LECTURE SLIDES

on

LOGIC

for

MATHEMATICS

and

COMPUTER SCIENCE

This set of lecture slides is a companion to the textbook

Logic for Mathematics and Computer Science

by Stanley Burris, Prentice Hall, 1998.

At the top of each slide one sees LMCS, referring to the textbook, usually with a page number to indicate the page of the text that (more or less) corresponds to the slide.

ARISTOTLE (4th Century B.C.)

Invented Logic

All men are mortal.
Socrates is a man.
$\therefore$ Socrates is mortal.

Some students are clever.
Some clever people are lazy.
$\therefore$ Some students are lazy.

The **four kinds of statements** permitted in the categorical syllogisms of Aristotle.

A	All S is P.	universal affirmative
E	No S is P.	universal negative
I	Some S is P.	particular affirmative
O	Some S is not P.	particular negative

Mnemonic Device:

A ff **I** rmo   n **E** g **O**

Syllogisms

Syllogisms are 3 line arguments:

Major Premiss	— $\square$ — $\square$	(Use P and M)
Minor Premiss	— $\square$ — $\square$	(Use S and M)
Conclusion	— S — P	

Actually you can write the premisses in any order.

The **major premiss** is the one with the **predicate of the conclusion**.

The **minor premiss** is the one with the **subject of the conclusion**.

Now there are $2 \times 2 \times 2 \times 1 = 8$ possibilities for the major premiss $— \square — \square$

and likewise 8 possibilities for the minor premiss $— \square — \square$

but just $2 \times 2 = 4$ possibilities for the conclusion $— S — P$

So there are **256 different syllogisms**.

A main goal of Aristotelian logic was to determine the valid categorical syllogisms.

Classification of Syllogisms

The **mood** XYZ of a syllogism is the AEIO classification of the three statments in a syllogism, where the first letter X refers to the major premiss, etc.

For example the syllogism

All students are clever.
No clever people are lazy.
∴ No students are lazy.

has the mood **EAE**.

There are **64 distinct moods**.

The **figure** of a syllogism refers to whether or not the middle term M comes first or second in each of the premisses.

The **four figures** for syllogisms:

1st Figure

— *M* — *P*
— *S* — *M*

— *S* — *P*

2nd Figure

— *P* — *M*
— *S* — *M*

— *S* — *P*

3rd Figure

— *M* — *P*
— *M* — *S*

— *S* — *P*

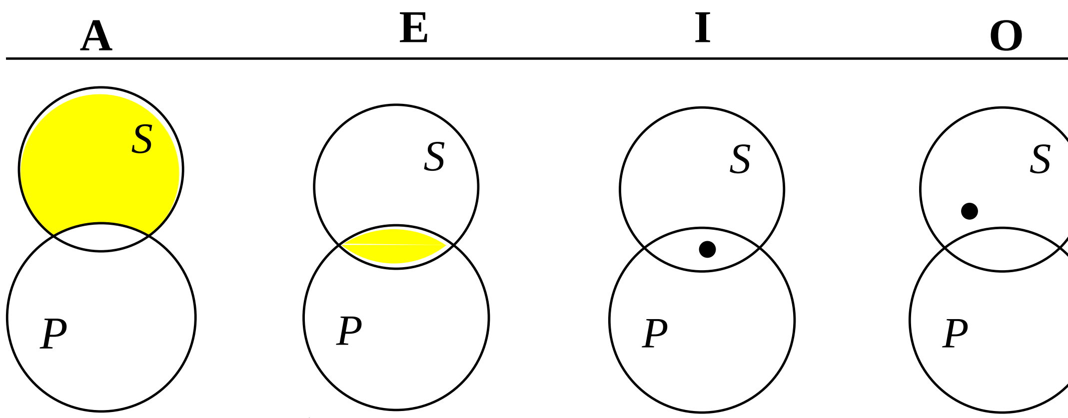
4th Figure

— *P* — *M*
— *M* — *S*

— *S* — *P*

Venn Diagrams for A, E, I, O statements:

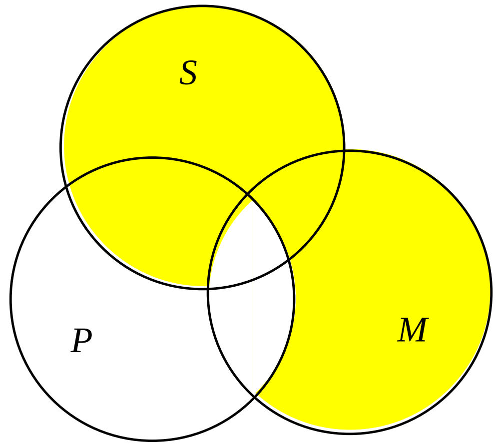
SHADED regions have NO ELEMENTS in them.



