HAILPERIN'S RESCUE OF BOOLE'S ALGEBRA OF LOGIC

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Abstract. Boole's numerical algebra approach to creating an Algebra of Logic was mysterious, in particular because of its use of uninterpretable terms and an undefined division. Nonetheless his system gave correct results. Just a decade after the appearance in 1854 of Boole's *Laws of Thought*, Jevons published an alternate approach that was basically modern Boolean Algebra. Boole's approach was abandoned. It was not until 1976 that Boole's numerical algebra approach was given an acceptable modern foundation, namely Hailperin embedded Boole's system in a certain first-order logic of rings with variables that range over idempotent elements. The goal of this note is to give a compact presentation of Hailperin's results using the familiar 1-sorted first-order logic.

1. First-order ring axioms for Boole's algebra of logic

A *non-trivial torsion-free commutative ring with unity* is an algebraic structure $\mathbf{R} = \langle R, +, \cdot, -, 0, 1\rangle$ with two binary operations *addition* $(+)$ and *multiplication* $(\cdot)$, a unary operation *minus* $(-)$, and two constants $0, 1$ such that the following collection $\mathcal{H}$ of axioms hold:[1]

$$1 \neq 0$$

$$(\forall x)\,[\,nx = 0 \rightarrow x = 0\,] \quad \text{for } n = 1, 2, \ldots$$

$$\begin{array}{ll}
(\forall x, y)\,[\,x + y = y + x\,] & (\forall x, y)\,[\,x \cdot y = y \cdot x\,] \\
(\forall x, y, z)\,[\,x + (y + z) = (x + y) + z\,] & (\forall x, y, z)\,[\,x \cdot (y \cdot z) = (x \cdot y) \cdot z\,] \\
(\forall x)\,[\,x + 0 = x\,] & (\forall x)\,[\,x \cdot 1 = x\,] \\
(\forall x)\,[\,x + (-x) = 0\,] &
\end{array}$$

$$(\forall x, y, z)\,[\,x \cdot (y + z) = x \cdot y + x \cdot z\,]$$

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[1] $\mathcal{H}$ is to remind us that these axioms are only a modification of those presented by Theodore Hailperin [7].

Models of $\mathcal{H}$ will be called $\mathcal{H}$-rings, the best known example being of course the *ring of integers* $\mathbf{Z}$. The torsion-free property ensures that every $\mathcal{H}$-ring has a copy of $\mathbf{Z}$ in it, namely the subring generated by 1.

2. Idempotent elements of the $\mathcal{H}$-rings $\mathbf{Z}^U$ and Boole's partial algebra models

The $\mathcal{H}$-rings $\mathbf{Z}^U$ of functions[2] from U to Z, for $U \neq \emptyset$, play a leading role in Hailperin's justification of Boole's work. This is for the following reason: the idempotent elements of $\mathbf{Z}^U$ are precisely the characteristic functions[3] χ_A for $A \subseteq U$, and for $A, B \subseteq U$ they satisfy

- $\chi_A \cdot \chi_B = \chi_{A\cap B}$
- $\chi_A + \chi_B = \chi_{A\cup B}$ provided $A \cap B = \emptyset$; otherwise $\chi_A + \chi_B$ is not idempotent
- $\chi_A - \chi_B = \chi_{A\smallsetminus B}$ provided $B \subseteq A$; otherwise $\chi_A - \chi_B$ is not idempotent.

This corresponds precisely to Boole's model for a given universe U, namely for $A, B \subseteq U$

- $A \cdot B = A \cap B$
- $A + B = A \cup B$ provided $A \cap B = \emptyset$; otherwise $A + B$ is not interpretable
- $A - B = A \smallsetminus B$ provided $B \subseteq A$; otherwise $A - B$ is not interpretable.

3. Relativizing quantifiers to idempotent elements

For φ a first-order formula φ_E means *the quantifiers are relativized to idempotent elements*, that is, φ_E is defined recursively as follows:

- $\varphi_E = \varphi$ if φ is an atomic formula;
- $[\neg\varphi]_E = \neg[\varphi_E]$;
- $\big(\varphi_1 \square \varphi_2\big)_E = (\varphi_1)_E \square (\varphi_2)_E$ for $\square$ a binary propositional connective;
- $[(\forall x)\varphi]_E = (\forall x)[\,(x^2 = x) \rightarrow \varphi_E]$;
- $[(\exists x)\varphi]_E = (\exists x)[\,(x^2 = x) \,\&\, \varphi_E]$.

