to primary propositions. We shall first consider the expression of terms; secondly, that of the propositions by which they are connected.

As 1 denotes the whole duration of time, and x that portion of it for which the proposition X is true, $1 - x$ will denote that portion of time for which the proposition X is false.

Again, as xy denotes that portion of time for which the propositions X and Y are both true, we shall, by combining this and the previous observation, be led to the following interpretations, viz.:

The expression $x(1 - y)$ will represent the time during which the proposition X is true, and the proposition Y false. The expression $(1 - x)(1 - y)$ will represent the time during which the propositions X and Y are simultaneously false.

The expression $x(1-y)+y(1-x)$ will express the time during which either X is true or Y true, but not both; for that time is the sum of the times in which they are singly and exclusively true. The expression $xy + (1 - x)(1 - y)$ will express the time during which X and Y are either both true or both false.

If another symbol z presents itself, the same principles remain applicable. Thus xyz denotes the time in which the propositions X, Y, and Z are simultaneously true; $(1 - x)(1 - y)(1 - z)$ the time in which they are simultaneously false; and the sum of these expressions would denote the time in which they are either true or false together.

The general principles of interpretation involved in the above examples do not need any further illustrations or more explicit statement.

11. The laws of the expression of propositions may now be exhibited and studied in the distinct cases in which they present themselves. There is, however, one principle of fundamental importance to which I wish in the first place to direct attention. Although the principles of expression which have been laid down are perfectly general, and enable us to limit our assertions of the truth or falsehood of propositions to any particular portions of that whole of time (whether it be an unlimited eternity, or a period whose beginning and whose end are definitely fixed, or the passing moment) which constitutes the universe of our discourse, yet, in the actual procedure of human reasoning, such limitation is not commonly employed. When we assert that a proposition is true, we generally mean that it is true throughout the whole duration of the time to which our discourse refers; and when different assertions of the unconditional truth or falsehood of propositions are jointly made as the premises of a logical demonstration, it is to the same universe of time that those assertions are referred, and not to particular and limited parts of it. In that necessary matter which is the object or field of the exact sciences every assertion of a truth may be the assertion of an "eternal truth." In reasoning upon transient phænomena (as of some social conjuncture) each assertion may be qualified by an immediate reference to the present time, "Now." But in both cases, unless there is a distinct expression to the contrary, it is to the same period of duration that each separate proposition relates. The cases which then arise for our consideration are the following:

Propositional logic in the exact sciences is two-valued (T/F).

1st. *To express the Proposition, "The proposition X is true."*

We are here required to express that within those limits of time to which the matter of our discourse is confined the proposition X is true. Now the time for which the proposition X is true is denoted by x, and the extent of time to which our discourse refers is represented by 1. Hence we have

$$\boxed{x = 1} \tag{4}$$

as the expression required.

2nd. *To express the Proposition, "The proposition X is false."*

We are here to express that within the limits of time to which our discourse relates, the proposition X is false; or that within those limits there is no portion of time for which it is true. Now the portion of time for which it is true is x. Hence the required equation will be

$$\boxed{x = 0}\,. \tag{5}$$

This result might also be obtained by equating to the whole duration of time 1, the expression for the time during which the proposition X is false, viz., $1 - x$. This gives

$$1 - x = 1\,,$$

whence

$$x = 0\,.$$

3rd. *To express the disjunctive Proposition, "Either the proposition*

X is true or the proposition Y is true;" it being thereby implied that the said propositions are mutually exclusive, that is to say, that one only of them is true.

The time for which either the proposition X is true or the proposition Y is true, but not both, is represented by the expression $x(1-y)+y(1-x)$. Hence we have

$$\boxed{x(1-y)+y(1-x) = 1}, \tag{6}$$

for the equation required.

If in the above Proposition the particles *either*, *or*, are supposed not to possess an absolutely disjunctive power, so that the possibility of the simultaneous truth of the propositions X and Y is not excluded, we must add to the first member of the above equations the term xy. We shall thus have

$$xy+x(1-y)+(1-x)y = 1,$$

or

$$\boxed{x+(1-x)y = 1}. \tag{7}$$

4th. *To express the conditional Proposition, "If the proposition Y is true, the proposition X is true."*

Since whenever the proposition Y is true, the proposition X is true, it is necessary and sufficient here to express, that the time in which the proposition Y is true is time in which the proposition X is true; that is to say, that it is some indefinite portion of the whole time in which the proposition X is true. Now the time in which the proposition Y is true is y, and the whole time in which the proposition X is true is x. Let v be a symbol of time indefinite, then will vx represent an indefinite portion of the whole time x. Accordingly, we shall have

$$\boxed{y = vx}$$

as the expression of the proposition given.

12. When v is thus regarded as a symbol of time indefinite, vx may be understood to represent the whole, or an indefinite part, or no part, of the whole time x; for any one of these meanings may be realized by a particular determination of the arbitrary symbol v. Thus, if v be determined to represent a time in which the whole time x is included, vx will represent the whole time x. If v be determined to represent a time, some part of which is included

Boole's use of v to indicate "some" for secondary propositions is not the same as for primary propositions; in the latter, v cannot be 0.

in the time x, but which does not fill up the measure of that time, vx will represent a part of the time x. If, lastly, v is determined to represent a time, of which no part is common with any part of the time x, vx will assume the value 0, and will be equivalent to "no time," or "never."

Now it is to be observed that the proposition, "If Y is true, X is true," contains no assertion of the truth of either of the propositions X and Y. It may equally consist with the supposition that the truth of the proposition Y is a condition indispensable to the truth of the proposition X, in which case we shall have $v = 1$; or with the supposition that although Y expresses a condition which, when realized, assures us of the truth of X, yet X may be true without implying the fulfilment of that condition, in which case v denotes a time, some part of which is contained in the whole time x; or, lastly, with the supposition that the proposition Y is not true at all, in which case v represents some time, no part of which is common with any part of the time x. All these cases are involved in the general supposition that v is a symbol of time indefinite.

5th. *To express a proposition in which the conditional and the disjunctive characters both exist.*

The general form of a conditional proposition is, "If Y is true, X is true," and its expression is, by the last section, $y = vx$. We may properly, in analogy with the usage which has been established in primary propositions, designate Y and X as the terms of the conditional proposition into which they enter; and we may further adopt the language of the ordinary Logic, which designates the term Y, to which the particle *if* is attached, the "antecedent" of the proposition, and the term X the "consequent."

Now instead of the terms, as in the above case, being simple propositions, let each or either of them be a disjunctive proposition involving different terms connected by the particles *either*, *or*, as in the following illustrative examples, in which X, Y, Z, &c. denote simple propositions.

1st. If either X is true or Y is true, then Z is true.

2nd. If X is true, then either Y is true or Z true.

3rd. If either X is true or Y is true, then either Z and W are both true, or they are both false.

It is evident that in the above cases the relation of the antecedent to the consequent is not affected by the circumstance that one of those terms or both are of a disjunctive character. Accordingly it is only necessary to obtain, in conformity with the principles already established, the proper expressions for the antecedent and the consequent, to affect the latter with the indefinite symbol v, and to equate the results. Thus for the propositions above stated we shall have the respective equation,

1st $$x(1-y)+(1-x)y = vz.$$
2nd. $$x = v\{y(1-z)+z(1-y)\}.$$
3rd. $$x(1-y)+y(1-x) = v\{zw+(1-z)(1-w)\}$$

The rule here exemplified is of general application.

Cases in which the disjunctive and the conditional elements enter in a manner different from the above into the expression of a compound proposition, are conceivable, but I am not aware that they are ever presented to us by the natural exigencies of human reason, and I shall therefore refrain from any discussion of them. No serious difficulty will arise from this omission, as the general principles which have formed the basis of the above applications are perfectly general, and a slight effort of thought will adapt them to any imaginable case.

13. In the laws of expression above stated those of interpretation are implicitly involved. The equation

$$x = 1$$

must be understood to express that the proposition X is true; the equation

$$x = 0,$$

that the proposition X is false. The equation

$$xy = 1$$

will express that the propositions X and Y are both true together; and the equation

$$xy = 0$$

that they are not both together true.

NOTE: One cannot express, by an equation, “X is always true or always false” in Boole’s algebra of logic.

In like manner the equations

$$x(1-y)+y(1-x) = 1,$$
$$x(1-y)+y(1-x) = 0,$$

will respectively assert the truth and the falsehood of the disjunctive Proposition, "Either X is true or Y is true." The equations

$$y = vx$$
$$y = v(1-x)$$

will respectively express the Propositions, "If the proposition Y is true, the proposition X is true." "If the proposition Y is true, the proposition X is false."

Examples will frequently present themselves, in the succeeding chapters of this work, of a case in which some terms of a particular member of an equation are affected by the indefinite symbol v, and others not so affected. The following instance will serve for illustration. Suppose that we have

$$y = xz + vx(1-z).$$

Here it is implied that the time for which the proposition Y is true consists of all the time for which X and Z are together true, together with an indefinite portion of the time for which X is true and Z false. From this it may be seen, 1st, That if Y is true, either X and Z are together true, or X is true and Z false; 2ndly, If X and Z are together true, Y is true. The latter of these may be called the reverse interpretation, and it consists in taking the antecedent out of the second member, and the consequent from the first member of the equation. The existence of a term in the second member, whose coefficient is unity, renders this latter mode of interpretation possible. The general principle which it involves may be thus stated:

14. PRINCIPLE.—*Any constituent term or terms in a particular member of an equation which have for their coefficient unity, may be taken as the antecedent of a proposition, of which all the terms in the other member form the consequent.*

Thus the equation

$$y = xz + vx(1-z) + (1-x)(1-z)$$

would have the following interpretations:

Direct Interpretation.—*If the proposition Y is true, then either X and Z are true, or X is true and Z false, or X and Z are both false.*

Reverse Interpretation.—*If either X and Z are true, or X and Z are false, Y is true.*

The aggregate of these partial interpretations will express the whole significance of the equation given.

15. We may here call attention again to the remark, that although the idea of time appears to be an essential element in the theory of the interpretation of secondary propositions, it may practically be neglected as soon as the laws of expression and of interpretation are definitely established. The forms to which those laws give rise seem, indeed, to correspond with the forms of a perfect language. Let us imagine any known or existing language freed from idioms and divested of superfluity, and let us express in that language any given proposition in a manner the most simple and literal,—the most in accordance with those principles of pure and universal thought upon which all languages are founded, of which all bear the manifestation, but from which all have more or less departed. The transition from such a language to the notation of analysis would consist of no more than the substitution of one set of signs for another, without essential change either of form or character. For the elements, whether things or propositions, among which relation is expressed, we should substitute letters; for the disjunctive conjunction we should write $+$; for the connecting copula or sign of relation, we should write $=$. This analogy I need not pursue. Its reality and completeness will be made more apparent from the study of those forms of expression which will present themselves in subsequent applications of the present theory, viewed in more immediate comparison with that imperfect yet noble instrument of thought—the English language.

16. Upon the general analogy between the theory of Primary and that of Secondary Propositions, I am desirous of adding a few remarks before dismissing the subject of the present chapter.

We might undoubtedly, have established the theory of Primary Propositions upon the simple notion of space, in the same way as that of secondary propositions has been established upon the notion of time. Perhaps, had this been done, the analogy which we are contemplating would have been in somewhat closer accordance with the view of those who regard space and time as merely "forms of the human understanding," conditions of knowledge imposed by the very constitution of the mind upon all that is submitted to its apprehension. But this view, while on the one hand it is incapable of demonstration, on the other hand ties us down to the recognition of "place," τὸ πο῎ν, as an essential category of existence. The question, indeed, whether it is so or not, lies, I apprehend, beyond the reach of our faculties; but it may be, and I conceive has been, established, that the formal processes of reasoning in primary propositions do not require, as an essential condition, the manifestation in space of the things about which we reason; that they would remain applicable, with equal strictness of demonstration, to forms of existence, if such there be, which lie beyond the realm of sensible extension. It is a fact, perhaps, in some degree analogous to this, that we are able in many known examples in geometry and dynamics, to exhibit the formal analysis of problems founded upon some intellectual conception of space different from that which is presented to us by the senses, or which can be realized by the imagination.[1] I conceive, therefore, that the idea of space is not

[1]Space is presented to us in perception, as possessing the three dimensions of length, breadth, and depth. But in a large class of problems relating to the properties of curved surfaces, the rotations of solid bodies around axes, the vibrations of elastic media, &c., this limitation appears in the analytical investigation to be of an arbitrary character, and if attention were paid to the processes of solution alone, no reason could be discovered why space should not exist in four or in any greater number of dimensions. The intellectual procedure in the imaginary world thus suggested can be apprehended by the clearest light of analogy.

The existence of space in three dimensions, and the views thereupon of the religious and philosophical mind of antiquity, are thus set forth by Aristotle:— Μεγέθος δὲ τὸ μὲν ἐφ ̈εν, γραμμή τὸ δ′ έπὶ δ′νο έπίπεδον, τὸ δ′ ὲπὶ τρία σῶμα′ Καί παρὰ τα῎ντα ο′νκ ̈εστιν ̈αλλο μέγεθος, διὰ τὸ τριά πάντα εὶναι καὶ τὸ τρὶς πάντη. Κάθαπερ γάρ φασι καὶ οὶ Πνθαγόρειοι, τὸ πᾶν καὶ τὰ πάντα τοῖς τρισὶν ̈ωρισται. Τελεντὴ γὰρ καὶ μέσον καὶ άρκὴ τὸν ὰριθμὸν ̈εκει τὸν το῎ν παντός′ τα῎ντα δὲ τὸν τῆς τριάδος. Διὸ παρὰ τῆς φ′νσεως είληφότες ̈ωσπερ νόμονς έκείνης, καὶ πρὸς τὰς ὰγιστείας κρώμεθα τῶν θεῶν τ῎ψ άριθμ῎ψ το′ντ῎ψ.—*De Caelo*, 1.

essential to the development of a theory of primary propositions, but am disposed, though desiring to speak with diffidence upon a question of such extreme difficulty, to think that the idea of time is essential to the establishment of a theory of secondary propositions. There seem to be grounds for thinking, that without any change in those faculties which are concerned in *reasoning*, the manifestation of space to the human mind might have been different from what it is, but not (at least the same) grounds for supposing that the manifestation of time could have been otherwise than we perceive it to be. Dismissing, however, these speculations as possibly not altogether free from presumption, let it be affirmed that the real ground upon which the symbol 1 represents in primary propositions the universe of things, and not the space they occupy, is, that the sign of identity $=$ connecting the members of the corresponding equations, implies that the things which they represent are identical, not simply that they are found in the same portion of space. Let it in like manner be affirmed, that the reason why the symbol 1 in secondary propositions represents, not the universe of events, but the eternity in whose successive moments and periods they are evolved, is, that the same sign of identity connecting the logical members of the corresponding equations implies, not that the events which those members represent are identical, but that the times of their occurrence are the same. These reasons appear to me to be decisive of the immediate question of interpretation. In a former treatise on this subject (Mathematical Analysis of Logic, p. 49), following the theory of Wallis respecting the Reduction of Hypothetical Propositions, I was led to interpret the symbol 1 in secondary propositions as the universe of "cases" or "conjunctures of circumstances;" but this view involves the necessity of a definition of what is meant by a "case," or "conjuncture of circumstances;" and it is certain, that whatever is involved in the term beyond the notion of time is alien to the objects, and restrictive of the processes, of formal Logic.