[Note: the shading for the Venn diagram for A is not correct in the textbook — this mistake occurred when, shortly before going to press, all the figures in the text needed to be redrawn with heavier lines. For a few other items that need to be changed see the Errata sheet on the web site. – S.B.]

The **first figure AAI** syllogism:

All M is P.
All S is M.
$\therefore$ Some S is P.



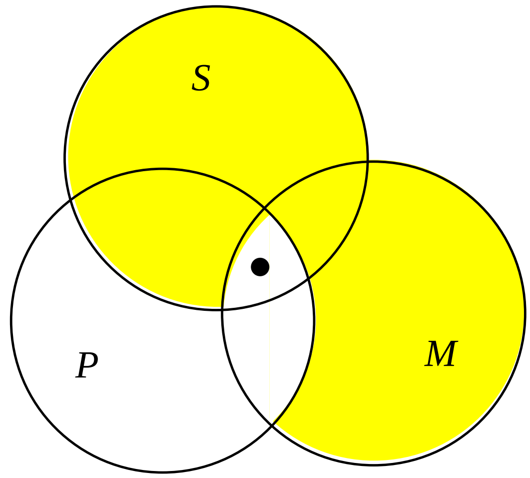
This is not a valid syllogism by **modern standards**, for consider the example:

All animals are mobile.
Unicorns are animals.
$\therefore$ Some unicorns are mobile.

[In this case **modern** means subsequent to C.S. Peirce's paper of 1880 called "The Algebra of Logic".]

But by **Aristotle's standards** the first figure AAI syllogism is valid:

All M is P.
All S is M.
$\therefore$ Some S is P.



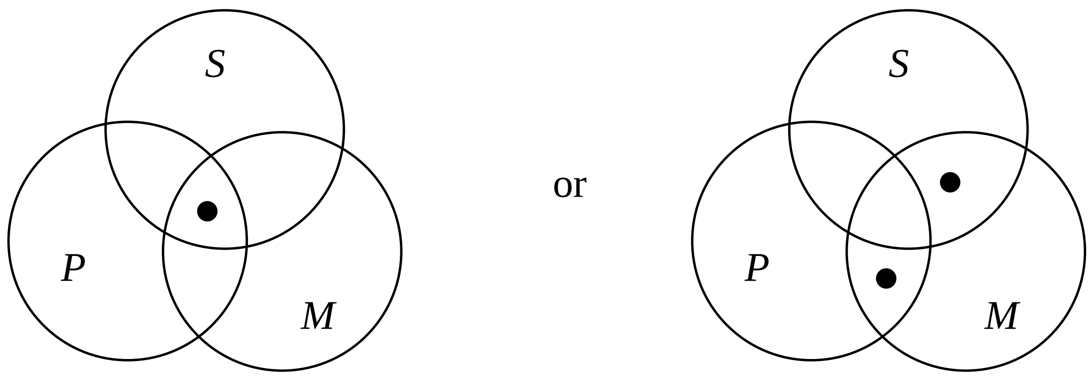
The previous example about unicorns would not be considered by Aristotle.

After all, why argue about something that doesn't even exist.

Third figure III syllogism:

Some M is P.
Some M is S.
$\therefore$ Some S is P.

There are two situations to consider:



The second diagram gives a **counterexample**. This is not a valid syllogism. To be a valid syllogism the conclusion must be true in all cases that make the premisses true.

The Valid Syllogisms

Major premiss	⟶	A	A	A	A	E	E	E	E
Minor premiss	⟶	A	E	I	O	A	E	I	O
	Conclusion ↓								
First figure	A	•							
	E					•			
	I	□		•					
	O					□		•	
Second figure	A								
	E		•			•			
	I								
	O		□		•	□		•	
Third figure	A								
	E								
	I	□		•					
	O					□		•	
Fourth figure	A								
	E		•						
	I	□							
	O		□			□		•	

etc.

□ means we assume the classes S, P, M are not empty.

George Boole (1815 – 1864)

Boole's Key Idea: **Use Equations**

For the **universal** statements:

The statement	*becomes the equation*
All S is P.	$S \cap P' = 0$ or just $SP' = 0$.
No S is P.	$S \cap P = 0$ or just $SP = 0$.

Boole also had equations for the **particular** statements. But by the end of the 1800s they were considered a bad idea.

