BOOLE'S CHAPTER XV: SYLLOGISM DETAILS

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Abstract. In Boole's famous 1854 book *The Laws of Thought* the mathematical analysis of Aristotelian logic was relegated to Chapter XV, the last chapter before his treatment of probability theory. This chapter is Boole's tour de force to show that he had a uniform method to obtain all valid syllogisms in his version of Aristotelian logic, namely he applied *reduction*, *elimination* and *solution* in that order to equational expressions for the premises. The premises of a syllogism were expressed as a pair of equations in 7 variables, but then all algebraic steps between this and the final expressions for x, vx and $1-x$ were omitted. The somewhat tedious details of those missing steps are given in this note. It is assumed that the reader is familiar with Boole's reduction, elimination and solution theorems.

1. Boole's two pairs of equations

In LT [3] (*The Laws of Thought*), pp. 232–236, in order to justify the simple rules describing valid syllogisms that were stated in his 1848 paper [2], Boole presented three solutions

$$\begin{aligned} x &= f_1(v,v',w,w')z + f_0(v,v',w,w')(1-z)\\ vx &= g_1(v,v',w,w')z + g_0(v,v',w,w')(1-z)\\ 1-x &= h_1(v,v',w,w')z + h_0(v,v',w,w')(1-z) \end{aligned}$$

for each of the two pairs of equations

$$\text{I.}\ \begin{cases} vx = v'y \\ wz = w'y \end{cases} \qquad \text{II.}\ \begin{cases} vx = v'y \\ wz = w'(1-y), \end{cases}$$

after eliminating y.

By independently substituting $1-x$ for x, $1-y$ for y and $1-z$ for z in I and II, carrying the substitutions over to the three solutions, Boole said that one would have the equational expressions for all possible arguments with categorical premises. However each of the three conclusions obtained by solution would not correspond to a categorical proposition unless either the coefficient of z or the coefficient of $1-z$ vanished.

In order for these equational arguments to be applicable to the determination of valid syllogisms he needed, in each of the three solutions in each of the two cases, to find all *permissible* substitutions of 1 for some of the variables v, v', w, w' such that the coefficient of z or the coefficient of $1-z$ would be 0. By a permissible substitution is meant one where not both v, v' nor both w, w' could be 1.

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2. The Details

2.1. Details for case I. Given the two equations

$$vx = v'y \tag{1}$$

$$wz = w'y, \tag{2}$$

reduce the system to a single equation $R(x, y, z, v, v', w, w') = 0$ where

$$R(x, y, z, v, v', w, w') := (vx - v'y)^2 + (wz - w'y)^2. \tag{3}$$

Eliminating y from $R = 0$ yields $E(x, z, v, v', w, w') = 0$ where

$$\begin{aligned} E(x, z, v, v', w, w') &:= R(x, 1, z, v, v', w, w')R(x, 0, z, v, v', w, w') \\ &= \Big((vx - v')^2 + (wz - w')^2\Big)(vx + wz). \end{aligned} \tag{4}$$

The goal is to solve $E = 0$ for x, for $1 - x$ and for vx.

2.2. Solving for *x*. To solve $E = 0$ for x, first express E in the form $E_1x + E_0(1-x)$ by expanding it about x:

$$\begin{aligned} &E(x, z, v, v', w, w') \\ = \; &E(1, z, v, v', w, w')x + E(0, z, v, v', w, w')(1 - x) \\ = \; &\Big(\Big((v - v')^2 + (wz - w')^2\Big)(v + wz)\Big)x + \Big(\Big(v' + (wz - w')^2\Big)wz\Big)(1 - x). \end{aligned}$$

From $E_1x + E_0(1 - x) = 0$ one has $(E_0 - E_1)x = E_0$, and thus one has the solution

$$\begin{aligned} x &= \frac{E(0, z, v, v', w, w')}{E(0, z, v, v', w, w') - E(1, z, v, v', w, w')} \\ &= \frac{\Big(v' + (wz - w')^2\Big)wz}{\Big(v' + (wz - w')^2\Big)wz - \Big((v - v')^2 + (wz - w')^2\Big)(v + wz)}. \end{aligned} \tag{5}$$

Now expand the right side of (5) into a linear combination of constituents using the following table—this is Boole's method of solution.[1]

[1] The coefficient 0/0 represents an *indefinite class*. ∞ is used as the value of any coefficient not in the form $0/b$ or b/b—Boole preferred to use $1/0$ instead of the symbol ∞.