Chapter XII

Of the Methods and Processes to be Adopted in the Treatment of Secondary Propositions.

1. It has appeared from previous researches (XI. 7) that the laws of combination of the literal symbols of Logic are the same, whether those symbols are employed in the expression of primary or in that of secondary propositions, the sole existing difference between the two cases being a difference of interpretation. It has also been established (V. 6), that whenever distinct systems of thought and interpretation are connected with the same system of formal laws, i.e., of laws relating to the combination and use of symbols, the attendant processes, intermediate between the expression of the primary conditions of a problem and the interpretation of its symbolical solution, are the same in both. Hence, as between the systems of thought manifested in the two forms of primary and of secondary propositions, this community of formal law exists, the processes which have been established and illustrated in our discussion of the former class of propositions will, without any modification, be applicable to the latter.

2. Thus the laws of the two fundamental processes of elimination and development are the same in the system of secondary as in the system of primary propositions. Again, it has been seen (Chap. VI. Prop. 2) how, in primary propositions, the interpretation of any proposed equation devoid of fractional forms may be effected by developing it into a series of constituents, and equating to 0 every constituent whose coefficient does not vanish. To the equations of secondary propositions the same method is applicable, and the interpreted result to which it finally conducts us is, as in the former case (VI. 6), a system of coexistent denials. But while in the former case the force of those denials is expended upon the existence of certain classes of things, in the latter it relates to the truth of certain combinations of the elementary propositions involved in the *terms* of the given premises. And as in primary propositions it was seen that the system of denials admitted of conversion into various other forms of propositions (VI. 7), &c., such conversion will be found to be possible here also, the sole difference consisting not in the forms of the equations, but in the nature of their interpretation.

3. Moreover, as in primary propositions, we can find the expression of any element entering into a system of equations, in terms of the remaining elements (VI. 10), or of any selected number of the remaining elements, and interpret that expression into a logical inference, the same object can be accomplished by the same means, difference of interpretation alone excepted, in the system of secondary propositions. The elimination of those elements which we desire to banish from the final solution, the reduction of the system to a single equation, the algebraic solution and the mode of its development into an interpretable form, differ in no respect from the corresponding steps in the discussion of primary propositions.

To remove, however, any possible difficulty, it may be desirable to collect under a general Rule the different cases which present themselves in the treatment of secondary propositions.

Rule.—*Express symbolically the given propositions (XI. 11).*

Eliminate separately from each equation in which it is found the indefinite symbol v (VII. 5).

This applies only to premiss equations of the form $x = vy$. Boole had a rule about *not* eliminating the indefinite symbol v when a premiss is 'particular'. See p. 124. Boole did not consider secondary propositions expressed by $vx = vy$; see page 163.

Eliminate the remaining symbols which it is desired to banish from the final solution: always before elimination reducing to a single equation those equations in which the symbol or symbols to be eliminated are found (VIII. 7). Collect the resulting equations into a single equation $V = 0$.

Then proceed according to the particular form in which it is desired to express the final relation, as—

How to express the conclusion in any of four different forms.

1st. *If in the form of a denial, or system of denials, develop the function* V, *and equate to 0 all those constituents whose coefficients do not vanish.*

2ndly. *If in the form of a disjunctive proposition, equate to 1 the sum of those constituents whose coefficients vanish.*

3rdly. *If in the form of a conditional proposition having a simple* *element, as x or $1 - x$, for its antecedent, determine the algebraic expression of that element, and develop that expression.*

4thly. *If in the form of a conditional proposition having a compound expression, as xy, $xy+(1-x)(1-y)$, &c., for its antecedent, equate that expression to a new symbol t, and determine t as a developed function of the symbols which are to appear in the consequent, either by ordinary methods or by the special method* (IX. 9).

5thly. *Interpret the results by* (XI. 13, 14).

If it only be desired to ascertain whether a particular elementary proposition x is true or false, we must eliminate all the symbols but x; then the equation $x = 1$ will indicate that the proposition is true, $x = 0$ that it is false, $0 = 0$ that the premises are insufficient to determine whether it is true or false.

An argument from Cicero.

4. Ex. 1.—The following prediction is made the subject of a curious discussion in Cicero's fragmentary treatise, De Fato:—"Si quis (Fabius) natus est oriente Canicula, is in mari non morietur." I shall apply to it the method of this chapter. Let y represent the proposition, "Fabius was born at the rising of the dogstar;" x the proposition, "Fabius will die in the sea." In saying that x *represents* the proposition, "Fabius, &c.," it is only meant that x is a symbol so appropriated (XI. 7) to the above proposition, that the equation $x = 1$ declares, and the equation $x = 0$ denies, the truth of that proposition. The equation we have to discuss will be

$$y = v(1 - x). \tag{1}$$

And, first, let it be required to reduce the given proposition to a negation or system of negations (XII. 3). We have, on transposition,

$$y - v(1 - x) = 0.$$

Eliminating v,

$$y\{y - (1 - x)\} = 0,$$

or,

$$y - y(1 - x) = 0,$$

or,

$$yx = 0. \tag{2}$$

The interpretation of this result is:—"It is not true that Fabius was born at the rising of the dogstar, and will die in the sea."

Cicero terms this form of proposition, "Conjunctio ex repugnantibus;" and he remarks that Chrysippus thought in this way to evade the difficulty which he imagined to exist in contingent assertions respecting the future: "Hoc loco Chrysippus aestuans falli sperat Chaldaeos casterosque divinos, neque eos usuros esse conjunctionibus ut ita sua percepta pronuntient: Si quis natus est oriente Canicula is in mari non morietur; sed potius ita dicant: Non et natus est quis oriente Caniculâ, et in mari morietur. O licentiam jocularem! ... Multa genera sunt enuntiandi, nec ullum distortius quam hoc quo Chrysippus sperat Chaldaeos contentos Stoicorum causa fore."—*Cic. De Fato*, 7, 8.

5. To reduce the given proposition to a disjunctive form.

The constituents not entering into the first member of (2) are

$$x(1-y), \quad (1-x)y, \quad (1-x)(1-y).$$

Whence we have

$$y(1-x) + x(1-y) + (1-x)(1-y) \;=\; 1. \tag{3}$$

The interpretation of which is:—*Either Fabius was born at the rising of the dogstar, and will not perish in the sea; or he was not born at the rising of the dogstar, and will perish in the sea; or he was not born at the rising of the dogstar, and will not perish in the sea.*

In cases like the above, however, in which there exist constituents differing from each other only by a single factor, it is, as we have seen (VII. 15), most convenient to collect such constituents into a single term. If we thus connect the first and third terms of (3), we have

$$(1-y)x + 1 - x \;=\; 1\,;$$

and if we similarly connect the second and third, we have

$$y(1-x) + 1 - y \;=\; 1.$$

These forms of the equation severally give the interpretations—

Either Fabius was not born under the day star, and will die in the sea, or he will not die in the sea.

Either Fabius was born under the day star, and will not die in the sea, or he was not born under the dogstar.

It is evident that these interpretations are strictly equivalent to the former one.

Let us ascertain, in the form of a conditional proposition, the consequences which flow from the hypothesis, that "Fabius will perish in the sea."

In the equation (2), which expresses the result of the elimination of v from the original equation, we must seek to determine x as a function of y.

We have

$$x = \frac{0}{y} = 0y + \frac{0}{0}(1 - y) \text{ on expansion,}$$

or

$$x = \frac{0}{0}(1 - y)\,;$$

the interpretation of which is,—*If Fabius shall die in the sea, he was not born at the rising of the dogstar.*

These examples serve in some measure to illustrate the connexion which has been established in the previous sections between primary and secondary propositions, a connexion of which the two distinguishing features are identity of process and analogy of interpretation.

An argument from Plato.

6. Ex. 2.—There is a remarkable argument in the second book of the Republic of Plato, the design of which is to prove the immutability of the Divine Nature. It is a very fine example both of the careful induction from familiar instances by which Plato arrives at general principles, and of the clear and connected logic by which he deduces from them the particular inferences which it is his object to establish. The argument is contained in the following dialogue:

"Must not that which departs from its proper form be changed either by itself or by another thing? Necessarily so. Are not things which are in the best state least changed and disturbed, as the body by meats and drinks, and labours, and every species of plant by heats and winds, and such like affections? Is not the healthiest and strongest the least changed? Assuredly. And does not any trouble from without least disturb and change that soul which is strongest and wisest? And as to all made vessels, and furnitures, and garments, according to the same principle, are not those which are well wrought, and in a good condition, least changed by time and other accidents? Even so. And whatever is in a right state, either by nature or by art, or by both these, admits of the smallest change from any other thing. So it seems. But God and things divine are in every sense in the best state. Assuredly. In this way, then, God should least of all bear many forms? Least, indeed, of all. Again, should He transform and change Himself? Manifestly He must do so, if He is changed at all. Changes He then Himself to that which is more good and fair, or to that which is worse and baser? Necessarily to the worse, if he be changed. For never shall we say that God is indigent of beauty or of virtue. You speak most rightly, said I, and the matter being so, seems it to you, O Adimantus, that God or man *willingly* makes himself in any sense worse? Impossible, said he. Impossible, then, it is, said I, that a god should wish to change himself; but ever being fairest and best, each of them ever remains absolutely in the same form."

The premises of the above argument are the following:

1st. If the Deity suffers change, He is changed either by Himself or by another.

2nd. If He is in the best state, He is not changed by another.

3rd. The Deity is in the best state.

4th. If the Deity is changed by Himself, He is changed to a worse state.

5th. If He acts willingly, He is not changed to a worse state.

6th. The Deity acts willingly.

Let us express the elements of these premises as follows:

Let x represent the proposition, "The Deity suffers change."

y, He is changed by Himself.

z, He is changed by another.

s, He is in the best state.

t, He is changed to a worse state.

w, He acts willingly.

Then the premises expressed in symbolical language yield, after elimination of the indefinite class symbols v, the following equations:

$$xyz + x(1-y)(1-z) = 0, \quad (1)$$
$$sz = 0, \quad (2)$$
$$s = 1, \quad (3)$$
$$y(1-t) = 0, \quad (4)$$
$$wt = 0, \quad (5)$$
$$w = 1. \quad (6)$$

Retaining x, I shall eliminate in succession z, s, y, t, and w (this being the order in which those symbols occur in the above system), and interpret the successive results.

Eliminating z from (1) and (2), we get

$$xs(1-y) = 0. \quad (7)$$

Eliminating s from (3) and (7),

$$x(1-y) = 0. \quad (8)$$

Eliminating y from (4) and (8),

$$x(1-t) = 0. \quad (9)$$

Eliminating t from (5) and (9),

$$xw = 0. \quad (10)$$

Eliminating w from (6) and (10),

$$x = 0. \quad (11)$$

These equations, beginning with (8), give the following results:

From (8) we have $x = \frac{0}{0}y$, therefore, *If the Deity suffers change, He is changed by Himself.*

From (9), $x = \frac{0}{0}t$, *If the Deity suffers change, He is changed to a worse state.*

From (10), $x = \frac{0}{0}(1-w)$. *If the Deity suffers change, He does not act willingly.*

From (11), *The Deity does not suffer change.* This is Plato's result.

Now I have before remarked, that the order of elimination is indifferent. Let us in the present case seek to verify this fact by eliminating the same symbols in a reverse order, beginning with w. The resulting equations are, Remarked, but not proved.

$$t = 0, \quad y = 0, \quad x(1-x) = 0, \quad z = 0, \quad x = 0;$$

yielding the following interpretations:

God is not changed to a worse state.
He is not changed by Himself.
If He suffers change, He is changed by another.
He is not changed by another.
He is not changed.

We thus reach by a different route the same conclusion.

Though as an exhibition of the *power* of the method, the above examples are of slight value, they serve as well as more complicated instances would do, to illustrate its nature and character.

7. It may be remarked, as a final instance of analogy between the system of primary and that of secondary propositions, that in the latter system also the fundamental equation,

$$x(1-x) = 0,$$

admits of interpretation. It expresses the axiom, *A proposition cannot at the same time be true and false.* Let this be compared with the corresponding interpretation (III. 15). Solved under the form

$$x = \frac{0}{1-x} = \frac{0}{0}x,$$

by development, it furnishes the respective axioms: "A thing is what it is:" "If a proposition is true, it is true:" forms of what has been termed "The principle of identity." Upon the nature and the value of these axioms the most opposite opinions have been entertained. Some have regarded them as the very pith and marrow of philosophy. Locke devoted to them a chapter, headed, "On Trifling Propositions."[1] In both these views there seems to have been a mixture of truth and error. Regarded as supplanting experience, or as furnishing materials for the vain and wordy janglings of the schools, such propositions are worse than trifling. Viewed, on the other hand, as intimately allied with the very laws and conditions of thought, they rise into at least a speculative importance.

[1] Essay on the Human Understanding, Book IV. Chap. viii.

Chapter XIII

Analysis of a Portion of Dr. Samuel Clarke's "Demonstration of the Being and Attributes of God," and of a Portion of ohe "Ethica Ordine Geometrico Demonstrata" of Spinoza.

1. The general order which, in the investigations of the following chapter, I design to pursue, is the following. I shall examine what are the actual premises involved in the demonstrations of some of the general propositions of the above treatises, whether those premises be expressed or implied. By the actual premises I mean whatever propositions are assumed in the course of the argument, without being proved, and are employed as parts of the foundation upon which the final conclusion is built. The premises thus determined, I shall express in the language of symbols, and I shall then deduce from them by the methods developed in the previous chapters of this work, the most important inferences which they involve, in addition to the particular inferences actually drawn by the authors. I shall in some instances modify the premises by the omission of some fact or principle which is contained in them, or by the addition or substitution of some new proposition, and shall determine how by such change the ultimate conclusions are affected. In the pursuit of these objects it will not devolve upon me to inquire, except incidentally, how far the metaphysical principles laid down in these celebrated productions are worthy of confidence, but only to ascertain what conclusions may justly be drawn from given premises; and in doing this, to exemplify the perfect liberty which we possess as concerns both the choice and the order of the elements of the final or concluding propositions, viz., as to determining what elementary propositions are true or false, and what are true or false under given restrictions, or in given combinations.

2. The chief practical difficulty of this inquiry will consist,

This observation is similar to one in the Port Royal Logic (5th ed., 1683), that it is not the application of the rules of logic that create the difficulties, but determining what are the correct premisses.

not in the application of the method to the premises once determined, but in ascertaining what the premises are. In what area regarded as the most rigorous examples of reasoning applied to metaphysical questions, it will occasionally be found that different trains of thought are blended together; that particular but essential parts of the demonstration are given parenthetically, or out of the main course of the argument; that the meaning of a premiss may be in some degree ambiguous; and, not unfrequently, that arguments, viewed by the strict laws of formal reasoning, are incorrect or inconclusive. The difficulty of determining and distinctly exhibiting the true premises of a demonstration may, in such cases, be very considerable. But it is a difficulty which must be overcome by all who would ascertain whether a particular conclusion is proved or not, whatever form they may be prepared or disposed to give to the ulterior process of reasoning. It is a difficulty, therefore, which is not peculiar to the method of this work, though it manifests itself more distinctly in connexion with this method than with any other. So intimate, indeed, is this connexion, that it is impossible, employing the method of this treatise, to form even a conjecture as to the validity of a conclusion, without a distinct apprehension and exact statement of all the premises upon which it rests. In the more usual course of procedure, nothing is, however, more common than to examine some of the steps of a train of argument, and thence to form a vague general impression of the scope of the whole, without any such preliminary and thorough analysis of the premises which it involves.