Example

The first figure AAA syllogism

All M is P.

All S is M.

$\therefore$ All S is P.

becomes the equational argument

$$\begin{aligned} MP' &= 0 \\ SM' &= 0 \\ \therefore SP' &= 0. \end{aligned}$$

We see that the equational argument (about classes)

$$MP' = 0, \quad SM' = 0 \qquad \therefore SP' = 0$$

is correct as

$$\begin{aligned} SP' &= S1P' \\ &= S(M \cup M')P' \\ &= SMP' \cup SM'P' \\ &= 0 \cup 0 \\ &= 0. \end{aligned}$$

For equational arguments you can use the fundamental identities.

Fundamental Identities

for the Calculus of Classes

1.	$X \cup X$	$=$	X	idempotent
2.	$X \cap X$	$=$	X	idempotent
3.	$X \cup Y$	$=$	$Y \cup X$	commutative
4.	$X \cap Y$	$=$	$Y \cap X$	commutative
5.	$X \cup (Y \cup Z)$	$=$	$(X \cup Y) \cup Z$	associative
6.	$X \cap (Y \cap Z)$	$=$	$(X \cap Y) \cap Z$	associative
7.	$X \cap (X \cup Y)$	$=$	X	absorption
8.	$X \cup (X \cap Y)$	$=$	X	absorption

9.	$X \cap (Y \cup Z)$	$=$	$(X \cap Y) \cup (X \cap Z)$	distributive
10.	$X \cup (Y \cap Z)$	$=$	$(X \cup Y) \cap (X \cup Z)$	distributive
11.	$X \cup X'$	$=$	1	
12.	$X \cap X'$	$=$	0	
13.	X''	$=$	X	
14.	$X \cup 1$	$=$	1	
15.	$X \cap 1$	$=$	X	
16.	$X \cup 0$	$=$	X	
17.	$X \cap 0$	$=$	0	
18.	$(X \cup Y)'$	$=$	$X' \cap Y'$	De Morgan's law
19.	$(X \cap Y)'$	$=$	$X' \cup Y'$	De Morgan's law.

Boole applied the algebra of equations to arguments with **many premisses**, and **many variables**, leading to:

- Many Equations with Many Variables

$$
\begin{aligned}
F_1(A_1, \ldots, A_m, B_1, \ldots, B_n) &= 0 \\
&\vdots \\
F_k(A_1, \ldots, A_m, B_1, \ldots, B_n) &= 0 \\
\therefore F(B_1, \ldots, B_n) &= 0.
\end{aligned}
$$

Boole's work marks the end of the focus on Aristotle's syllogisms, and the beginning of Mathematical Logic.

Chapter 1 of LMCS gives four different methods for analyzing such equational arguments:

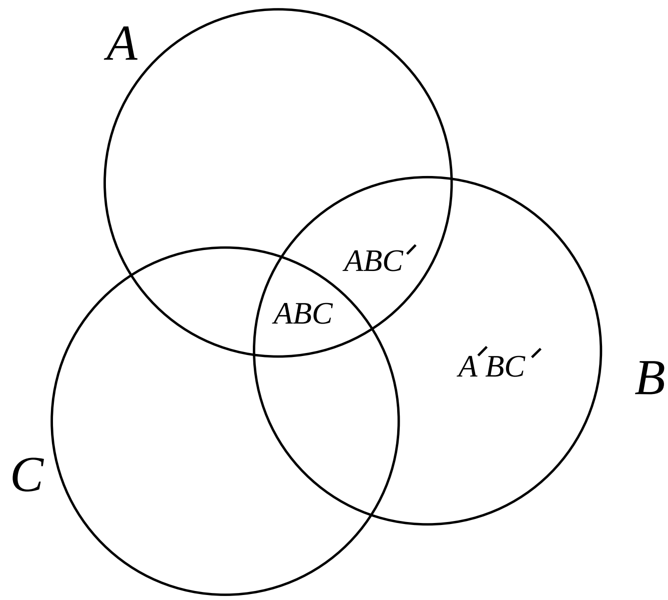
- Fundamental Identities
  for algebraic manipulations

- Venn Diagrams

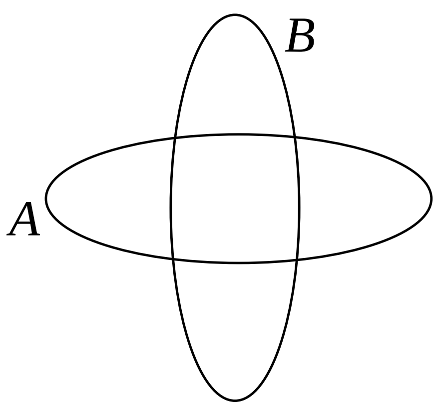
- The Elimination Method of Boole

- The Tree Method of Lewis Carroll

Venn Diagrams

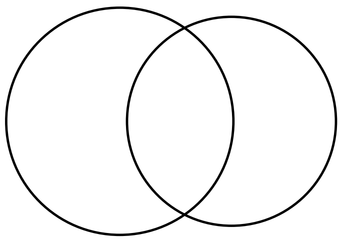


subdivide the plane into connected **constituents**.