The sum of the constituents with coefficient ∞ set equal to 0 gives the expansion of the constraint equation $E_1 \cdot E_0 = 1$.

	z	v	v'	w	w'	$Coeff$	$Value$	$Constituent$
1.	0	0	0	0	0	0/0	0/0	$(1-z)\cdot(1-v)\cdot(1-v')\cdot(1-w)\cdot(1-w')$
2.	0	0	0	0	1	0/0	0/0	$(1-z)\cdot(1-v)\cdot(1-v')\cdot(1-w)\cdot w'$
3.	0	0	0	1	0	0/0	0/0	$(1-z)\cdot(1-v)\cdot(1-v')\cdot w\cdot(1-w')$
4.	0	0	0	1	1	0/0	0/0	$(1-z)\cdot(1-v)\cdot(1-v')\cdot w\cdot w'$
5.	0	0	1	0	0	0/0	0/0	$(1-z)\cdot(1-v)\cdot v'\cdot(1-w)\cdot(1-w')$
6.	0	0	1	0	1	0/0	0/0	$(1-z)\cdot(1-v)\cdot v'\cdot(1-w)\cdot w'$
7.	0	0	1	1	0	0/0	0/0	$(1-z)\cdot(1-v)\cdot v'\cdot w\cdot(1-w')$
8.	0	0	1	1	1	0/0	0/0	$(1-z)\cdot(1-v)\cdot v'\cdot w\cdot w'$
9.	0	1	0	0	0	0/1	0	$(1-z)\cdot v\cdot(1-v')\cdot(1-w)\cdot(1-w')$
10.	0	1	0	0	1	0/2	0	$(1-z)\cdot v\cdot(1-v')\cdot(1-w)\cdot w'$
11.	0	1	0	1	0	0/1	0	$(1-z)\cdot v\cdot(1-v')\cdot w\cdot(1-w')$
12.	0	1	0	1	1	0/2	0	$(1-z)\cdot v\cdot(1-v')\cdot w\cdot w'$
13.	0	1	1	0	0	0/0	0/0	$(1-z)\cdot v\cdot v'\cdot(1-w)\cdot(1-w')$
14.	0	1	1	0	1	0/1	0	$(1-z)\cdot v\cdot v'\cdot(1-w)\cdot w'$
15.	0	1	1	1	0	0/0	0/0	$(1-z)\cdot v\cdot v'\cdot w\cdot(1-w')$
16.	0	1	1	1	1	0/1	0	$(1-z)\cdot v\cdot v'\cdot w\cdot w'$
17.	1	0	0	0	0	0/0	0/0	$z\cdot(1-v)\cdot(1-v')\cdot(1-w)\cdot(1-w')$
18.	1	0	0	0	1	0/0	0/0	$z\cdot(1-v)\cdot(1-v')\cdot(1-w)\cdot w'$
19.	1	0	0	1	0	$-1/0$	∞	$z\cdot(1-v)\cdot(1-v')\cdot w\cdot(1-w')$
20.	1	0	0	1	1	0/0	0/0	$z\cdot(1-v)\cdot(1-v')\cdot w\cdot w'$
21.	1	0	1	0	0	0/0	0/0	$z\cdot(1-v)\cdot v'\cdot(1-w)\cdot(1-w')$
22.	1	0	1	0	1	0/0	0/0	$z\cdot(1-v)\cdot v'\cdot(1-w)\cdot w'$
23.	1	0	1	1	0	$-2/0$	∞	$z\cdot(1-v)\cdot v'\cdot w\cdot(1-w')$
24.	1	0	1	1	1	$-1/0$	∞	$z\cdot(1-v)\cdot v'\cdot w\cdot w'$
25.	1	1	0	0	0	0/1	0	$z\cdot v\cdot(1-v')\cdot(1-w)\cdot(1-w')$
26.	1	1	0	0	1	0/2	0	$z\cdot v\cdot(1-v')\cdot(1-w)\cdot w'$
27.	1	1	0	1	0	$-1/3$	∞	$z\cdot v\cdot(1-v')\cdot w\cdot(1-w')$
28.	1	1	0	1	1	0/2	0	$z\cdot v\cdot(1-v')\cdot w\cdot w'$
29.	1	1	1	0	0	0/0	0/0	$z\cdot v\cdot v'\cdot(1-w)\cdot(1-w')$
30.	1	1	1	0	1	0/1	0	$z\cdot v\cdot v'\cdot(1-w)\cdot w'$
31.	1	1	1	1	0	$-2/0$	∞	$z\cdot v\cdot v'\cdot w\cdot(1-w')$
32.	1	1	1	1	1	$-1/-1$	1	$z\cdot v\cdot v'\cdot w\cdot w'$

Case I: Table for Equation (5)

The solution for x is that it equals the sum of the *coefficient* $\times$ *constituent* for which the coefficient is 1 or 0/0. The same comment applies to the subsequent solutions for $1-x$ and vx.