It is interesting to note that Boole's complaints, about the existing literature on logical arguments, often apply to his own writing. It has taken more that 150 years to properly understand Boole's algebra of logic.

The necessity of a rigorous determination of the real premises of a demonstration ought not to be regarded as an evil; especially as, when that task is accomplished, every source doubt or ambiguity is removed. In employing the method of this treatise, the order in which premises are arranged, the mode of connexion which they exhibit, with every similar circumstance may be esteemed a matter of indifference, and the process inference is conducted with a precision which might almost termed mechanical.

3. The "Demonstration of the Being and Attributes of God," consists of a series of propositions or theorems, each of them proved by means of premises resolvable, for the most part, into two distinct classes, viz., facts of observation, such as the existence of a material world, the phenomenon of motion, &c., and hypothetical principles, the authority and universality of which are supposed to be recognised *à priori*. It is, of course, upon the truth of the latter, assuming the correctness of the reasoning, that the validity of the demonstration really depends. But whatever may be thought of its claims in this respect, it is unquestionable that, as an intellectual performance, its merits are very high. Though the trains of argument of which it consists are not in general very clearly arranged, they are almost always specimens of correct Logic, and they exhibit a subtlety of apprehension and a force of reasoning which have seldom been equalled, never perhaps surpassed. We see in them the consummation of those intellectual efforts which were awakened in the realm of metaphysical inquiry, at a period when the dominion of hypothetical principles was less questioned than it now is, and when the rigorous demonstrations of the newly risen school of mathematical physics seemed to have furnished a model for their direction. They appear to me for this reason (not to mention the dignity of the subject of which they treat) to be deserving of high consideration; and I do not deem it a vain or superfluous task to expend upon some of them a careful analysis.

4. The Ethics of Benedict Spinoza is a treatise, the object of which is to prove the identity of God and the universe, and to establish, upon this doctrine, a system of morals and of philosophy. The analysis of its main argument is extremely difficult, owing not to the complexity of the separate propositions which it involves, but to the use of vague definitions, and of axioms which, through a like defect of clearness, it is perplexing to determine whether we ought to accept or to reject. While the reasoning of Dr. Samuel Clarke is in part verbal, that of Spinoza is so in a much greater degree; and perhaps this is the reason why, to some minds, it has appeared to possess a formal cogency, to which in reality it possesses no just claim. These points will, however, be considered in the proper place.

One can apply much the same criticism to Boole's chapters on his algebra of logic.

CLARKE'S DEMONSTRATION.

Proposition I.

5. "*Something has existed from eternity.*"

The proof is as follows:—

"For since something now is, 'tis manifest that something always was. Otherwise the things that now are must have risen out of nothing, absolutely and without cause. Which is a plain contradiction in terms. For to say a thing is produced, and yet that there is no cause at all of that production, is to say that something is effected when it is effected by nothing, that is, at the same time when it is not effected at all. Whatever exists has a cause of its existence, either in the necessity of its own nature, and thus it must have been of itself eternal: or in the will of some other being, and then that other being must, at least in the order of nature and causality, have existed before it."

Let us now proceed to analyze the above demonstration. Its first sentence is resolvable into the following propositions:

1st. Something is.

2nd. If something is, either something always was, or the things that now are must have risen out of nothing.

The next portion of the demonstration consists of a proof that the second of the above alternatives, viz., "The things that now are have risen out of nothing," is impossible, and it may formally be resolved as follows:

3rd. If the things that now are have risen out of nothing, something has been effected, and at the same time that something has been effected by nothing.

4th. If that something has been effected by nothing, it has not been effected at all.

The second portion of this argument appears to be a mere assumption of the point to be proved, or an attempt to make that point clearer by a different verbal statement.

The third and last portion of the demonstration contains a distinct proof of the truth of either the original proposition to be proved, viz., "Something always was," or the point proved in the second part of the demonstration, viz., the untenable nature of the hypothesis, that "the things that now are have risen out of nothing." It is resolvable as follows:—

5th. If something is, either it exists by the necessity of its own nature, or it exists by the will of another being.

6th. If it exists by the necessity of its own nature, something always was.

7th. If it exists by the will of another being, then the proposition, that the things which exist have arisen out of nothing, is false.

The last proposition is not expressed in the same form in the text of Dr. Clarke; but his expressed conclusion of the prior existence of another Being is clearly meant as equivalent to a denial of the proposition that the things which now are have risen out of nothing.

It appears, therefore, that the demonstration consists of two distinct trains of argument: one of those trains comprising what I have designated as the *first* and *second* parts of the demonstration; the other comprising the *first* and *third* parts. Let us consider the latter train.

The premises are:—

1st. Something is.

2nd. If something is, either something always was, or the things that now are have risen out of nothing.

3rd. If something is, either it exists in the necessity of its own nature, or it exists by the will of another being.

4th. If it exists in the necessity of its own nature, something always was.

5th. If it exists by the will of another being, then the hypothesis, that the things which now are have risen out of nothing, is false.

We must now express symbolically the above proposition.

Let	$x =$	Something is.
	$y =$	Something always was.
	$z =$	The things which now are have risen from nothing.
	$p =$	It exists in the necessity of its own nature (i.e. the *something* spoken of above).
	$q =$	It exists by the will of another Being.

It must be understood, that by the expression, Let $x =$ "Something is," is meant no more than that x is the representative symbol of that proposition (XI. 7), the equations $x = 1$, $x = 0$, respectively declaring its truth and its falsehood.

The equations of the premises are:—

$$\begin{aligned} &\text{1st.} & x &= 1\,; \\ &\text{2nd.} & x &= v\{y(1-x) + z(1-y)\}\,; \\ &\text{3rd.} & x &= v\{p(1-q) + q(1-p)\}\,; \\ &\text{4th.} & p &= vy\,; \\ &\text{5th.} & q &= v(1-z)\,; \end{aligned}$$

and on eliminating the several indefinite symbols v, we have

$$1 - x = 0\,; \tag{1}$$

$$x\{yz + (1-y)(1-z)\} = 0\,; \tag{2}$$

$$x\{pq + (1-p)(1-q)\} = 0\,; \tag{3}$$

$$p(1-y) = 0\,; \tag{4}$$

$$qz = 0\,. \tag{5}$$

6. First, I shall examine whether any conclusions are deducible from the above, concerning the truth or falsity of the single propositions represented by the symbols y, z, p, q, viz., of the propositions, "Something always was;" "The things which now are have risen from nothing;" "The something which is exists by the necessity of its own nature;" "The something which is exists by the will of another being."

For this purpose we must separately eliminate all the symbols but y, all these but z, &c. The resulting equation will determine whether any such separate relations exist.

To eliminate x from (1), (2), and (3), it is only necessary to substitute in (2) and (3) the value of x derived from (1). We find as the results,

$$yz + (1-y)(1-z) = 0. \tag{6}$$

$$pq + (1-p)(1-q) = 0. \tag{7}$$

To eliminate p we have from (4) and (7), by addition,

$$p(1-y) + pq + (1-p)(1-q) = 0; \tag{8}$$

whence we find,

$$(1-y)(1-q) = 0. \tag{9}$$

To eliminate q from (5) and (9), we have

$$qz + (1-y)(1-q) \ = \ 0 \ ;$$

whence we find

$$x(1-y) \ = \ 0. \tag{10}$$

There now remain but the two equations (6) and (10), which, on addition, give

$$yz + 1 - y \ = \ 0.$$

Eliminating from this equation z, we have

$$1 - y = 0\,, \quad \text{or,} \quad y = 1. \tag{11}$$

Eliminating from the same equation y, we have

$$z = 0. \tag{12}$$

The interpretation of (11) is

Something always was.

The interpretation of (12) is

The things which are have not risen from nothing.

Next resuming the system (6), (7), with the two equations (4), (5), let us determine the two equations involving p and q respectively.

To eliminate y we have from (4) and (6),

$$p(1-y) + yz + (1-y)(1-z) \ = \ (0) \ ;$$

whence

$$(p + 1 - z)z \ = \ 0\,, \quad \text{or,} \quad pz \ = \ 0. \tag{13}$$

To eliminate z from (5) and (13), we have

$$qz + pz = 0 \ ;$$

whence we get,

$$0 \ = \ 0.$$

There remains then but the equation (7), from which eliminating q, we have $0 \ = \ 0$ for the final equation, in p.

Hence there is no conclusion derivable from the premises affirming the simple truth or falsehood of the proposition, "The something which is exists in the necessity of its own nature." And as, on eliminating p, there is the same result, $0 \ = \ 0$, for the ultimate equation in q, it also follows, that *there is no conclusion deducible from the premises as to the simple truth or falsehood of the proposition, "The something which is exists by the will of another Being."*

Of relations connecting more than one of the propositions represented by the elementary symbols, it is needless to consider any but that which is denoted by the equation (7) connecting p and q, inasmuch as the propositions represented by the remaining symbols are absolutely true or false independently of any connexion of the kind here spoken of. The interpretation of (7), placed under the form

$$p(1-q)+q(1-p) = 1, \quad \text{is,}$$

The something which is, either exists in the necessity of its own nature, or by the will of another being.

I have exhibited the details of the above analysis with a, perhaps, needless fulness and prolixity, because in the examples which will follow, I propose rather to indicate the steps by which results are obtained, than to incur the danger of a wearisome frequency of repetition. The conclusions which have resulted from the above application of the method are easily verified by ordinary reasoning.

The reader will have no difficulty in applying the method to the other train of premises involved in Dr. Clarke's first Proposition, and deducing from them the two first of the conclusions to which the above analysis has led.

Proposition II.

7. *Some one unchangeable and independent Being has existed from eternity.*

The premises from which the above proposition is prove are the following:

1st. Something has always existed.

2nd. If something has always existed, either there has existed some one unchangeable and independent being, or the whole of existing things has been comprehended in a succession of changeable and dependent beings.

3rd. If the universe has consisted of a succession of changeable and dependent beings, either that series has had a cause from without, or it has had a cause from within.

4th. It has not had a cause from without (because it includes, by hypothesis, all things that exist).

5th. It has not had a cause from within (because no part is necessary, and if no part is necessary, the whole cannot be necessary).

Omitting, merely for brevity, the subsidiary proofs contained in the parentheses of the fourth and fifth premiss, we may represent the premises as follows:

Let x = Something has always existed.
y = There has existed some one unchangeable and independent being.
z = There has existed a succession of changeable and dependent beings.
p = That series has had a cause from without.
q = That series has had a cause from within.

Then we have the following system of equations, viz.:

$$\begin{array}{lrcl} \text{1st.} & x & = & 1\,; \\ \text{2nd.} & x & = & v\big\{y(1-z)+z(1-y)\big\}\,; \\ \text{3rd.} & z & = & v\big\{p(1-q)+(1-p)q\big\}\,; \\ \text{4th.} & p & = & 0\,; \\ \text{5th.} & q & = & 0\,: \end{array}$$

which, on the separate elimination of the indefinite symbols v, gives

$$1-x = 0\,; \quad (1)$$
$$x\big\{yz+(1-y)(1-z)\big\} = 0\,; \quad (2)$$
$$z\big\{pq+(1-p)(1-q)\big\} = 0\,; \quad (3)$$
$$p = 0\,; \quad (4)$$
$$q = 0\,. \quad (5)$$

The elimination from the above system of x, p, q, and y, conducts to the equation

$$z = 0\,.$$

And the elimination of x, p, q, and z, conducts in a similar manner to the equation

$$y = 1.$$

Of which equations the respective interpretations are:

1st. *The whole of existing things has not been comprehended in a succession of changeable and dependent beings.*

2nd. *There has existed some one unchangeable and independent being.*

The latter of these is the proposition which Dr. Clarke proves. As, by the above analysis, all the propositions represented by the literal symbols x, y, z, p, q, are determined as absolutely true or false, it is needless to inquire into the existence of any further relations connecting those propositions together.

Another proof is given of Prop. II., which for brevity I pass over. It may be observed, that the "impossibility of infinite succession," the proof of which forms a part of Clarke's argument, has commonly been assumed as a fundamental principle of metaphysics, and extended to other questions than that of causation. Aristotle applies it to establish the necessity of first principles of demonstration;[1] the necessity of an end (the good), in human actions, &c.[2] There is, perhaps, no principle more frequently referred to in his writings. By the schoolmen it was similarly applied to prove the impossibility of an infinite subordination of genera and species, and hence the necessary existence of universals. Apparently the impossibility of our forming a definite and complete conception of an infinite series, i.e. of comprehending it as a *whole*, has been confounded with a logical inconsistency, or contradiction in the idea itself.

8. The analysis of the following argument depends upon the theory of Primary Propositions.

Proposition III.

That unchangeable and independent Being must be self-existent.

The premises are:—

1. Every being must either have come into existence out of nothing, or it must have been produced by some external cause, or it must be self-existent.

2. No being has come into existence out of nothing.

3. The unchangeable and independent Being has not been produced by an external cause.

For the symbolical expression of the above, let us assume,

[1] Metaphysics, III. 4; Anal. Post. I, 19, *et seq.*
[2] Nic. Ethics, Book I. Cap. II.

x = Beings which have arisen out of nothing.
y = Beings which have been produced by an external cause.
z = Beings which are self-existent.
w = The unchangeable and independent Being.

Then we have

$$x(1-y)(1-z) + y(1-x)(1-z) + z(1-x)(1-y) = 1\,, \qquad (1)$$
$$x = 0\,, \qquad (2)$$
$$w = v(1-y), \qquad (3)$$

from the last of which eliminating v,

$$wy = 0\,. \qquad (4)$$

Whenever, as above, the value of a symbol is given as 0 or 1, it is best eliminated by simple substitution. Thus the elimination of x gives

$$y(1-z) + z(1-y) = 1\,; \qquad (5)$$
$$\text{or,} \quad yz + (1-y)(1-z) = 0\,. \qquad (6)$$

Now adding (4) and (6), and eliminating y, we get

$$w(1-z) = 0\,,$$
$$\therefore\ w = vz\,;$$

the interpretation of which is,—*The unchangeable and independent being is necessarily self-existing.*

Of (5), in its actual form, the interpretation is,—*Every being has either been produced by an external cause, or it is self-existent.*

9. In Dr. Samuel Clarke's observations on the above proposition occurs a remarkable argument, designed to prove that the material world is not the self-existent being above spoken of. The passage to which I refer is the following:

"If matter be supposed to exist necessarily, then in that necessary existence there is either included the power of gravitation, or not. If not, then in a world merely material, and in which no intelligent being presides, there never could have been any motion; because motion, as has been already shown, and is now granted in the question, is not necessary of itself. But if the power of gravitation be included in the pretended necessary existence of matter: then, it following necessarily that there must be a vacuum (as the incomparable Sir Isaac Newton has abundantly demonstrated that there must, if gravitation be an universal quality or affection of matter), it follows likewise, that matter is not a necessary being. For if a vacuum actually be, then it is plainly more than possible for matter not to be."—(pp. 25, 26).

It will, upon attentive examination, be found that the actual premises involved in the above demonstration are the following:

1st. If matter is a necessary being, either the property of gravitation is necessarily present, or it is necessarily absent.

2nd. If gravitation is necessarily absent, and the world is not subject to any presiding Intelligence, motion does not exist.