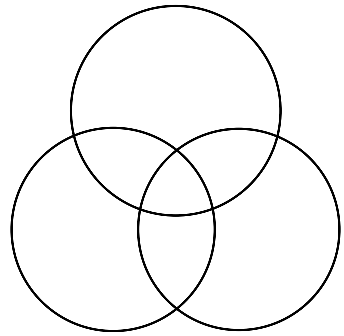


is not a Venn diagram.

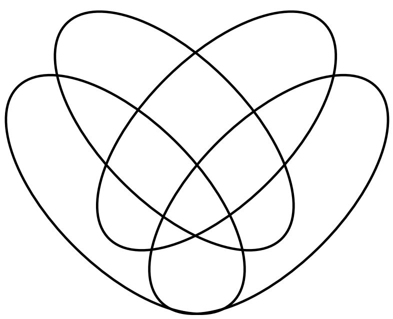
Venn's Venn Diagrams



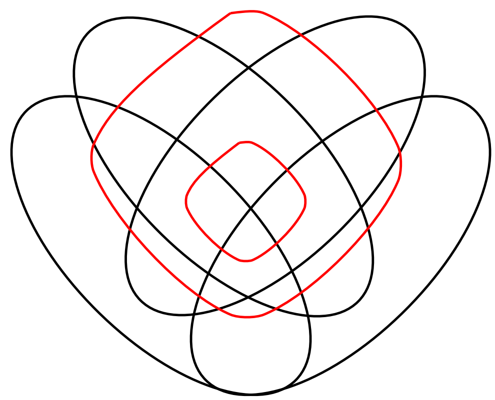
Two Classes



Three Classes

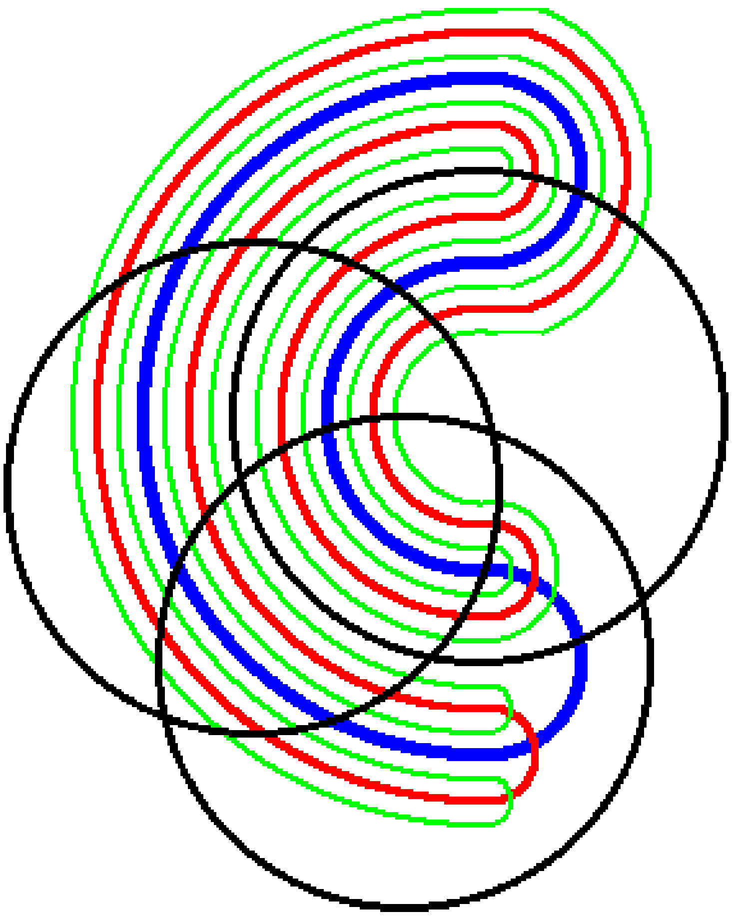


Four Classes



Five Classes

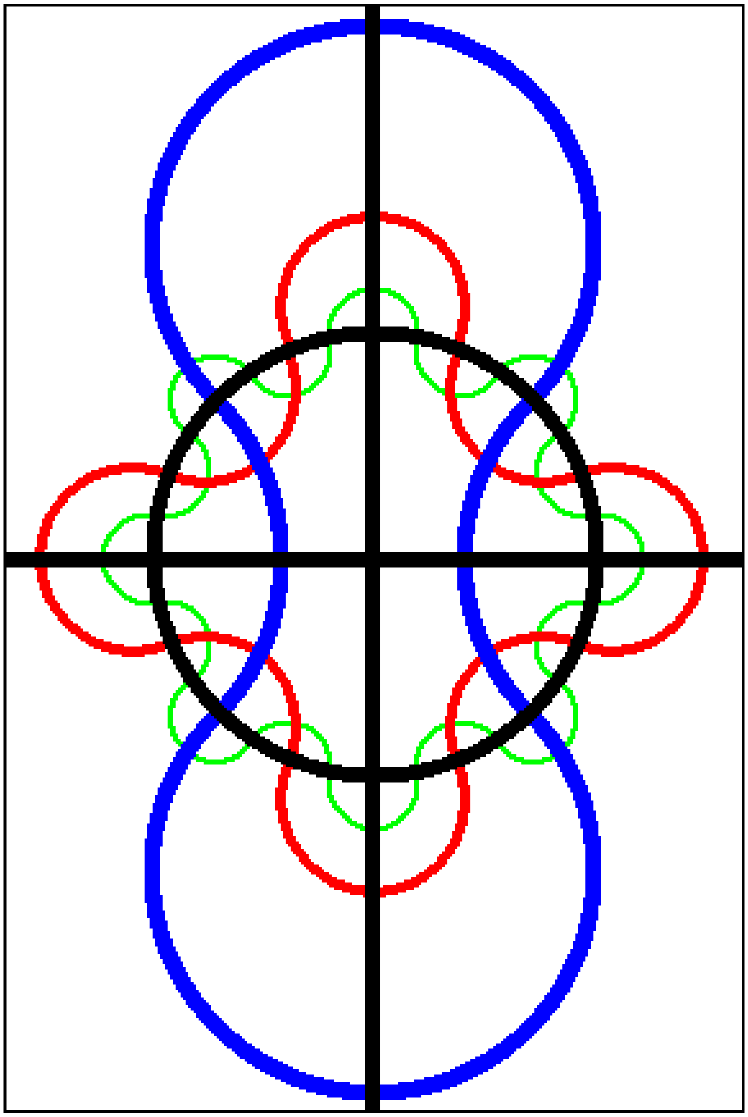
Venn's Construction for 6 Regions*



Draw the three circles first, then add: (4) the blue region, (5) the red region, and finally (6) the green region. (This can be continued for any number of regions.)

*This diagram is courtesy of Frank Ruskey from his *Survey of Venn Diagrams*: www.combinatorics.org/Surveys/ds5/VennEJC.html