[2] Halperin called these functions *signed multisets*.

[3] In 1933 Hassler Whitney [13] translated the modern Boolean algebra of sets into the algebra of numbers using characteristic functions.

Unlike Boole's system, $\mathcal{H}$ has no axiom stating that variables are idempotent.[4] Instead, Boole's results, when formulated for $\mathcal{H}$-rings, take the form

$$\mathcal{H} \vdash \psi_E$$

for ψ a first-order sentence. For the case that such a ψ has the form $(\forall\vec{x})\varphi(\vec{x})$ the assertion $\mathcal{H} \vdash \psi_E$ takes the form

$$\mathcal{H} \vdash (\forall\vec{x})[I(\vec{x}) \rightarrow \varphi(\vec{x})_E],$$

or equivalently,

$$\mathcal{H}, I(\vec{x}) \vdash \varphi(\vec{x})_E,$$

where

- $\vec{x}$ is $x_1, \ldots, x_n$,
- $(\forall\vec{x})$ means $(\forall x_1)\cdots(\forall x_n)$,
- $I(\vec{x})$ is $(x_1^2 = x_1)\ \&\ \cdots\ \&\ (x_n^2 = x_n)$.

When dealing with a quantifier-free formula $\omega(\vec{x})$, for example as in equational logic, note that $\omega(\vec{x})_E$ is just $\omega(\vec{x})$, and $\mathcal{H}, I(\vec{x}) \vdash \omega(\vec{x})$ can be viewed as using Boole's axiom that variables are idempotent.

4. Boole's main results

Items (1) through (12) below enumerate the main definitions and results from Boole's Algebra of Logic, expressed in the first-order logic of $\mathcal{H}$-rings. Item (13) is a strong version of Boole's Rule of 0 and 1.

(1) **Definition** A term $t(\vec{x})$ is *E-idempotent* if

$$\mathcal{H}, I(\vec{x}) \vdash\ t(\vec{x})^2 = t(\vec{x}).$$

Simple examples of E-idempotent terms (written as polynomials) are

$$0,\ 1,\ x,\ 1-x,\ xy,\ x+y-xy,\ x+y-2xy.$$

$x+y$ and $x-y$ are not E-idempotent.

[4] Boole did not state that his law $x^2 = x$ only applied to variables; this only becomes evident when Boole points out that certain terms are not idempotent.

(2) **Definition** The *constituents* of a list $\vec{x}$ of variables are the terms $C_\sigma(\vec{x})$, where σ is a sequence of 0s and 1s of the same length as $\vec{x}$, defined by

$$C_\sigma(\vec{x}) \coloneqq \prod_i C_{\sigma_i}(x_i)$$

where $C_1(x) \coloneqq x$ and $C_0(x) \coloneqq 1 - x$.
For example, $C_{1101}(x_1, x_2, x_3, x_4)$ is $x_1 x_2(1 - x_3)x_4$.

(3) **Constituents are E-idempotent**

$$\mathcal{H}, I(\vec{x}) \vdash\ C_\sigma(\vec{x})^2 = C_\sigma(\vec{x}).$$

(4) **Constituents are E-pairwise disjoint**

$$\mathcal{H}, I(\vec{x}) \vdash\ C_\sigma(\vec{x}) \cdot C_\tau(\vec{x}) = 0 \quad \text{for } \sigma \neq \tau.$$

(5) **Constituents E-sum to 1**

$$\mathcal{H}, I(\vec{x}) \vdash\ \sum_\sigma C_\sigma(\vec{x}) = 1.$$

(6) **Value of $C_\sigma(\tau)$**

$$\mathcal{H} \vdash\ C_\sigma(\sigma) = 1, \qquad \mathcal{H} \vdash\ C_\sigma(\tau) = 0 \quad \text{if} \quad \sigma \neq \tau.$$

(7) **Expansion** (or **Development**) **Theorem**

$$\mathcal{H}, I(\vec{x}, \vec{y}) \vdash\ t(\vec{x}, \vec{y}) = \sum_\sigma t(\sigma, \vec{y}) \cdot C_\sigma(\vec{x}).$$

The *complete expansion* of a term $t(\vec{x})$ is given by

$$\mathcal{H}, I(\vec{x}) \vdash\ t(\vec{x}) = \sum_\sigma t(\sigma) \cdot C_\sigma(\vec{x}).$$

In this case the coefficients $t(\sigma)$ are integers.