3rd. If the property of gravitation is necessarily present, existence of a vacuum is necessary.

4th. If the existence of a vacuum is necessary, matter is not necessary being.

5th. If matter is a necessary being, the world is not subject to a presiding Intelligence.

6th. Motion exists.

Of the above premises the first four are expressed in the demonstration; the fifth is implied in the connexion of its first and second sentences; and the sixth expresses a fact, which the author does not appear to have thought it necessary to state, but which is obviously a part of the ground of his reasoning. Let us represent the elementary propositions in the following manner:

Let x = Matter is a necessary being.
y = Gravitation is necessarily present.
t = Gravitation is necessarily absent.
z = The world is merely material, and not subject to any presiding Intelligence.
w = Motion exists.
v = A vacuum is necessary.

Then the system of premises will be represented by the following equations, in which q is employed as the symbol of time indefinite:

Here q rather than v is used for "time indefinite", and v denotes a proposition.

$$\begin{aligned} x &= q\{y(1-t)+(1-y)t\}. \\ tz &= q(1-w). \\ y &= qv. \\ v &= q(1-x). \\ x &= qz. \\ w &= 1. \end{aligned}$$

From which, if we eliminate the symbols q, we have the following system, viz.:

$$x\{yt+(1-y)(1-t)\} = 0. \quad (1)$$
$$tzw = 0. \quad (2)$$
$$y(1-v) = 0. \quad (3)$$
$$vx = 0. \quad (4)$$
$$x(1-z) = 0. \quad (5)$$
$$1-w = 0. \quad (6)$$

Now if from these equations we eliminate w, v, z, y, and t, we obtain the equation

$$x = 0,$$

which expresses the proposition, *Matter is not a necessary being.* This is Dr. Clarke's conclusion. If we endeavour to eliminate any other set of five symbols (except the set v, z, y, t, and x, which would give $w = 1$), we obtain a result of the form $0 = 0$. It hence appears that *there are no other conclusions expressive of the absolute truth or falsehood of any of the elementary propositions designated by single symbols.*

Of conclusions expressed by equations involving two symbols, there exists but the following, viz.:— *If the world is merely material, and not subject to a presiding Intelligence, gravitation is not necessarily absent.* This conclusion is expressed by the equation

$$tz = 0, \quad \text{whence} \quad z = q(1-t).$$

If in the above analysis we suppress the concluding premiss, expressing the fact of the existence of motion, and leave the hypothetical principles which are embodied in the remaining premises untouched, some remarkable conclusions follow. To these I shall direct attention in the following chapter.

10. Of the remainder of Dr. Clarke's argument I shall briefly state the substance and connexion, dwelling only on certain portions of it which are of a more complex character than the others, and afford better illustrations of the method of this work.

In Prop. IV. it is shown that the substance or essence of the self-existent being is incomprehensible. The tenor of the reasoning employed is, that we are ignorant of the essential nature of all other things,—much more, then, of the essence of the self-existent being.

In Prop. V. it is contended that "though the substance or essence of the self-existent being is itself absolutely incomprehensible to us, yet many of the essential attributes of his nature are strictly demonstrable, as well as his existence."

In Prop. VI. it is argued that "the self-existent being must of necessity be infinite and omnipresent;" and it is contended that his infinity must be "an infinity of fulness as well as of immensity." The ground upon which the demonstration proceeds is, that an absolute necessity of existence must be independent of time, place, and circumstance, free from limitation, and therefore excluding all imperfection. And hence it is inferred that the self-existent being must be "a most simple, unchangeable, incorruptible being, without parts, figure, motion, or any other such properties as we find in matter."

The premises actually employed may be exhibited as follows:

1. If a finite being is self-existent, it is a contradiction to suppose it not to exist.

2. A finite being may, without contradiction, be absent from one place.

3. That which may without contradiction be absent from one place may without contradiction be absent from all places.

4. That which may without contradiction be absent from all places may without contradiction be supposed not to exist.

Let us assume

x	=	Finite beings.
y	=	Things self-existent.
z	=	Things which it is a contradiction to suppose not to exist.
w	=	Things which may be absent without contradiction from one place.
t	=	Things which without contradiction may be absent from every place.

We have on expressing the above, and eliminating the indefinite symbols,

$$
\begin{aligned}
xy(1-z) &= 0. \quad (1)\\
x(1-w) &= 0. \quad (2)\\
w(1-t) &= 0. \quad (3)\\
tz &= 0. \quad (4)
\end{aligned}
$$

Eliminating in succession t, w, and z, we get

$$
\begin{aligned}
xy &= 0,\\
\therefore\ y &= \frac{0}{0}(1-x);
\end{aligned}
$$

the interpretation of which is,—*Whatever is self-existent is infinite.*

In Prop. VII. it is argued that the self-existent being must of necessity be One. The order of the proof is, that the self-existent being is "necessarily existent," that "necessity absolute in itself is simple and uniform, and without any possible difference or variety," that all "variety or difference of existence" implies dependence; and hence that "whatever exists necessarily is the one simple essence of the self-existent being."

The conclusion is also made to flow from the following premises:—

1. If there are two or more necessary and independent beings, either of them may be supposed to exist alone.

2. If either may be supposed to exist alone, it is not a contradiction to suppose the other not to exist.

3. If it is not a contradiction to suppose this, there are not two necessary and independent beings.

Let us represent the elementary propositions as follows:—

x = there exist two necessary independent beings.
y = either may be supposed to exist alone.
z = it is not a contradiction to suppose the other not to exist.

We have then, on proceeding as before,

$$
\begin{aligned}
x(1-y) &= 0. \quad (1)\\
y(1-z) &= 0. \quad (2)\\
zx &= 0. \quad (3)
\end{aligned}
$$

Eliminating y and z, we have

$$x = 0.$$

Whence, *There do not exist two necessary and independent beings.*

11. To the premises upon which the two previous propositions rest, it is well known that Bishop Butler, who at the time of the publication of the "Demonstration," was a student in a non-conformist academy, made objection in some celebrated letters, which, together with Dr. Clarke's replies to them, are usually appended to editions of the work. The real question at issue is the validity of the principle, that "whatsoever is absolutely necessary at all is absolutely necessary in every part of space, and in every point of duration,"—a principle assumed in Dr. Clarke's reasoning, and explicitly stated in his reply to Butler's first letter. In his second communication Butler says: "I do not conceive that the idea of ubiquity is contained in the idea of self-existence, or *directly follows from it*, any otherwise than as whatever exists must exist *somewhere*." That is to say, necessary existence implies existence in some part of space, but not in every part. It does not appear that Dr. Clarke was ever able to dispose effectually of this objection. The whole of the correspondence is extremely curious and interesting. The objections of Butler are precisely those which would occur to an acute mind impressed with the conviction, that upon the sifting of first principles, rather than upon any mechanical dexterity of reasoning, the successful investigation of truth mainly depends. And the replies of Dr. Clarke, although they cannot be admitted as satisfactory, evince, in a remarkable degree, that peculiar intellectual power which is manifest in the work from which the discussion arose.

12. In Prop. VIII. it is argued that the self-existent and original cause of all things must be an Intelligent Being.

The main argument adduced in support of this proposition is, that as the cause is more excellent than the effect, the self-existent being, as the cause and original of all things, must contain in itself the perfections of all things; and that Intelligence is one of the perfections manifested in a part of the creation. It is further argued that this perfection is not a modification of figure, divisibility, or any of the known properties of matter; for these are not perfections, but *limitations*. To this is added the *à posteriori* argument from the manifestation of design in the frame of the universe.

There is appended, however, a distinct argument for the existence of an intelligent self-existent being, founded upon the phænomenal existence of motion in the universe. I shall briefly exhibit this proof, and shall apply to it the method of the present treatise.

The argument, omitting unimportant explanations, is as follows:—

"'Tis evident there is some such a thing as motion in the world; which either began at some time or other, or was eternal. If it began in time, then the question is granted that the first cause is an intelligent being.... On the contrary, if motion was eternal, either it was eternally caused by some eternal intelligent being, or it must of itself be necessary and self-existent, or else, without any necessity in its own nature, and without any external necessary cause, it must have existed from eternity by an endless successive communication. If motion was eternally caused by some eternal intelligent being, this also is granting the question as to the present dispute. If it was of itself necessary and self-existent, then it follows that it must be a contradiction in terms to suppose any matter to be at rest. And yet, at the same time, because the determination of this self-existent motion must be every way at once, the effect of it would be nothing else but a perpetual rest.... But if it be said that motion, without any necessity in its own nature, and without any external necessary cause, has existed from eternity merely by an endless successive communication, as Spinoza inconsistently enough seems to assert, this I have before shown (in the proof of the second general proposition of this discourse) to be a plain contradiction. It remains, therefore, that motion must of necessity be originally caused by something that is intelligent."

The premises of the above argument may be thus disposed:

1. If motion began in time, the first cause is an intelligent being.

2. If motion has existed from eternity, either it has been eternally caused by some eternal intelligent being, or it is self-existent, or it must have existed by endless successive communication.

3. If motion has been eternally caused by an eternal intelligent being, the first cause is an intelligent being.

4. If it is self-existent, matter is at rest and not at rest.

5. That motion has existed by endless successive communication, and that at the same time it is not self-existent, and has not been eternally caused by some eternal intelligent being, is false.

To express these propositions, let us assume—

x	=	Motion began in time (and therefore)
$1-x$	=	Motion has existed from eternity.
y	=	The first cause is an intelligent being.
p	=	Motion has been eternally caused by some eternal intelligent being.
q	=	Motion is self-existent.
r	=	Motion has existed by endless successive communication.
s	=	Matter is at rest.

The equations of the premises then are—

$$\begin{aligned} x &= vy. \\ 1-x &= v\{p(1-q)(1-r)+q(1-p)(1-r)+r(1-p)(1-q)\}. \\ p &= vy. \\ q &= vs(1-s) = 0. \\ r(1-q)(1-p) &= 0. \end{aligned}$$

Since, by the fourth equation, $q=0$, we obtain, on substituting for q its value in the remaining equations, the system

$$x = vy, \qquad 1-x = v\{p(1-r)+r(1-p)\},$$
$$p = vy, \qquad r(1-p) = 0,$$

from which eliminating the indefinite symbols v, we have the final reduced system,

$$x(1-y) = 0, \tag{1}$$
$$(1-x)\{pr+(1-p)(1-r)\} = 0, \tag{2}$$
$$p(1-y) = 0. \tag{3}$$
$$r(1-p) = 0. \tag{4}$$

We shall first seek the value of y, the symbol involved in Dr. Clarke's conclusion. First, eliminating x from (1) and (2), we have

$$(1-y)\{pr+(1-p)(1-r)\} = 0. \quad (5)$$

Next, to eliminate r from (4) and (5), we have

$$r(1-p)+(1-y)\{pr+(1-p)(1-r)\} = 0,$$
$$\therefore \{1-p+(1-y)p\}\times(1-y)(1-p) = 0;$$

whence
$$(1-y)(1-p) = 0. \quad (6)$$

Lastly, eliminating p from (3) and (6), we have

$$1-y = 0,$$
$$\therefore y = 1,$$

which expresses the required conclusion, *The first cause is an intelligent being.*

Let us now examine what other conclusions are deducible from the premises.

If we substitute the value just found for y in the equations (1), (2), (3), (4), they are reduced to the following pair of equations, viz.,

$$(1-x)\{pr+(1-p)(1-r)\} = 0, \quad r(1-p) = 0. \quad (7)$$

Eliminating from these equations x, we have

$$r(1-p) = 0, \quad \text{whence} \quad r = vp,$$

which expresses the conclusion, *If motion has existed by endless successive communication, it has been eternally caused by an eternal intelligent being.*

Again eliminating, from the given pair, r, we have

$$(1-x)(1-p) = 0,$$

or,
$$1-x = vp,$$

which expresses the conclusion, *If motion has existed from eternity, it has been eternally caused by some eternal intelligent being.*

Lastly, from the same original pair eliminating p, we get

$$(1-x)r = 0,$$

which, solved in the form

$$1-x = r(1-r),$$

gives the conclusion, *If motion has existed from eternity, it has not existed by an endless successive communication.*

Solved under the form

$$r = vx,$$

the above equation leads to the equivalent conclusion, *If motion exists by an endless successive communication, it began in time.*

13. Now it will appear to the reader that the first and last of the above four conclusions are inconsistent with each other. The two consequences drawn from the hypothesis that motion exists by an endless successive communication, viz., 1st, that it has been eternally caused by an eternal intelligent being; 2ndly, that it began in time,—are plainly at variance. Nevertheless, they are both rigorous deductions from the original premises. The opposition between them is not of a *logical*, but of what is technically termed a *material*, character. This opposition might, however, have been formally stated in the premises. We might have added to them a formal proposition, asserting that "whatever is *externally* caused by an eternal intelligent being, does not begin in time." Had this been done, no such opposition as now appears in our conclusions could have presented itself. Formal logic can only take account of relations which are formally expressed (VI. 16); and it may thus, in particular instances, become necessary to express, in a formal manner, some connexion among the premises which, without actual statement, is involved in the very meaning of the language employed.

To illustrate what has been said, let us add to the equations (2) and (4) the equation

$$px = 0,$$

which expresses the condition above adverted to. We have

$$(1 - x)\{pr + (1 - p)(1 - r)\} + r(1 - p) + px = 0. \tag{8}$$

Eliminating p from this, we find simply

$$r = 0,$$

which expresses the proposition, *Motion does not exist by an endless successive communication.* If now we substitute for r its value in (8), we have

$$(1 - x)(1 - p) + px = 0, \quad \text{or,} \quad 1 - x = p;$$

whence we have the interpretation, *If motion has existed from eternity, it has been eternally caused by an eternal intelligent being;* together with the converse of that proposition.

In Prop. IX. it is argued, that "the self-existent and original cause of all things is not a necessary agent, but a being endued with liberty and choice." The proof is based mainly upon his possession of intelligence, and upon the existence of final causes, implying design and choice. To the objection that the supreme cause operates by necessity for the production of what is best, it is replied, that this is a necessity of fitness and wisdom, and not of nature.

14. In Prop. X. it is argued, that "the self-existent being, the supreme cause of all things, must of necessity have infinite power." The ground of the demonstration is, that as "all the powers of all things are derived from him, nothing can make any difficulty or resistance to the execution of his will." It is defined that the infinite power of the self-existent being does not extend to the "making of a thing which implies a contradiction," or the doing of that "which would imply imperfection (whether natural or moral) in the being to whom such power is ascribed," but that it does extend to the creation of matter, and of an immaterial, cogitative substance, endued with a power of beginning motion, and with a liberty of will or choice. Upon this doctrine of liberty it is contended that we are able to give a satisfactory answer to "that ancient and great question, πόθεν τὸ κακὸν, what is the cause and original of evil?" The argument on this head I shall briefly exhibit,

"All that we call evil is either an evil of imperfection, as the want of certain faculties or excellencies which other creatures have; or natural evil, as pain, death, and the like; or moral evil, as all kinds of vice. The first of these is not properly an evil; for every power, faculty, or perfection, which any creature enjoys, being the free gift of God,... it is plain the want of any certain faculty or perfection in any kind of creatures, which never belonged to their natures is no more an evil to them, than their never having been created or brought into being at all could properly have been called an evil. The second kind of evil, which we call natural evil, is either a necessary consequence of the former, as death to a creature on whose nature immortality was never conferred; and then it is no more properly an evil than the former. Or else it is counterpoised on the whole with as great or greater good, as the afflictions and sufferings of good men, and then also it is not properly an evil; or else, lastly, it is a punishment, and then it is a necessary consequence of the third and last kind of evil, viz., moral evil. And this arises wholly from the abuse of liberty which God gave to His creatures for other purposes, and which it was reasonable and fit to give them for the perfection and order of the whole creation. Only they, contrary to God's intention and command, have abused what was necessary to the perfection of the whole, to the corruption and depravation of themselves. And thus all sorts of evils have entered into the world without any diminution to the infinite goodness of the Creator and Governor thereof."—p. 112.