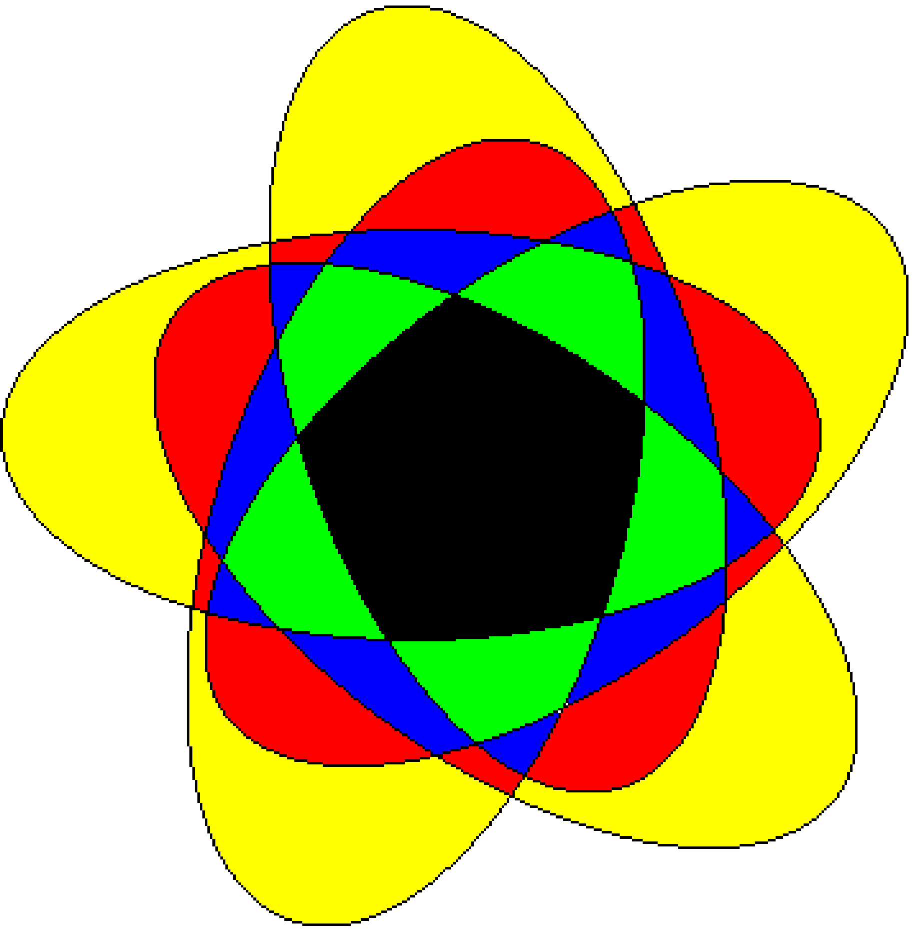
Edward's Construction for 6 Regions*



Draw the perpendicular lines and the circle first. Then follow the circle with: (4) the blue region, (5) the red region, and (6) the green region. Join the endpoints of the perpendicular lines to make closed regions.

*This diagram is courtesy of Frank Ruskey from his *Survey of Venn Diagrams*: www.combinatorics.org/Surveys/ds5/VennEJC.html

A Symmetric Venn Diagram*



Venn diagrams with n regions that admit a symmetry of rotation by $2\pi/n$ are **symmetric**. This can hold only if the regions are congruent and n is prime. Such are known for $n = 2, 3, 5, 7$, but not for $n \geq 11$.

*This diagram, using 5 congruent ellipses, is courtesy of Frank Ruskey from his *Survey of Venn Diagrams*: www.combinatorics.org/Surveys/ds5/VennEJC.html

Simplification of the Premisses

(Useful before shading a Venn diagram.)

Write each premiss as a union of intersections of classes or their complements.

Then put each of the intersections equal to 0.

Example

Express the premiss $A(B'C)' = 0$ as

$$AB \cup AC' = 0$$

and then break this up into:

$$AB = 0 \quad \text{and} \quad AC' = 0.$$

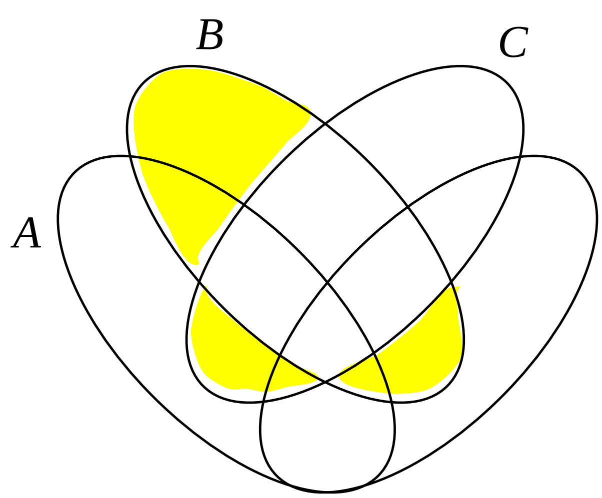
Example

Given $(AC \cup B)(AB' \cup C') = 0$,

for the Venn diagram first simplify this to

$$AB'C = 0 \quad \text{and} \quad BC' = 0$$

Now proceed to shade the intersections $AB'C$ and BC' :



Two methods for such simplification:

- **Use Fundamental Identities**

(We have already discussed this.)

- **Boole's Expansion Theorem**

For two variables A, B this looks like:

$$F(A,B) = F(1,1)AB \cup F(1,0)AB' \cup F(0,1)A'B \cup F(0,0)A'B'$$

or just expanding on A gives

$$F(A,B) = F(1,B)A \cup F(0,B)A'$$

Example

For $F(A,B) = (A' \cap B)'$

$$\begin{aligned} F(1,1) &= (1' \cap 1)' = 1 \\ F(1,0) &= (1' \cap 0)' = 1 \\ F(0,1) &= (0' \cap 1)' = 0 \\ F(0,0) &= (0' \cap 0)' = 1 \end{aligned}$$

Thus

$$F(A,B) = AB \cup AB' \cup A'B'.$$

Reducing the Number of Premiss Equations to One

One can replace the premiss equations

$$F_1 = 0$$
$$\vdots$$
$$F_k = 0$$

by the single equation

$$F_1 \cup \ldots \cup F_k = 0.$$

This follows from the fact that $A \cup B = 0$ holds iff $A = 0$ and $B = 0$ hold.