(8) **Definition** For $t(\vec{x})$ a term let

$$t^\star(\vec{x}) \coloneqq \sum_{\sigma\ :\ t(\sigma) \neq 0} C_\sigma(\vec{x}).$$

(9) **Properties of $t^\star(\vec{x})$**
- $t^\star(\vec{x})$ is E-idempotent.
- If $t(\vec{x})$ is E-idempotent then $\mathcal{H}, I(\vec{x}) \vdash\ t^\star(\vec{x}) = t(\vec{x})$.
- $\mathcal{H}, I(\vec{x}) \vdash\ t(\vec{x}) = 0 \ \leftrightarrow\ t^\star(\vec{x}) = 0$.
- $\mathcal{H}, I(\vec{x}) \vdash\ [s(\vec{x}) \cdot t(\vec{x})]^\star \ = \ s(\vec{x})^\star \cdot t(\vec{x})^\star$.

(10) **Reduction Theorem**

$$\mathcal{H}, I(\vec{x}) \vdash \bigwedge_i \big[\, t_i(\vec{x}) = 0 \,\big] \;\leftrightarrow\; \sum_i t_i(\vec{x})^2 = 0.$$

(11) **Elimination Theorem**

$$\mathcal{H}, I(\vec{x}) \vdash (\exists \vec{y}) \big[\, I(\vec{y}) \,\&\, t(\vec{x}, \vec{y}) = 0 \,\big] \;\leftrightarrow\; \prod_\tau t(\vec{x}, \tau) = 0.$$

If $\vec{y}$ is a single variable y then one has

$$\mathcal{H}, I(\vec{x}) \vdash (\exists y) \big[\, I(y) \,\&\, t(\vec{x}, y) = 0 \,\big] \;\leftrightarrow\; t(\vec{x}, 1) \cdot t(\vec{x}, 0) = 0.$$

(12) **Solution Theorem**

$$\begin{aligned}\mathcal{H}, I(\vec{x}, y) \vdash \; t(\vec{x}, y) = 0 \;\leftrightarrow\; \Big(& t(\vec{x}, 1) \cdot t(\vec{x}, 0) = 0 \\ & \&\; (\exists v) \big[\, I(v) \,\&\, y = t^\star(\vec{x}, 0) \;+\; v \cdot \overline{t^\star(\vec{x}, 0)} \cdot \overline{t^\star(\vec{x}, 1)} \;\big] \Big).\end{aligned}$$

where $\overline{t^\star(\vec{x}, i)}$ is $1 - t^\star(\vec{x}, i)$ for $i = 0, 1$.

Applying the Expansion Theorem to the terms in the right side of the iff symbol in the Solution Theorem gives a version of the Solution Theorem used by Boole, namely in terms of constituents:

$$\begin{aligned}\mathcal{H}, I(\vec{x}, y) \vdash \; t(\vec{x}, y) = 0 \;\leftrightarrow\; \Big[& \sum_{\sigma \in J_{\mathrm{IND}}} C_\sigma(\vec{x}) = 0 \\ & \&\quad (\exists v) \Big(I(v) \;\&\; y = \sum_{\sigma \in J_1} C_\sigma(\vec{x}) + v \cdot \sum_{\sigma \in J_{0/0}} C_\sigma(\vec{x}) \Big) \Big],\end{aligned}$$

where

$$\begin{aligned} J_{\mathrm{IND}} &\coloneqq \big\{\sigma : t^\star(\sigma, 1) = t^\star(\sigma, 0) = 1\big\} &&= \big\{\sigma : t(\sigma, 1) \neq 0,\; t(\sigma, 0) \neq 0\big\} \\ J_1 &\coloneqq \big\{\sigma : t^\star(\sigma, 1) = 0,\; t^\star(\sigma, 0) = 1\big\} &&= \big\{\sigma : t(\sigma, 1) = 0,\; t(\sigma, 0) \neq 0\big\} \\ J_{0/0} &\coloneqq \big\{\sigma : t^\star(\sigma, 1) = t^\star(\sigma, 0) = 0\big\} &&= \big\{\sigma : t(\sigma, 1) = 0,\; t(\sigma, 0) = 0\big\}. \end{aligned}$$