The main premises of the above argument may be thus stated:

1st. All reputed evil is either evil of imperfection, or natural evil, or moral evil.

2nd. Evil of imperfection is not absolute evil.

3rd. Natural evil is either a consequence of evil of imperfection, or it is compensated with greater good, or it is a consequence of moral evil.

4th. That which is either a consequence of evil of imperfection, or is compensated with greater good, is not absolute evil.

5th. All absolute evils are included in reputed evils.

To express these premises let us assume—

$w =$ reputed evil.
$x =$ evil of imperfection.
$y =$ natural evil.
$z =$ moral evil.
$p =$ consequence of evil of imperfection.
$q =$ compensated with greater good.
$r =$ consequence of moral evil.
$t =$ absolute evil.

Then, regarding the premises as Primary Propositions, of which all the predicates are particular, and the conjunctions *either, or*, as absolutely disjunctive, we have the following equations:

$$w = v\{x(1-y)(1-q) + y(1-x)(1-z) + z(1-x)(1-y)\}$$
$$x = v(1-t).$$
$$y = v\{p(1-q)(1-r) + q(1-p)(1-r) + r(1-p)(1-q)\}$$
$$p(1-q) + q(1-p) = v(1-t).$$
$$t = vw.$$

From which, if we separately eliminate the symbol v, we have

$$w\{1 - x(1-y)(1-z) - y(1-x)(1-z) - z(1-x)(1-y)\} = 0, \qquad (1)$$
$$xt = 0, \qquad (2)$$
$$y\{1 - p(1-q)(1-r) - q(1-p)(1-r) - r(1-p)(1-q)\} = 0, \qquad (3)$$
$$\{p(1-q) + q(1-p)\}t = 0, \qquad (4)$$
$$t(1-w) = 0. \qquad (5)$$

Let it be required, first, to find what conclusion the premises warrant us in forming respecting absolute evils, as concerns their dependence upon moral evils, and the consequences of moral evils.

For this purpose we must determine t in terms of z and r.

The symbols w, x, y, p, q must therefore be eliminated. The process is easy, as any set of the equations is reducible to a single equation by addition.

Eliminating w from (1) and (5), we have

$$t\{1 - x(1-y)(1-z) - y(1-x)(1-z) - z(1-x)(1-y)\} = 0. \qquad (6)$$

The elimination of p from (3) and (4) gives

$$yqr + yqt + yt(1-r)(1-q) = 0. \qquad (7)$$

The elimination of q from this gives

$$yt(1-r) = 0. \qquad (8)$$

The elimination of x between (2) and (6) gives

$$t\{yz + (1-y)(1-z)\} = 0. \qquad (9)$$

The elimination of y from (8) and (9) gives

$$t(1-z)(1-r) = 0.$$

This is the only relation existing between the elements t, z, and r.

We hence get

$$\begin{aligned} t &= \frac{0}{(1-z)(1-r)} \\ &= \frac{0}{0}zr + \frac{0}{0}z(1-r) + \frac{0}{0}(1-z)r + 0(1-z)(1-r) \\ &= \frac{0}{0}z + \frac{0}{0}(1-z)r\,; \end{aligned}$$

the interpretation of which is, *Absolute evil is either moral evil, or it is, if not moral evil, a consequence of moral evil.*

Any of the results obtained in the process of the above solution furnish us with interpretations. Thus from (8) we might deduce

$$\begin{aligned} t = \frac{0}{y(1-r)} &= \frac{0}{0}yr + \frac{0}{0}(1-y)r + \frac{0}{0}(1-y)(1-r) \\ &= \frac{0}{0}yr + \frac{0}{0}(1-y)\,; \end{aligned}$$

whence, *Absolute evils are either natural evils, which are the consequences of moral evils, or they are not natural evils at all.*

A variety of other conclusions may be deduced from the given equations in reply to questions which may be arbitrarily proposed. Of such I shall give a few examples, without exhibiting the intermediate processes of solution.

Quest. 1.—Can any relation be deduced from the premises connecting the following elements, viz.: absolute evils, consequences of evils of imperfection, evils compensated with greater good?

Ans.—*No relation exists.* If we eliminate all the symbols but z, p, q, the result is $0 = 0$.

Quest. 2.—Is any relation implied between absolute evils, evils of imperfection, and consequences of evils of imperfection.

Ans.—The final relation between x, t, and p is

$$xt + pt = 0\,;$$

whence

$$t = \frac{0}{p+x} = \frac{0}{0}(1-p)(1-x).$$

Therefore, *Absolute evils are neither evils of imperfection, nor consequences of evils of imperfection.*

Quest. 3. — Required the relation of natural evils to evils of imperfection and evils compensated with greater good.

We find

$$pqy = 0,$$

$$\therefore\ y = \frac{0}{pq} = \frac{0}{0}p(1-q) + \frac{0}{0}(1-p).$$

Therefore, *Natural evils are either consequences of evils of imperfection which are not compensated with greater good, or they are not consequences of evils of imperfection at all.*

Quest. 4. — In what relation do those natural evils which are not moral evils stand to absolute evils and the consequences of moral evils?

If $y(1-z) = s$, we find, after elimination,

$$ts(1-r) = 0;$$

$$\therefore\ s = \frac{0}{t(1-r)} = \frac{0}{0}tr + \frac{0}{0}(1-t).$$

Therefore, *Natural evils, which are not moral evils, are either absolute evils, which are the consequences of moral evils, or they are not absolute evils at all.*

The following conclusions have been deduced in a similar manner. The subject of each conclusion will show of what particular things a description was required, and the predicate will show what elements it was designed to involve: —

Absolute evils, which are not consequences of moral evils, are moral and not natural evils.

Absolute evils which are not moral evils are natural evils, which are the consequences of moral evils.

Natural evils which are not consequences of moral evils are not absolute evils.

Lastly, let us seek a description of evils which are not absolute, expressed in terms of natural and moral evils.

We obtain as the final equation,

$$1 - t = yz + \frac{0}{0}y(1-z) + \frac{0}{0}(1-y)z + (1-y)(1-z).$$

The direct interpretation of this equation is a necessary truth, but the reverse interpretation is remarkable. *Evils which are both* *natural and moral, and evils which are neither natural nor moral, are not absolute evils.*

This conclusion, though it may not express a truth, is certainly involved in the given premises, as *formally* stated.

15. Let us take from the same argument a somewhat fuller system of premises, and let us in those premises suppose that the particles, *either, or,* are not absolutely disjunctive, so that in the meaning of the expression, "either evil of imperfection, or natural evil, or moral evil," we include whatever possesses one or more of these qualities.

Let the premises be —

1. All evil (w) is either evil of imperfection (x), or natural evil (y), or moral evil (z).

2. Evil of imperfection (x) is not absolute evil (t).

3. Natural evil (y) is either a consequence of evil of imperfection (p), or it is compensated with greater good (q), or it is a consequence of moral evil (r).

4. Whatever is a consequence of evil of imperfection (p) is not absolute evil (t).

5. Whatever is compensated with greater good (q) is not absolute evil (t).

6. Moral evil (z) is a consequence of the abuse of liberty (u).

7. That which is a consequence of moral evil (r) is a consequence of the abuse of liberty (u).

8. Absolute evils are included in reputed evils.

The premises expressed in the usual way give, after the elimination of the indefinite symbols v, the following equations:

$$w(1-x)(1-y)(1-z) = 0, \tag{1}$$
$$xt = 0, \tag{2}$$
$$y(1-p)(1-q)(1-r) = 0, \tag{3}$$
$$pt = 0, \tag{4}$$
$$qt = 0, \tag{5}$$
$$z(1-u) = 0, \tag{6}$$
$$r(1-u) = 0, \tag{7}$$
$$t(1-w) = 0. \tag{8}$$

Each of these equations satisfies the condition $V(1-V) = 0$.

The following results are easily deduced —

Natural evil is either absolute evil, which is a consequence of moral evil, or it is not absolute evil at all.

All evils are either absolute evils, which are consequences of the abuse of liberty, or they are not absolute evils.

Natural evils are either evils of imperfection, which are not absolute evils, or they are not evils of imperfection at all.

Absolute evils are either natural evils, which are consequences of the abuse of liberty, or they are not natural evils, and at the same time not evils of imperfection.

Consequences of the abuse of liberty include all natural evils which are absolute evils, and are not evils of imperfection, with an indefinite remainder of natural evils which are not absolute, and of evils which are not natural.

16. These examples will suffice for illustration. The reader can easily supply others if they are needed. We proceed now to examine the most essential portions of the demonstration of Spinoza.

DEFINITIONS.

1. By a *cause of itself (causa sui)*, I understand that of which the essence involves existence, or that of which the nature cannot be conceived except as existing.

2. That thing is said to be finite or bounded in its own kind *(in suo genere finita)* which may be bounded by another thing of the same kind; e. g. Body is said to be finite, because we can always conceive of another body greater than a given one. So thought is bounded by other thought. But body is not bounded by thought, nor thought by body.

3. By substance, I understand that which is in itself *(in se)*, and is conceived by itself *(per se concipitur)*, i.e., that whose conception does not require to be formed from the conception of another thing.

4. By attribute, I understand that which the intellect perceives in substance, as constituting its very essence.

5. By mode, I understand the affections of substance, or that which is in another thing, by which thing also it is conceived.

6. By God, I understand the Being absolutely infinite, that

is the substance consisting of infinite attributes, each of which expresses an eternal and infinite essence.

Explanation.—I say absolutely infinite, not infinite in its own kind. For to whatever is only infinite in its own kind we may deny the possession of (some) infinite attributes. But when a thing is absolutely infinite, whatsoever expresses essence and involves no negation belongs to its essence.

7. That thing is termed *free*, which exists by the sole necessity of its own nature, and is determined to action by itself alone; *necessary*, or rather constrained, which is determined by another thing to existence and action, in a certain and determinate manner.

8. By eternity, I understand existence itself, in so far as it is conceived necessarily to follow from the sole definition of the eternal thing.

Explanation.—For such existence, as an eternal truth, is conceived as the essence of the thing, and therefore cannot be explained by mere duration or time, though the latter should be conceived as without beginning and without end.

AXIOMS.

1. All things which exist are either in themselves *in se* or in another thing.

2. That which cannot be conceived by another thing ought to be conceived by itself.

3. From a given determinate cause the effect necessarily follows, and, contrariwise, if no determinate cause be granted, it is impossible that an effect should follow.

4. The knowledge of the effect depends upon, and involves, the knowledge of the cause.

5. Things which have nothing in common cannot be understood by means of each other; or the conception of the one does not involve the conception of the other.

6. A true idea ought to agree with its own object. (*Idea vera debet cum suo ideato convenire.*)

7. Whatever can be conceived as non-existing does not involve existence in its essence.

Other definitions are implied, and other axioms are virtually assumed, in some of the demonstrations. Thus, in Prop. I., "Substance is prior in nature to its affections," the proof of which consists in a mere reference to Defs. 3 and 5, there seems to be an assumption of the following axiom, viz., "That by which a thing is conceived is prior in nature to the thing conceived." Again, in the demonstration of Prop. V. the converse of this axiom is assumed to be true. Many other examples of the same kind occur. It is impossible, therefore, by the mere processes of Logic, to deduce the whole of the conclusions of the first book of the Ethics from the axioms and definitions which are prefixed to it, and which are given above. In the brief analysis which will follow, I shall endeavour to present in their proper order what appear to me to be the real premises, whether formally stated or implied, and shall show in what manner they involve the conclusions to which Spinoza was led.

17. I conceive, then, that in the course of his demonstration, Spinoza effects several parallel divisions of the universe of possible existence, as,

1st. Into things which are in themselves, x, and things which are in some other thing, x'; whence, as these classes of thing together make up the universe, we have

$$x + x' = 1; \qquad \text{(Ax. I.)}$$

or,

$$x = 1 - x'.$$

2nd. Into things which are conceived by themselves, y, and things which are conceived through some other thing, y'; whence

$$y = 1 - y'. \qquad \text{(Ax. II)}$$

3rd. Into substance, z, and modes, z'; whence

$$z = 1 - z'. \qquad \text{(Def. III. V.)}$$

4th. Into things free, f, and things necessary, f'; whence

$$f = 1 - f'. \qquad \text{(Def. VII.)}$$

5th. Into things which are causes and self-existent, e, and things caused by some other thing, e'; whence

$$e = 1 - e'. \qquad \text{(Def. I. Ax. VII.)}$$

And his reasoning proceeds upon the expressed or assumed principle, that these divisions are not only parallel, but equivalent. Thus in Def. III., Substance is made equivalent with that which is conceived by itself; whence

$$z = y.$$

Again, Ax. IV., as it is actually applied by Spinoza, establishes the identity of cause with that by which a thing is conceived; whence

$$y = e.$$

Again, in Def. VII., things free are identified with things self-existent; whence

$$f = e.$$

Lastly, in Def. V mode is made identical with that which is in another thing; whence $z' = x'$, and therefore,

$$z = x.$$

All these results may be collected together into the following series of equations, viz.:

$$x = y = z = f = e = 1-x' = 1-y' = 1-f' = 1-z' = 1-e'.$$

And any two members of this series connected together by the sign of equality express a conclusion, whether drawn by Spinoza or not, which is a legitimate consequence of his system. Thus the equation

$$z = 1-e',$$

expresses the sixth proposition of his system, viz., One substance cannot be produced by another. Similarly the equation

$$z = e,$$

expresses his seventh proposition, viz., "It pertains to the nature of substance to exist." This train of deduction it is unnecessary to pursue. Spinoza applies it chiefly to the deduction according to his views of the properties of the Divine Nature, having first endeavoured to prove that the only substance is God. In the steps of this process, there appear to me to exist some fallacies, dependent chiefly upon the ambiguous use of words, to which it will be necessary here to direct attention.

18. In Prop. v. it is endeavoured to show, that "There cannot exist two or more substances of the same nature or attribute." The proof is virtually as follows: If there are more substances than one, they are distinguished either by attributes or modes; if by attributes, then there is only one substance of the same attribute; if by modes, then, laying aside these as non-essential, there remains no *real* ground of distinction. Hence there exists but one substance of the same attribute. The assumptions here involved are inconsistent with those which are found in other parts of the treatise. Thus substance, Def. iv., is apprehended by the intellect through the means of attribute. By Def. vi. it may have many attributes. One substance may, therefore, *conceivably* be distinguished from another by a difference in some of its attributes, while others remain the same.

In Prop. viii. it is attempted to show that, All substance is necessarily infinite. The proof is as follows. There exists but one substance, of one attribute, Prop. v.; and it pertains to its nature to exist, Prop. vii. It will, therefore, be of its nature to exist either as finite or infinite. But not as finite, for, by Def. ii. it would require to be bounded by another substance of the same nature, which also ought to exist *necessarily*, Prop. vii. Therefore, there would be two substances of the same attribute, which is absurd, Prop. v. Substance, therefore, is infinite.