Example

The two premisses

$$A(B'C)' = 0$$
$$(A \cup B)C' = 0$$

become

$$\Big(A(B'C)'\Big) \cup \Big((A \cup B)C'\Big) = 0.$$

Boole's Main Result
The Elimination Theorem

Given the single premiss

$$E(A_1, \ldots, A_m, B_1, \ldots, B_n) \ = \ 0$$

what is the most general conclusion

$$F(B_1, \ldots, B_n) \ = \ 0$$

involving only the classes $B_1, \ldots, B_n$ **?**

Answer: F is the intersection of instances of E obtained by putting 0s and 1s in for the A_i, in all possible ways. So F is:

$$E(0, \ldots, 0, B_1, \ldots, B_n) \cdots E(1, \ldots, 1, B_1, \ldots, B_n)$$

Example

Find the most general conclusion involving only P and S that follows from

$$\frac{PQ' = 0 \qquad QR' = 0 \qquad RS' = 0}{}$$

First collapse the premisses into a single premiss $E = 0$ by setting

$$E(P,Q,R,S) = PQ' \cup QR' \cup RS'.$$

The most general conclusion for P and S is

$$E(P,0,0,S)\,E(P,0,1,S)\,E(P,1,0,S)\,E(P,1,1,S) = 0.$$

This is $P(P \cup S')1S' = 0$, and simplifies to $PS' = 0$.

Lewis Carroll's TREE METHOD

Showing $F = 0$ reduces to showing

$$FX = 0 \quad \text{and} \quad FX' = 0$$

since

$$F = FX \cup FX'.$$

To show a conclusion $F = 0$ is valid simply build an **(upside down) tree**

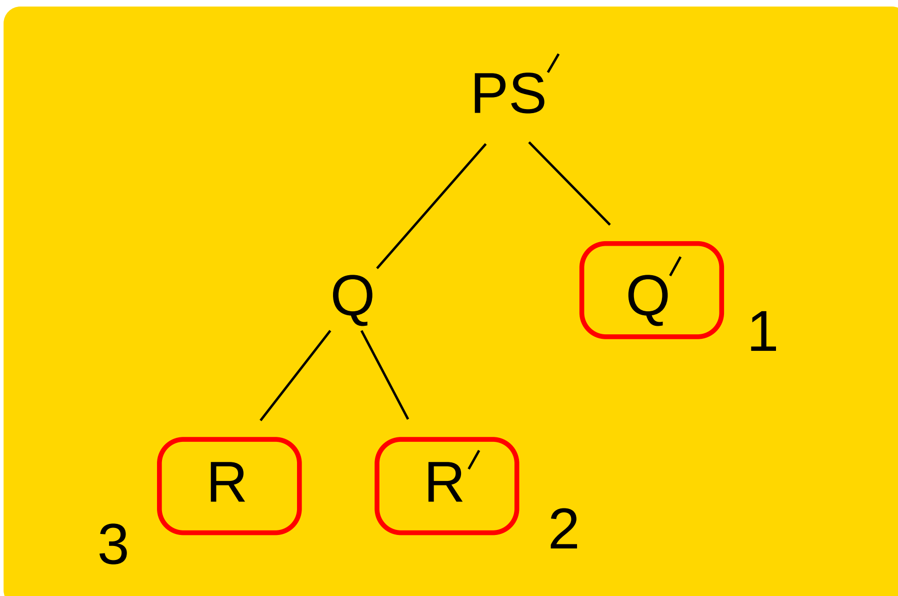
starting with the conclusion
with each branch multiplying out to 0.