(13) **Rule of 0 and 1** (See [6])

For φ a Horn sentence one has

$$\mathcal{H} \vdash \varphi_E \quad \text{iff} \quad \mathbf{Z} \models \varphi_E.$$

5. Observations and useful facts

5.1. **Regarding the axioms $\mathcal{H}$.** The only axioms from $\mathcal{H}$ that Boole explicitly listed were the commutative and distributive laws—his distributive laws included $x \cdot (y - z) = x \cdot y - x \cdot z$. He also included, in a round-about way, the axioms $x + 0 = x$ and $x \cdot 1 = x$. Of course his most famous axiom was $x^2 = x$, which is not included in $\mathcal{H}$.

Boole used the property

$$nC_\sigma(\vec{x}) = 0 \text{ implies } C_\sigma(\vec{x}) = 0 \quad \text{for } n = 1, 2, \ldots$$

without explicitly saying where it came from—it is a consequence of his Rule of 0 and 1. This is the source of the torsion-free property, a property that I suspect many did not realize applied to Boole's Algebra.

5.2. **Regarding the embedding of Boole's model in $\mathbf{Z}^U$.** The fundamental operation *minus* is a unary operation in rings, but Boole had important reasons for working with the binary *subtraction*, namely to express the complement of A by the totally defined term $1 - A$. It is not possible to express the complement in Boole's model by a totally defined term if one replaces subtraction with the unary minus because $-A$ would only be defined for $A = \emptyset$.

5.3. **Regarding E-idempotents.** Suppose $s(\vec{x})$ and $t(\vec{x})$ are E-idempotents. Then

- $s(\vec{x}) \cdot t(\vec{x})$ is an E-idempotent;
- $s(\vec{x}) + t(\vec{x})$ is an E-idempotent iff $\mathcal{H}, I(\vec{x}) \vdash s(\vec{x}) \cdot t(\vec{x}) = 0$.
- $t(\vec{x}) - s(\vec{x})$ is an E-idempotent iff $\mathcal{H}, I(\vec{x}) \vdash s(\vec{x}) \cdot t(\vec{x}) = s(\vec{x})$.

5.4. **Regarding Properties of $t^\star$.** The third property allowed Boole to claim that every equation was equivalent to a totally interpretable equation.

5.5. **Regarding the Reduction Theorem.** This allowed Boole to reduce a finite collection of equations to a single equation. Thus the Elimination and Solution Theorems could be applied to a collection of equations.

5.6. **Regarding the Solution Theorem.** The first version of the Solution Theorem given here, adapted to modern Boolean Algebra, can be found in Ernst Schröder's 1877 work [11].

5.7. **Regarding the Rule of 0 and 1.** For $\mathbf{R}$ an $\mathcal{H}$-ring let $\mathbf{R}_E$ be the subring of $\mathbf{R}$ generated by the idempotent elements. Of course $\mathbf{Z}_E = \mathbf{Z}$. An easy observation is that for a sentence φ one has $\mathbf{R} \models \varphi_E$ iff $\mathbf{R}_E \models \varphi_E$. Now it is known ([10], [12]) that each $\mathbf{R}_E$ is isomorphic to a bounded Boolean power of $\mathbf{Z}$. Furthermore, bounded Boolean powers preserve Horn sentences.

Thus by the completeness theorem for first-order logic it follows that for a Horn sentence φ, $\mathcal{H} \vdash \varphi_E$ iff $\mathbf{Z} \models \varphi_E$, that is, iff φ holds in $\mathbf{Z}$ when the variables are restricted to the two idempotents 0, 1.

Each result in (3) through (12) can be expressed as $\mathcal{H} \vdash \varphi_E$ where φ is logically equivalent to a Horn sentence, and thus φ_E is also logically equivalent to a Horn sentence. Then to prove $\mathcal{H} \vdash \varphi_E$ it suffices to show that $\mathbf{Z} \models \varphi_E$. (This is the essence of the proof of Boole's results given in [6].)

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