In this demonstration the word "finite" is confounded with the expression, "Finite in its own kind," Def. ii. It is thus assumed that nothing can be finite, unless it is bounded by another thing of the same kind. This is not consistent with the ordinary meaning of the term. Spinoza's use of the term finite tends to make space the only form of substance, and all existing things but affections of space, and this, I think, is really one of the ultimate foundations of his system.

The first scholium applied to the above Proposition is remarkable. I give it in the original words: "Quum finitum esse revera sit ex parte negatio, et infinitum absoluta affirmatio existentiae alicujus naturae, sequitur ergo ex sola Prop. vii. omnem substantiam debere esse infinitam." Now this is in reality an assertion of the principle affirmed by Clarke, and controverted by

Butler (XIII. 11), that necessary existence implies existence in every part of space. Probably this principle will be found to lie at the basis of every attempt to demonstrate, *à priori*, the existence of an Infinite Being.

From the general properties of substance above stated, and the definition of God as the substance consisting of infinite attributes, the peculiar doctrines of Spinoza relating to the Divine Nature necessarily follow. As substance is self-existent, free, causal in its very nature, the thing in which other things are, and by which they are conceived; the same properties are also asserted of the Deity. He is self-existent, Prop. XI.; indivisible, Prop. XIII.; the only substance, Prop. XIV.; the Being in which all things are, and by which all things are conceived, Prop. XV.; free, Prop. XVII.; the immanent cause of all things, Prop. XVIII. The proof that God is the only substance is drawn from Def. VI., which is interpreted into a declaration that "God is the Being absolutely infinite, of whom no attribute which expresses the essence of substance can be denied." Every conceivable attribute being thus assigned by definition to Him, and it being determined in Prop. V. that there cannot exist two substances of the same attribute, it follows that God is the only substance.

Though the "Ethics" of Spinoza, like a large portion of his other writings, is presented in the geometrical form, it does not afford a good praxis for the symbolical method of this work. Of course every train of reasoning admits, when its ultimate premises are truly determined, of being treated by that method; but in the present instance, such treatment scarcely differs, except in the use of letters for words, from the processes employed in the original demonstrations. Reasoning which consists so largely of a play upon terms defined as equivalent, is not often met with; and it is rather on account of the interest attaching to the subject, than of the merits of the demonstrations, highly as by some they are esteemed, that I have devoted a few pages here to their exposition.

Boole was impressed by the subtlety of Spinoza's argument.

19. It is not possible, I think, to rise from the perusal of the arguments of Clarke and Spinoza without a deep conviction of the futility of all endeavours to establish, entirely *à priori*, the existence

Boole said reasoning alone cannot establish the existence and properties of a supreme being.

of an Infinite Being, His attributes, and His relation to the universe. The fundamental principle of all such speculations, viz., that whatever we can clearly conceive, must exist, fails to accomplish its end, even when its truth is admitted. For how shall the finite comprehend the infinite? Yet must the possibility of such conception be granted, and in something more than the sense of a mere withdrawal of the limits of phaenomenal existence, before any solid ground can be established for the knowledge, *à priori*, of things infinite and eternal. Spinoza's affirmation of the reality of such knowledge is plain and explicit: "Mens humana adaequatum habet cognitionem aeternae et infinitae essentiae Dei" (Prop. XLVII., Part 2nd). Let this be compared with Prop. XXXIV., Part 2nd: "Omnis idea quae in nobis est absoluta sive adaequata et perfecta, vera est;" and with Axiom VI., Part 1st, "Idea vera debet cum suo ideato convenire." Moreover, this species of knowledge is made the essential constituent of all other knowledge: "De natura rationis est res sub quadam aeternitatis specie percipere" (Prop. XLIV., Cor. II., Part 2nd). Were it said, that there is a tendency in the human mind to rise in contemplation from the particular towards the universal, from the finite towards the infinite, from the transient towards the eternal; and that this tendency suggests to us, with high probability, the existence of more than sense perceives or understanding comprehends; the statement might be accepted as true for at least a large number of minds. There is, however, a class of speculations, the character of which must be explained in part by reference to other causes,—impatience of probable or limited knowledge, so often all that we can really attain to; a desire for absolute certainty where intimations sufficient to mark out before us the path of duty, but not to satisfy the demands of the speculative intellect, have alone been granted to us; perhaps, too, dissatisfaction with the present scene of things. With the undue predominance of these motives, the more sober procedure of analogy and probable induction falls into neglect. Yet the latter is, beyond all question, the course most adapted to our present condition. To infer the existence of an intelligent cause from the teeming evidences of surrounding design, to rise to the conception of a moral Governor of the world, from the study of

Seems Boole favoured the Intelligent Design argument for the existence of God. (LT was published 15 years before Darwin's first publication on evolution.)

the constitution and the moral provisions of our own nature;— these, though but the feeble steps of an understanding limited in its faculties and its materials of knowledge, are of more avail than the ambitious attempt to arrive at a certainty unattainable on the ground of natural religion. And as these were the most ancient, so are they still the most solid foundations, Revelation being set apart, of the belief that the course of this world is not abandoned to chance and inexorable fate.

Chapter XIV

Example of the Analysis of a System of Equations by the Method of Reduction to a Single Equivalent Equation $\mathbf{V = 0}$, wherein $\mathbf{V}$ satisfies the condition $\mathbf{V(1 - V) = 0}$.

1. Let us take the remarkable system of premises employed in the previous Chapter, to prove that "Matter is not a necessary being;" and suppressing the 6th premiss, viz., Motion exists,—examine some of the consequences which flow from the remaining premises. This is in reality to accept as true Dr. Clarke's hypothetical principles; but to suppose ourselves ignorant of the fact of the existence of motion. Instances may occur in which such a selection of a portion of the premises of an argument may lead to interesting consequences, though it is with other views that the present example has been resumed. The premises actually employed will be—

Boole's book LT has the word misspelled, as "ignororant".

1. If matter is a necessary being, either the property of gravitation is necessarily present, or it is necessarily absent.

2. If gravitation is necessarily absent, and the world is not subject to any presiding intelligence, motion does not exist.

3. If gravitation is necessarily present, a vacuum is necessary.

4. If a vacuum is necessary, matter is not a necessary being.

5. If matter is a necessary being, the world is not subject to a presiding intelligence.

If, as before, we represent the elementary propositions by the following notation, viz.:

x = Matter is a necessary being.
y = Gravitation is necessarily present.
w = Motion exists.
t = Gravitation is necessarily absent.
z = The world is merely material, and not subject to a presiding intelligence.
v = A vacuum is necessary.

We shall on expression of the premises and elimination of the indefinite class symbols (q), obtain the following system of equations:

$$\begin{aligned} xyt + x\bar{y}\bar{t} &= 0\,, \\ tzw &= 0\,, \\ y\bar{v} &= 0\,, \\ vx &= 0\,, \\ x\bar{z} &= 0\,; \end{aligned}$$

in which for brevity $\bar{y}$ stands for $1-y$, $\bar{t}$ for $1-t$, and so on; whence, also, $1-\bar{t}=t$, $1-\bar{y}=y$, &c.

As the first members of these equations involve only positive terms, we can form a single equation by adding them together (VIII. Prop. 2), viz.:

$$xyt + x\bar{y}\bar{t} + y\bar{v} + vx + x\bar{z} + tzw \ = \ 0\,,$$

and it remains to reduce the first member so as to cause it to satisfy the condition $V(1-V)=0$.

For this purpose we will first obtain its development with reference to the symbols x and y. The result is—

$$\begin{aligned} (t+\bar{v}+v+\bar{z}+tzw)xy + (\bar{t}+v+\bar{z}+tzw)x\bar{y} \\ +(\bar{v}+tzw)\bar{x}y + tzw\bar{x}\bar{y} \ = \ 0\,. \end{aligned}$$

And our object will be accomplished by reducing the four coefficients of the development to equivalent forms, themselves satisfying the condition required.

Now the first coefficient is, since $v+\bar{v}=1$,

$$1+t+\bar{z}+tzw,$$

which reduces to unity (IX. Prop. 1).

The second coefficient is

$$\bar{t}+v+\bar{z}+tzw\,;$$

and its reduced form (X. 3) is

$$\bar{t}+tv+t\bar{v}\bar{z}+t\bar{v}zw.$$

The third coefficient, $\bar{v}+tzw$, reduces by the same method to $\bar{v}+tzwv$; and the last coefficient tzw needs no reduction. Hence the development becomes

$$xy + (\bar{t} + tv + t\bar{v}\bar{z} + t\bar{v}zw)\, x\bar{y} + (\bar{v} + tzwv)\, \bar{x}y + tzw\bar{x}\bar{y} \;=\; 0\,; \qquad (1)$$

and this is the form of reduction sought.

2. Now according to the principle asserted in Prop. III., Chap. X., the whole relation connecting any particular set of the symbols in the above equation may be deduced by developing that equation with reference to the particular symbols in question, and retaining in the result only those constituents whose coefficients are unity. Thus, if x and y are the symbols chosen, we are immediately conducted to the equation

$$xy \;=\; 0\,,$$

whence we have

$$y \;=\; \frac{0}{0}(1-x),$$

with the interpretation, *If gravitation is necessarily present, matter is not a necessary being.*

Let us next seek the relation between x and w. Developing (1) with respect to those symbols, we get

$$(y + \bar{t}\bar{y} + tv\bar{y} + t\bar{v}\bar{z}\bar{y} + t\bar{v}z\bar{y})\, xw + (y + \bar{t}\bar{y} + tv\bar{y} + t\bar{v}\bar{z}\bar{y})\, x\bar{w} \\ + (\bar{v}y + tzvy + tz\bar{y})\, \bar{x}w + \bar{v}y\bar{x}\bar{w} \;=\; 0\,.$$

The coefficient of xw, and it alone, reduces to unity. For $t\bar{v}\bar{z}\bar{y} + t\bar{v}z\bar{y} \;=\; t\bar{v}\bar{y}$, and $tv\bar{y} + t\bar{v}\bar{y} \;=\; t\bar{y}$, and $\bar{t}\bar{y} + t\bar{y} \;=\; \bar{y}$, and lastly, $y + \bar{y} \;=\; 1$. This is always the mode in which such reductions take place. Hence we get

$$xw \;=\; 0\,,$$
$$\therefore\; w \;=\; \frac{0}{0}(1-x),$$

of which the interpretation is, *If motion exists, matter is not a necessary being.*

If, in like manner, we develop (1) with respect to x and z, we get the equation

$$x\bar{z} \;=\; 0\,,$$
$$\therefore\; x \;=\; \frac{0}{0}z,$$

with the interpretation, *If matter is a necessary being, the world is merely material, and without a presiding intelligence.*

This, indeed, is only the fifth premiss reproduced, but it shows that there is no other relation connecting the two elements which it involves.

If we seek the whole relation connecting the elements x, w, and y, we find, on developing (1) with reference to those symbols, and proceeding as before,

$$xy + xw\bar{y} = 0\,.$$

Suppose it required to determine hence the consequences of the hypothesis, "Motion does not exist," relatively to the questions of the necessity of matter, and the necessary presence of gravitation. We find

$$w = \frac{-xy}{x\bar{y}},$$

$$\therefore\ 1 - w = \frac{x}{x\bar{y}} = \frac{1}{0}xy + x\bar{y} + \frac{0}{0}\bar{x}\,;$$

or,

$$1 - w = x\bar{y} + \frac{0}{0}\bar{x}, \quad \text{with} \quad xy = 0\,.$$

The direct interpretation of the first equation is, *If motion does not exist, either matter is a necessary being, and gravitation is not necessarily present, or matter is not a necessary being.*

The reverse interpretation is, *If matter is a necessary being, and gravitation not necessary, motion does not exist.*

In exactly the same mode, if we sought the full relation between x, z, and w, we should find

$$xzw + x\bar{z} = 0\,.$$

From this we may deduce

$$z = x\bar{w} + \frac{0}{0}\bar{x}, \quad \text{with} \quad xw = 0\,.$$

Therefore, *If the world is merely material, and not subject to any presiding intelligence, either matter is a necessary being, and motion does not exist, or matter is not a necessary being.*

Also, *reversely, If matter is a necessary being, and there is no such thing as motion, the world is merely material.*

3. We might, of course, extend the same method to the determination of the consequences of any complex hypothesis u, such as, "The world is merely material, and without any presiding intelligence (z), but motion exists" (w), with reference to any other elements of doubt or speculation involved in the original premises, such as, "Matter is a necessary being" (x), "Gravitation is a necessary quality of matter," (y). We should, for this purpose, connect with the general equation (1) a new equation,

$$u = wz,$$

reduce the system thus formed to a single equation, $V = 0$, in which V satisfies the condition $V(1-V) = 0$, and proceed as above to determine the relation between u, x, and y, and finally u as a developed function of x and y. But it is very much better to adopt the methods of Chapters VIII. and IX. I shall here simply indicate a few results, with the leading steps of their deduction, and leave their verification to the reader's choice.

In the problem last mentioned we find, as the relation connecting x, y, w, and z,

$$xw + x\bar{w}y + x\bar{w}\bar{y}\bar{z} = 0 .$$

And if we write $u = xy$, and then eliminate the symbols x and y by the general problem, Chap. IX., we find

$$xu + xy\bar{u} = 0 ,$$

whence

$$u = \frac{1}{0}xy + 0x\bar{y} + \frac{0}{0}\bar{x} ;$$

wherefore

$$wz = \frac{0}{0}\bar{x} \text{ with } xy = 0 .$$

Hence, *If the world is merely material, and without a presiding intelligence, and at the same time motion exists, matter is not a necessary being.*

Now it has before been shown that *if motion exists, matter is not a necessary being*, so that the above conclusion tells us even less than we had before ascertained to be (inferentially) true. Nevertheless, that conclusion is the proper and complete answer to the question which was proposed, which was, to determine simply the consequences of a certain complex hypothesis.

4. It would thus be easy, even from the limited system of premises before us, to deduce a great variety of additional inferences, involving, in the conditions which are given, any proposed combinations of the elementary propositions. If the condition is one which is inconsistent with the premises, the fact will be indicated by the form of the solution. The value which the method will assign to the combination of symbols expressive of the proposed condition will be 0. If, on the other hand, the fulfilment of the condition in question imposes no restriction upon the propositions among which relation is sought, so that every combination of those propositions is equally possible,—the fact will also be indicated by the form of the solution. Examples of each of these cases are subjoined.

If in the ordinary way we seek the consequences which would flow from the condition that *matter is a necessary being*, and at the same time that *motion exists*, as affecting the Propositions, *The world is merely material, and without a presiding intelligence*, and, *Gravitation is necessarily present*, we shall obtain the equation

$$xw = 0,$$

which indicates that the condition proposed is inconsistent with the premises, and therefore cannot be fulfilled.