Example

To show that

$$\begin{aligned} 1.\ PQ' &= 0 \\ 2.\ QR' &= 0 \\ 3.\ RS' &= 0 \\ \therefore PS' &= 0 \end{aligned}$$

is valid:



Translating the lengthy argument in Example 1.3.4 into equations:

1. Good-natured tenured mathematics professors are dynamic.

$$ABC \subseteq D \quad \text{or} \quad ABCD' = 0.$$

2. Grumpy student advisors play slot machines.

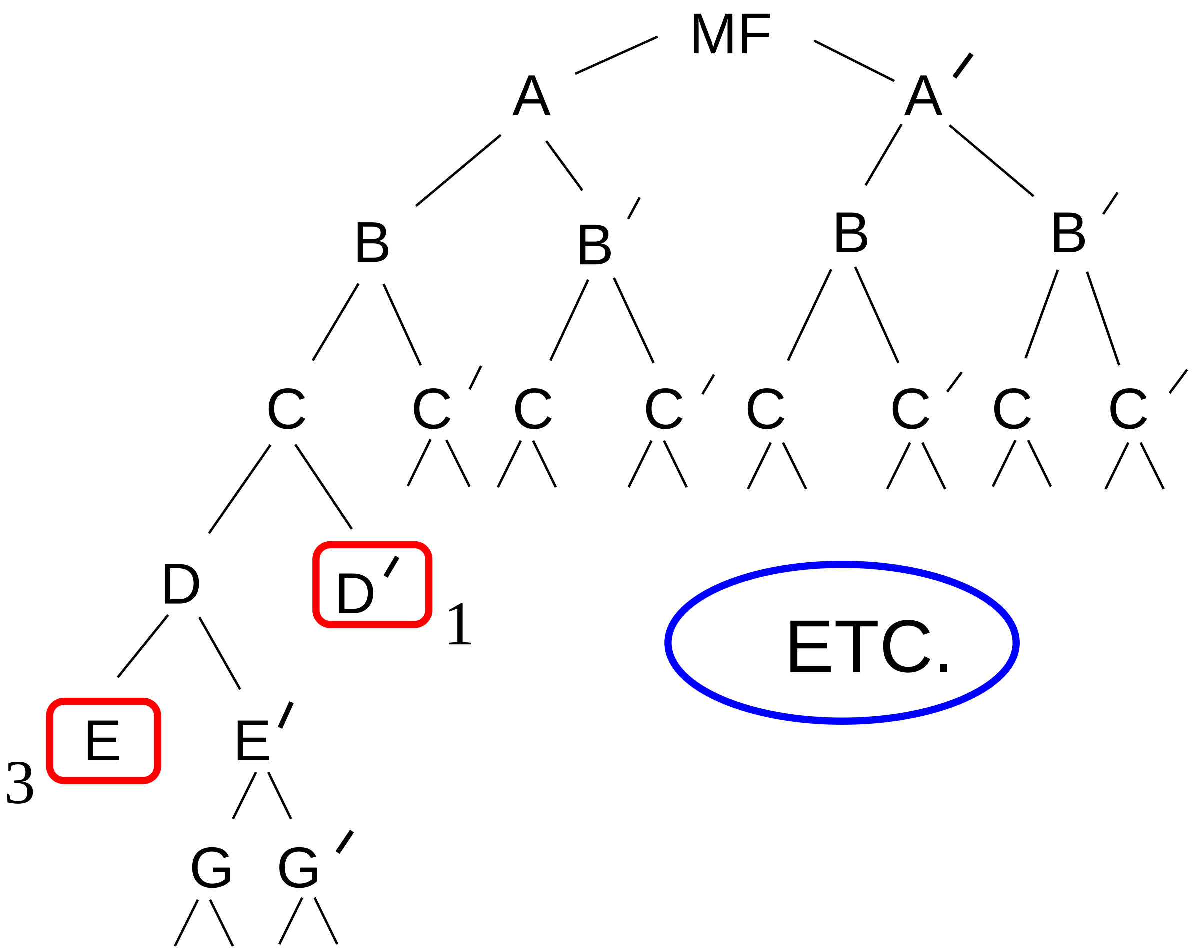
$$A'M \subseteq L \quad \text{or} \quad A'ML' = 0.$$

Etc.

A Naive Approach to

1. $ABCD' = 0$	2. $A'ML' = 0$	3. $FED = 0$
4. $GMC' = 0$	5. $B'FH = 0$	6. $D'BEG' = 0$
7. $MI'J' = 0$	8. $HMK' = 0$	9. $KJL'E' = 0$
10. $H'FL' = 0$	11. $MLF = 0$	12. $KIAE' = 0$

$\therefore MF = 0$



A Smart Approach

1.	$ABCD' = 0$	2.	$A'ML' = 0$	3.	$FED = 0$
4.	$GMC' = 0$	5.	$B'FH = 0$	6.	$D'BEG' = 0$
7.	$MI'J' = 0$	8.	$HMK' = 0$	9.	$KJL'E' = 0$
10.	$H'FL' = 0$	11.	$MLF = 0$	12.	$KIAE' = 0$

$\therefore MF = 0$