If we seek the consequences which would flow from the condition that *Matter is not a necessary being*, and at the same time that *Motion does exist*, with reference to the same elements as above, viz., *the absence of a presiding intelligence*, and the *necessity of gravitation*,—we obtain the following result,

$$(1-x)w = \frac{0}{0}yz + \frac{0}{0}y(1-z) + \frac{0}{0}(1-y)z + \frac{0}{0}(1-y)(1-z),$$

which might literally be interpreted as follows:

If matter is not a necessary being, and motion exists, then either the world is merely material and without a presiding intelligence, and gravitation is necessary, or one of these two results follows without the other, or they both fail of being true. Wherefore of the four possible combinations, of which some one is true of necessity, and of which of necessity one only can be true, it is affirmed that any one may be true. Such a result is a truism— a mere *necessary* truth. Still it contains the only answer which can be given to the question proposed.

I do not deem it necessary to vindicate against the charge of laborious trifling these applications. It may be requisite to enter with some fulness into details useless in themselves, in order to establish confidence in general principles and methods. When this end shall have been accomplished in the subject of the present inquiry, let all that has contributed to its attainment, but has afterwards been found superfluous, be forgotten.

Chapter XV

The Aristotelian Logic and its Modern Extensions, Examined by the Method of this Treatise.

1. The logical system of Aristotle, modified in its details, but unchanged in its essential features, occupies so important a place in academical education, that some account of its nature, and some brief discussion of the leading problems which it presents, seem to be called for in the present work. It is, I trust, in no narrow or harshly critical spirit that I approach this task. My object, indeed, is not to institute any direct comparison between the time-honoured system of the schools and that of the present treatise; but, setting truth above all other considerations, to endeavour to exhibit the real nature of the ancient doctrine, and to remove one or two prevailing misapprehensions respecting its extent and sufficiency.

That which may be regarded as essential in the spirit and procedure of the Aristotelian, and of all cognate systems of Logic, is the attempted classification of the allowable forms of inference, and the distinct reference of those forms, collectively or individually, to some general principle of an axiomatic nature, such as the "dictum of Aristotle": Whatsoever is affirmed or denied of the genus may in the same sense be affirmed or denied of any species included under that genus. Concerning such general principles it may, I think, be observed, that they either state directly, but in an abstract form, the argument which they are supposed to elucidate, and, so stating that argument, affirm its validity; or involve in their expression technical terms which, after definition, conduct us again to the same point, viz., the abstract statement of the supposed allowable forms of inference. The idea of classification is thus a pervading element in those systems. Furthermore, they exhibit Logic as resolvable into two great branches, the one of which is occupied with the treatment of categorical, the other with that of hypothetical or

By "modern extensions" Boole meant the minor modifications De Morgan, Hamilton, etc., made in the collection of categorical propositions. One direction was the 'quantification of the predicate', that is, putting 'all' or 'some' in front of the predicate. Another modification was to allow converses of simple subjects and predicates. Combining these one has the 16 forms:

(All/Some) (S/not-S)
is
(all/some) (P/not-P).

Note that allowing converses permits one to dispense with the 'is not', since, for example, "Some S is not P" can be expressed by "Some S is not-P".

The rather heated and public dispute between Augustus De Morgan and the Scottish philosopher Sir William Hamilton in early 1847 was the spark that led Boole to write MAL, his 1847 booklet on logic.

conditional propositions. The distinction is nearly identical with that of primary and secondary propositions in the present work. The discussion of the theory of categorical propositions is, in all the ordinary treatises of Logic, much more full and elaborate than that of hypothetical propositions, and is occupied partly with ancient scholastic distinctions, partly with the canons of deductive inference. To the latter application only is it necessary to direct attention here.

2. Categorical propositions are classed under the four following heads, viz.:

The four kinds of Aristotelian categorical propositions.

			TYPE
1st.	Universal affirmative Propositions:		All Y's are X's.
2nd.	Universal negative	"	No Y's are X's.
3rd.	Particular affirmative	"	Some Y's are X's.
4th.	Particular negative	"	Some Y's are not X's.

To these forms, four others have recently been added, so as to constitute in the whole eight forms (see the next article) susceptible, however, of reduction to six, and subject to relations which have been discussed with great fulness and ability by Professor De Morgan, in his Formal Logic. A scheme somewhat different from the above has been given to the world by Sir W. Hamilton, and is made the basis of a method of syllogistic inference, which is spoken of with very high respect by authorities on the subject of Logic.[1]

The processes of Formal Logic, in relation to the above system of propositions, are described as of two kinds, viz., "Conversion" and "Syllogism." By Conversion is meant the expression of any proposition of the above kind in an equivalent form, but with a reversed order of terms. By Syllogism is meant the deduction from two such propositions having a common term, whether subject or predicate, of some third proposition inferentially involved in the two, and forming the "conclusion." It is maintained by most writers on Logic, that these processes, and according to some, the single process of Syllogism, furnish the universal types of reasoning, and that it is the business of the mind, in any train of demonstration, to conform itself, whether

Here we see two examples, Conversion and Syllogism, of what Boole meant by "processes" in logic; they are expressed symbolically as equational arguments, or one could say as quasi-equations (a conjunction of premiss equations implying a conclusion equation). Boole's work focused on the processes of reduction, elimination, solution and expansion.

[1]Thomson's Outlines of the Laws of Thought, p. 177.

consciously or unconsciously, to the particular models of the processes which have been classified in the writings of logicians.

3. The course which I design to pursue is to show how these processes of Syllogism and Conversion may be conducted in the most general manner upon the principles of the present treatise, and, viewing them thus in relation to a system of Logic, the foundations of which, it is conceived, have been laid in the ultimate laws of thought, to seek to determine their true place and essential character.

The expressions of the eight fundamental types of proposition in the language of symbols are as follows:

By allowing the subject and predicate to be either simple classes X or their converses not-X, one has eight categorical propositions. The second choice, using the converses, was not allowed in Aristotelian logic.

1. All Y's are X's,	$y = vx.$	
2. No Y's are X's,	$y = v(1-x).$	
3. Some Y's are X's,	$vy = vx.$	
4. Some Y's are not-X's,	$vy = v(1-x).$	
5. All not-Y's are X's,	$1-y = vx.$	(1)
6. No not-Y's are X's,	$1-y = v(1-x).$	
7. Some not-Y's are X's,	$v(1-y) = vx.$	
8. Some not-Y's are not-X's,	$v(1-y) = v(1-x).$	

In referring to these forms, it will be convenient to apply, in a sense shortly to be explained, the epithets of logical quantity, "universal" and "particular," and of quality, "affirmative" and "negative," to the terms of propositions, and not to the propositions themselves. We shall thus consider the term "All Y's," as universal-affirmative; the term "Y's," or "Some Y's," as particular-affirmative; the term "All not-Y's," as universal-negative; the term "Some not-Y's," as particular-negative. The expression "No Y's," is not properly a *term* of a proposition, for the true meaning of the proposition, "No Y's are X's," is "All Y's are not-X's." The subject of that proposition is, therefore, universal-affirmative, the predicate particular-negative. That there is a real distinction between the conceptions of "men" and "not men" is manifest. This distinction is all that I contemplate when applying as above the designations of affirmative and negative, without, however, insisting upon the etymological propriety of the application to the terms of propositions. The designations positive and privative would have been more appropriate, but the former term is already employed in a fixed sense in other parts of this work.

Section 4 is about conversion.

4. From the symbolical forms above given the laws of conversion immediately follow. Thus from the equation

$$y = vx,$$

representing the proposition, "All Y's are X's," we deduce, on eliminating v,

$$y(1-x) = 0,$$

which gives by solution with reference to $1-x$,

$$1 - x = \frac{0}{0}(1-y);$$

the interpretation of which is,

All not-X's are not-Y's.

Boole's original translation of universal propositions involved a parameter that was $\neq 0$. Here he is using a more general parameter, 0/0, in the reverse direction.

This form is the contrapositive of the above form.

This is an example of what is called "negative conversion." In like manner, the equation

$$y = v(1-x),$$

representing the proposition, "No Y's are X's," gives

$$x = \frac{0}{0}(1-y),$$

the interpretation of which is, "No X's are Y's." This is an example of what is termed simple conversion; though it is in reality of the same kind as the conversion exhibited in the previous example. All the examples of conversion which have been noticed by logicians are either of the above kind, or of that which consists in the mere transposition of the terms of a proposition, without altering their quality, as when we change

"Mere transposition" is the *symmetry* rule of inference in equational logic.

$$vy = vx, \quad \text{representing, Some } Y\text{'s are } X\text{'s},$$

into

$$vx = vy, \quad \text{representing, Some } X\text{'s are } Y\text{'s};$$

or they involve a combination of those processes with some auxiliary process of limitation, as when from the equation

$$y = vx, \quad \text{representing, All } Y\text{'s are } X\text{'s},$$

we deduce on multiplication by v,

$$vy = vx, \quad \text{representing, Some } Y\text{'s are } X\text{'s},$$

and hence

$$vx = vy, \quad \text{representing, Some } X\text{'s are } Y\text{'s}.$$

The last example on this page is "conversion by limitation". This inference from Aristotelian logic requires that the subject in a universal categorical proposition be a non-empty class. Boole's algebra of logic is more elegant if one drops this kind of inference, allowing class-symbols to denote any subclass of the universe. Perhaps this is what motivated C.S. Peirce to initiate the modern semantics for class-symbols in 1880.

In this example, the process of limitation precedes that of transposition.

From these instances it is seen that conversion is a particular application of a much more general process in Logic, of which many examples have been given in this work. That process has for its object the determination of any element in any proposition, however complex, as a logical function of the remaining elements. Instead of confining our attention to the subject and predicate, regarded as simple terms, we can take any element or any combination of elements entering into either of them; make that element, or that combination, the "subject" of a new proposition; and determine what its predicate shall be, in accordance with the data afforded to us. It may be remarked, that even the simple forms of propositions enumerated above afford some ground for the application of such a method, beyond what the received laws of conversion appear to recognise. Thus the equation

Boole said conversion, viewed from the algebra side, was just a simple case of solving an equation for a variable.

$$y = vx, \text{ representing, All } Y\text{'s are } X\text{'s},$$

gives us, in addition to the proposition before deduced, the three following:

1st. $y(1-x) = 0\,.$ There are no Y's that are not-X's.

2nd. $1-y = \frac{0}{0}x + (1-x).$ Things that are not-Y's include all things that are not-X's, and an indefinite remainder of things that are X's.

3rd. $x = y + \frac{0}{0}(1-y).$ Things that are X's include all things that are Y's, and an indefinite remainder of things that are not-Y's.

These conclusions, it is true, merely place the given proposition in other and equivalent forms,—but such and no more is the office of the received mode of "negative conversion."

Furthermore, these processes of conversion are not elementary, but they are combinations of processes more simple than they, more immediately dependent upon the ultimate laws and axioms which govern the use of the symbolical instrument of reasoning. This remark is equally applicable to the case of Syllogism, which we proceed next to consider.

Section 5 is about syllogisms.

5. The nature of syllogism is best seen in the particular instance. Suppose that we have the propositions,

All X's are Y's,
All Y's are Z's.

From these we may deduce the conclusion,

All X's are Z's.

This is a syllogistic inference. The terms X and Z are called the extremes, and Y is called the middle term. The function of the syllogism generally may now be defined. Given two propositions of the kind whose species are tabulated in (1), and involving one middle or common term Y, which is connected in one of the propositions with an extreme X, in the other with an extreme Z; required the relation connecting the extremes X and Z. The term Y may appear in its affirmative form, as, All Y's, Some Y's; or in its negative form, as, All not-Y's, Some not-Y's; in either proposition, without regard to the particular form which it assumes in the other.

Nothing is easier than in particular instances to resolve the Syllogism by the method of this treatise. Its resolution is, indeed, a particular application of the process for the reduction of systems of propositions. Taking the examples above given, we have,

$$x = vy,$$
$$y = v'z\,;$$

This is the first time we see the word 'some' expressed using distinct parameters (v, v') in distinct premisses.

whence by substitution,

$$x = vv'z,$$

which is interpreted into

All X's are Z's.

Or, proceeding rigorously in accordance with the method developed in (VIII.7), we deduce

$$x(1-y) = 0\,, \qquad y(1-z) = 0\,.$$

Adding these equations, and eliminating y, we have

$$x(1-z) = 0\,;$$

whence $x = \frac{0}{0}z$, or, All X's are Z's.
And in the same way may any other case be treated.

6. Quitting, however, the consideration of special examples, let us examine the general forms to which all syllogism may be reduced.

Proposition I.

To deduce the general rules of Syllogism.

Boole was referring to the rules for syllogisms that he introduced in "The Calculus of Logic" in 1848.

By the general rules of Syllogism, I here mean the rules applicable to premises admitting of every variety both of quantity and of quality in their subjects and predicates, except the combination of two universal terms in the same proposition. The admissible forms of propositions are therefore those of which a tabular view is given in (1).

Let X and Y be the elements or things entering into the first premiss, Z and Y those involved in the second. Two cases, fundamentally different in character, will then present themselves. The terms involving Y will either be of *like* or of *unlike quality*, those terms being regarded as of like quality when they both speak of "Y's," or both of "Not-Y's," as of unlike quality when one of them speaks of "Y's," and the other of "Not-Y's." Any pair of premises, in which the former condition is satisfied, may be represented by the equations

$$vx = v'y, \quad (1)$$

$$wz = w'y; \quad (2)$$

Middle terms of 'like quality'.

for we can employ the symbol y to represent either "All Y's," or "All not-Y's," since the interpretation of the symbol is purely conventional. If we employ y in the sense of "All not-Y's," then $1 - y$ will represent "All Y's," and no other change will be introduced. An equal freedom is permitted with respect to the symbols x and z, so that the equations (1) and (2) may, by properly assigning the interpretations of x, y, and z, be made to represent all varieties in the combination of premises dependent upon the *quality* of the respective terms. Again, by assuming proper interpretations to the symbols v, v', w, w', in those equations, all varieties with reference to *quantity* may also be

Quantity is either universal (All) or particular (Some).

represented. Thus, if we take $v = 1$, and represent by v' a class indefinite, the equation (1) will represent a universal proposition according to the ordinary sense of that term, i. e., a proposition with universal subject and particular predicate. We may, in fact, give to subject and predicate in either premiss whatever *quantities* (using this term in the scholastic sense) we please, except that by hypothesis, they must not both be universal. The system (1), (2), represents, therefore, with perfect generality, the possible combinations of premises which have like middle terms.

7. That our analysis may be as general as the equations to which it is applied, let us, by the method of this work, eliminate y from (1) and (2), and seek the expressions for x, $1-x$, and vx, in terms of z and of the symbols v, v', w, w'. The above will include all the possible forms of the subject of the conclusion. The form $v(1-x)$ is excluded, inasmuch as we cannot from the interpretation $vx =$ Some X's, given in the premises, interpret $v(1-x)$ as Some not-X's. The symbol v, when used in the sense of "some", applies to that term only with which it is connected in the premises.

This is the first time Boole mentioned that a symbol like v, introduced in a premise to express 'some,' could subsequently only be read as 'some' when v was multiplied by the term with which it appeared in the premises.

The results of the analysis are as follows:

$$x = \left[vv'ww' + \frac{0}{0}\Big\{vv'(1-w)(1-w') + ww'(1-v)(1-v') + (1-v)(1-w)\Big\}\right]z + \frac{0}{0}\Big\{vv'(1-w') + 1 - v\Big\}(1-z), \qquad \text{(I.)}$$

$$1-x = \Big[v(1-v')\Big\{ww' + (1-w)(1-w')\Big\} + v(1-w)w' + \frac{0}{0}\Big\{vv'(1-w)(1-w') + ww'(1-v)(1-v') + (1-v)(1-w)\Big\}\Big]z + \Big[v(1-w)w' + \frac{0}{0}\Big\{vv'(1-w') + 1 - v\Big\}\Big](1-z), \qquad \text{(II.)}$$

Boole claimed to derive the syllogistic logic as a routine, if somewhat tedious, application of his elimination and solution theorems. He simply stated the final results of applying these two theorems, leaving it to the reader to verify that the equations I–VI are correct.

$$vx = \Big\{vv'ww' + \frac{0}{0}vv'(1-w)(1-w')\Big\}z + \frac{0}{0}(1-w')(1-z). \qquad \text{(III.)}$$

Each of these expressions involves in its second member two terms, of one of which z is a factor, of the other $1-z$. But syllogistic inference does not, as a matter of form, admit of contrary classes in its conclusion, as of Z's and not-Z's together.

We must, therefore, in order to determine the rules of that species of inference, ascertain under what conditions the second members of any of our equations are reducible to a single term.

The simplest form is (III.), and it is reducible to a single term if $w' = 1$. The equation then becomes

$$vx = vv'wz, \tag{3}$$

the first member is identical with the extreme in the first premiss; the second is of the same quantity and quality as the extreme in the second premiss. For since $w' = 1$, the second member of (2), involving the middle term y, is universal; therefore, by the hypothesis, the first member is particular, and therefore, the second member of (3), involving the same symbol w in its coefficient, is particular also. Hence we deduce the following law.

Condition of Inference.—One middle term, at least, universal.

Rule of Inference.—Equate the extremes.

From an analysis of the equations (I.) and (II.), it will further appear, that the above is the only condition of syllogistic inference when the middle terms are of like quality. Thus the second member of (I.) reduces to a single term, if $w' = 1$ and $v = 1$; and the second member of (II.) reduces to a single term, if $w' = 1$, $v = 1$, $w = 1$. In each of these cases, it is necessary that $w' = 1$, the solely sufficient condition before assigned.

Consider, secondly, the case in which the middle terms are of unlike quality. The premises may then be represented under the forms

Middle terms of 'unlike quality'.

$$vx = v'y, \tag{4}$$
$$wz = w'(1-y)\,; \tag{5}$$

and if, as before, we eliminate y, and determine the expressions of x, $1 - x$, and vx, we get

$$x = \left[vv'(1-w)w' + \frac{0}{0}\Big\{ww'(1-v) + (1-v)(1-v')(1-w) + v'(1-w)(1-w')\Big\}\right]z + \left[vv'w' + \frac{0}{0}\Big\{(1-v)(1-v') + v'(1-w')\Big\}\right](1-z). \quad \text{(IV.)}$$

$$\begin{aligned}1 - x \ = \ \Big[& ww'v + v(1-v')(1-w) + \frac{0}{0}\Big\{ ww'(1-v) \\ & + (1-v)(1-v')(1-w) + v'(1-w)(1-w')\Big\}\Big] z \\ & + \Big[v(1-v') + \frac{0}{0}\Big\{ v'(1-w') + (1-v)(1-v')\Big\}\Big](1-z).\end{aligned} \tag{V.}$$

$$\begin{aligned} vx \ = \ & \Big\{ vv'(1-w)w' + \frac{0}{0} vv'(1-w)(1-w')\Big\} z \\ & + \Big\{ vv'w' + \frac{0}{0} vv'(1-w')\Big\}(1-z). \end{aligned} \tag{VI.}$$

Now the second member of (VI.) reduces to a single term relatively to z, if $w = 1$, giving

$$vx \ = \ \Big\{ vv'w' + \frac{0}{0} vv'(1-w')\Big\}(1-z)\,;$$

the second member of which is opposite, both in quantity and quality, to the corresponding extreme, wz, in the second premiss. For since $w = 1$, wz is universal. But the factor vv' indicates that the term to which it is attached is particular, since by hypothesis v and v' are not both equal to 1. Hence we deduce the following law of inference in the case of like middle terms:

Should say "in the case of unlike middle terms:" Error in Boole's book.

FIRST CONDITION OF INFERENCE.—*At least one universal extreme.*

RULE OF INFERENCE.—*Change the quantity and quality of that extreme, and equate the result to the other extreme.*

Moreover, the second member of (V.) reduces to a single term if $v' = 1$, $w' = 1$; it then gives

$$1 - x \ = \ \Big\{ vw + \frac{0}{0}(1-v)w\Big\} z.$$

Now since $v' = 1$, $w' = 1$, the middle terms of the premises are both universal, therefore the extremes vx, wz, are particular. But in the conclusion the extreme involving x is opposite, both in quantity and quality, to the extreme vx in the first premiss, while the extreme involving z agrees both in quantity and quality with the corresponding extreme wz in the second premiss. Hence the following general law:

Second Condition of Inference.—*Two universal middle terms.*

Rule of Inference.—*Change the quantity and quality of either extreme, and equate the result to the other extreme unchanged.*

There are in the case of unlike middle terms no other conditions or rules of syllogistic inference than the above. Thus the equation (IV.), though reducible to the form of a syllogistic conclusion, when $w = 1$ and $v = 1$, does not thereby establish a new condition of inference; since, by what has preceded, the single condition $v = 1$, or $w = 1$, would suffice.

Section 8 applies the aforementioned rules of syllogistic inference to a few examples. Example 3 is not covered by the classical rules.

8. The following examples will sufficiently illustrate the general rules of syllogism above given:

1. All Y's are X's.
   All Z's are Y's.

This belongs to Case 1. All Y's is the universal middle term. The extremes equated give as the conclusion

All Z's are X's ;

the universal term, All Z's, becoming the subject; the particular term (some) X's, the predicate.

2. All X's are Y's.
   No Z's are Y's.

The proper expression of these premises is

All X's are Y's.
All Z's are not-Y's.

They belong to Case 2, and satisfy the first condition of inference. The middle term, Y's, in the first premiss, is particular-affirmative; that in the second premiss, not-Y's, particular-negative. If we take All Z's as the universal extreme, and change its quantity and quality according to the rule, we obtain the term Some not-Z's, and this equated with the other extreme, All X's, gives,

All X's are not-Z's, i.e., No X's are Z's.

If we commence with the other universal extreme, and proceed similarly, we obtain the equivalent result,

No Z's are X's.

3. All Y's are X's.
   All not-Y's are Z's.

Here also the middle terms are unlike in quality. The premises therefore belong to Case 2, and there being two universal middle terms, the second condition of inference is satisfied. If by the rule we change the quantity and quality of the first extreme, (some) X's, we obtain All not-X's, which, equated with the other extreme, gives

All not-X's are Z's.

The reverse order of procedure would give the equivalent result,

All not-Z's are X's.

The conclusions of the two last examples would not be recognised as valid in the scholastic system of Logic, which virtually requires that the subject of a proposition should be affirmative. They are, however, perfectly legitimate in themselves, and the rules by which they are determined form undoubtedly the most general canons of syllogistic inference. The process of investigation by which they are deduced will probably appear to be of needless complexity; and it is certain that they might have been obtained with greater facility, and without the aid of any symbolical instrument whatever. It was, however, my object to conduct the investigation in the most general manner, and by an analysis thoroughly exhaustive. With this end in view, the brevity or prolixity of the method employed is a matter of indifference. Indeed the analysis is not properly that of the syllogism, but of a much more general combination of propositions; for we are permitted to assign to the symbols v, v', w, w', any class-interpretations that we please. To illustrate this remark, I will apply the solution (I.) to the following imaginary case:

Suppose that a number of pieces of cloth striped with different colours were submitted to inspection, and that the two following observations were made upon them:

1st. That every piece striped with white and green was also striped with black and yellow, and *vice versâ*.

2nd. That every piece striped with red and orange was also striped with blue and yellow, and *vice versâ*.

Suppose it then required to determine how the pieces marked with green stood affected with reference to the colours white, black, red, orange, and blue.

Here if we assume v = white, x = green, v' = black, y = yellow, w = red, z = orange, w' = blue, the expression of our premises will be

$$\begin{aligned} vx &= v'y, \\ wz &= w'y, \end{aligned}$$

agreeing with the system (1) (2). The equation (I.) then leads to the following conclusion:

Pieces striped with green are either striped with orange, white, black, red, and blue, together, all pieces possessing which character are included in those striped with green; or they are striped with orange, white, and black, but not with red or blue; or they are striped with orange, red, and blue, but not with white or black; or they are striped with orange, but not with white or red; or they are striped with white and black, but not with blue or orange; or they are striped neither with white nor orange.

Considering the nature of this conclusion, neither the symbolical expression (I.) by which it is conveyed, nor the analysis by which that expression is deduced, can be considered as needlessly complex.

9. The form in which the doctrine of syllogism has been presented in this chapter affords ground for an important observation. We have seen that in each of its two great divisions the entire discussion is reducible, so far, at least, as concerns the determination of rules and methods, to the analysis of a pair of equations, viz., of the system (1), (2), when the premises have like middle terms, and of the system (4), (5), when the middle terms are unlike. Moreover, that analysis has been actually conducted by a method founded upon certain general laws deduced immediately from the constitution of language, Chap. II., confirmed by the study of the operations of the human mind, Chap. III., and proved to be applicable to the analysis of all systems of equations whatever, by which propositions, or combinations of propositions, can be represented, Chap. VIII. Here, then, we have the means of definitely resolving the question, whether syllogism is indeed the fundamental type of reasoning,—whether the study of its laws is co-extensive with the study of deductive logic. For if it be so, some indication of the fact must be given in the systems of equations upon the analysis of which we have been engaged. It cannot be conceived that syllogism should be the one essential process of reasoning, and yet the manifestation of that process present nothing indicative of this high quality of pre-eminence. No sign, however, appears that the discussion of all systems of equations expressing propositions is involved in that of the particular system examined in this chapter. And yet writers on Logic have been all but unanimous in their assertion, not merely of the supremacy, but of the universal sufficiency of syllogistic inference in deductive reasoning. The language of Archbishop Whately, always clear and definite, and on the subject of Logic entitled to peculiar attention, is very express on this point. "For Logic," he says, "which is, as it were, the Grammar of Reasoning, does not bring forward the regular Syllogism as a *distinct mode of argumentation*, designed to be *substituted* for any other mode; but as the form to which *all* correct reasoning may be ultimately reduced."[2] And Mr. Mill, in a chapter of his System of Logic, entitled, "Of Ratiocination or Syllogism," having enumerated the ordinary forms of syllogism, observes, "All valid ratiocination, all reasoning by which from general propositions previously admitted, other propositions, equally or less general, are inferred, may be exhibited in some of the above forms." And again: "We are therefore at liberty, in conformity with the general opinion of logicians, to consider the two elementary forms of the first figure as the universal types of all correct ratiocination." In accordance with these views it has been contended that the science of Logic enjoys an immunity from those conditions of imperfection and of progress to which all other sciences are subject;[3] and its origin from the travail of one mighty mind of old has, by a somewhat daring metaphor, been compared to the mythological birth of Pallas.

As Syllogism is a species of elimination, the question before us manifestly resolves itself into the two following ones:—1st. Whether all elimination is reducible to Syllogism; 2ndly.

[2]Elements of Logic, p. 13, ninth edition.

[3]Introduction to Kant's "Logik."

Whether deductive reasoning can with propriety be regarded as consisting only of elimination. I believe, upon careful examination, the true answer to the former question to be, that it is always theoretically possible so to resolve and combine propositions that elimination may subsequently be effected by the syllogistic canons, but that the process of reduction would in many instances be constrained and unnatural, and would involve operations which are not syllogistic. To the second question I reply, that reasoning cannot, except by an arbitrary restriction of its meaning, be confined to the process of elimination. No definition can suffice which makes it less than the aggregate of the methods which are founded upon the laws of thought, as exercised upon propositions; and among those methods, the process of elimination, eminently important as it is, occupies only a place.

Much of the error, as I cannot but regard it, which prevails respecting the nature of the Syllogism and the extent of its office, seems to be founded in a disposition to regard all those truths in Logic as *primary* which possess the character of simplicity and intuitive certainty, without inquiring into the relation which they sustain to other truths in the Science, or to general methods in the Art, of Reasoning. Aristotle's *dictum de omni et nullo* is a self-evident principle, but it is not found among those *ultimate* laws of the reasoning faculty to which all other laws, however plain and self-evident, admit of being traced, and from which they may in strictest order of scientific evolution be deduced. For though of every science the fundamental truths are usually the most simple of apprehension, yet is not that simplicity the *criterion* by which their title to be regarded as fundamental must be judged. This must be sought for in the nature and extent of the structure which they are capable of supporting. Taking this view, Leibnitz appears to me to have judged correctly when he assigned to the "principle of contradiction" a fundamental place in Logic;[4] for we have seen the consequences of that law of thought of which it is the axiomatic expression (III. 15). But enough has been said upon the nature of deductive inference and upon its constitutive elements. The subject of

[4]Nouveaux Essais sur l'entendement humain. Liv. IV. cap. 2. Theodicée Pt. I. sec. 44.

induction may probably receive some attention in another part of this work.

10. It has been remarked in this chapter that the ordinary treatment of hypothetical, is much more defective than that of categorical, propositions. What is commonly termed the hypothetical syllogism appears, indeed, to be no syllogism at all.

Let the argument—

If A is B, C is D,
But A is B,
Therefore C is D,

be put in the form—

If the proposition X is true, Y is true,
But X is true,
Therefore Y is true;

wherein by X is meant the proposition A is B, and by Y, the proposition C is D. It is then seen that the premises contain only two terms or elements, while a syllogism essentially involves three. The following would be a genuine hypothetical syllogism:

If X is true, Y is true;
If Y is true, Z is true;
$\therefore$ If X is true, Z is true.

After the discussion of secondary propositions in a former part of this work, it is evident that the forms of hypothetical syllogism must present, in every respect, an exact counterpart to those of categorical syllogism. *Particular* Propositions, such as, "Sometimes if X is true, Y is true," may be introduced, and the conditions and rules of inference deduced in this chapter for categorical syllogisms may, without abatement, be interpreted to meet the corresponding cases in hypotheticals.

The use of an indefinite v when working with secondary propositions is different from when working with categorical propositions. See page 170. This means that conversion by limitation does not hold for secondary propositions?

11. To what final conclusions are we then led respecting the nature and extent of the scholastic logic? I think to the following: that it is not a science, but a collection of scientific truths, too incomplete to form a system of themselves, and not sufficiently fundamental to serve as the foundation upon which a perfect system may rest. It does not, however, follow, that because the logic of the schools has been invested with attributes to which it has no just claim, it is therefore undeserving of regard. A system which has been associated with the very growth of language, which has left its stamp upon the greatest questions and the most famous demonstrations of philosophy, cannot be altogether unworthy of attention. Memory, too, and usage, it must be admitted, have much to do with the intellectual processes; and there are certain of the canons of the ancient logic which have become almost inwoven in the very texture of thought in cultured minds. But whether the mnemonic forms, in which the particular rules of conversion and syllogism have been exhibited, possess any real utility, —whether the very skill which they are supposed to impart might not, with greater advantage to the mental powers, be acquired by the unassisted efforts of a mind left to its own resources,—are questions which it might still be not unprofitable to examine. As concerns the particular results deduced in this chapter, it is to be observed, that they are solely designed to aid the inquiry concerning the nature of the ordinary or scholastic logic, and its relation to a more perfect theory of deductive reasoning.