Here is the expansion of (5) in the form $f_1(v,v',w,w')z+f_0(v,v',w,w')(1-z)$, where the numbers in parentheses (1),...,(32) give the row numbers for the constituents included in the expression for x. Next underbraces show which constituents give the indicated term. x, z and $1-z$ are displayed in boldface to assist the reader in parsing the expressions.

$$\begin{aligned}
\boldsymbol{x} &= \Big((32)+\frac{0}{0}\Big((17)+(18)+(20)+(21)+(22)+(29)\Big)\Big)\boldsymbol{z} \\
&+ \frac{0}{0}\Big((1)+\cdots+(8)+(13)+(15)\Big)\boldsymbol{(1-z)} \\
&= \Big(\underbrace{vv'ww'}_{(32)} + \frac{0}{0}\Big(\underbrace{(1-v)(1-v')ww'}_{(20)} + \underbrace{(1-v)(1-w)}_{(17)+(18)+(21)+(22)} + \underbrace{vv'(1-w)(1-w')}_{(29)}\Big)\Big)\boldsymbol{z} \\
&+ \frac{0}{0}\Big(\underbrace{(1-v)}_{(1)+\cdots+(8)} + \underbrace{vv'(1-w')}_{(13)+(15)}\Big)\boldsymbol{(1-z)} \\
&= \Big(vv'ww' + \frac{0}{0}\Big((1-v)(1-v')ww' + (1-v)(1-w) + vv'(1-w)(1-w')\Big)\Big)\boldsymbol{z} \\
&+ \frac{0}{0}\Big((1-v) + vv'(1-w')\Big)\boldsymbol{(1-z)}. \qquad (6)
\end{aligned}$$

Formula (6) is Boole's (I.) on p. 233 of LT.

2.3. **Solving for $\boldsymbol{1-x}$.** From $E_1x+E_0(1-x)=0$ one has $(E_1-E_0)(1-x)=E_1$, thus

$$\begin{aligned}
1-x &= \frac{E(1,z,v,v',w,w')}{E(1,z,v,v',w,w')-E(0,z,v,v',w,w')} \\
&= \frac{\Big((v-v')^2+(wz-w')^2\Big)(v+wz)}{\Big((v-v')^2+(wz-w')^2\Big)(v+wz)-\Big(v'+(wz-w')^2\Big)wz} \qquad (7)
\end{aligned}$$

so construct the table for (7):

	z	v	v'	w	w'	$Coeff$	$Value$	$Constituent$
1.	0	0	0	0	0	0/0	0/0	$(1-z)\cdot(1-v)\cdot(1-v')\cdot(1-w)\cdot(1-w')$
2.	0	0	0	0	1	0/0	0/0	$(1-z)\cdot(1-v)\cdot(1-v')\cdot(1-w)\cdot w'$
3.	0	0	0	1	0	0/0	0/0	$(1-z)\cdot(1-v)\cdot(1-v')\cdot w\cdot(1-w')$
4.	0	0	0	1	1	0/0	0/0	$(1-z)\cdot(1-v)\cdot(1-v')\cdot w\cdot w'$
5.	0	0	1	0	0	0/0	0/0	$(1-z)\cdot(1-v)\cdot v'\cdot(1-w)\cdot(1-w')$
6.	0	0	1	0	1	0/0	0/0	$(1-z)\cdot(1-v)\cdot v'\cdot(1-w)\cdot w'$
7.	0	0	1	1	0	0/0	0/0	$(1-z)\cdot(1-v)\cdot v'\cdot w\cdot(1-w')$
8.	0	0	1	1	1	0/0	0/0	$(1-z)\cdot(1-v)\cdot v'\cdot w\cdot w'$
9.	0	1	0	0	0	1/1	1	$(1-z)\cdot v\cdot(1-v')\cdot(1-w)\cdot(1-w')$
10.	0	1	0	0	1	2/2	1	$(1-z)\cdot v\cdot(1-v')\cdot(1-w)\cdot w'$
11.	0	1	0	1	0	1/1	1	$(1-z)\cdot v\cdot(1-v')\cdot w\cdot(1-w')$
12.	0	1	0	1	1	2/2	1	$(1-z)\cdot v\cdot(1-v')\cdot w\cdot w'$
13.	0	1	1	0	0	0/0	0/0	$(1-z)\cdot v\cdot v'\cdot(1-w)\cdot(1-w')$
14.	0	1	1	0	1	1/1	1	$(1-z)\cdot v\cdot v'\cdot(1-w)\cdot w'$
15.	0	1	1	1	0	0/0	0/0	$(1-z)\cdot v\cdot v'\cdot w\cdot(1-w')$
16.	0	1	1	1	1	1/1	1	$(1-z)\cdot v\cdot v'\cdot w\cdot w'$
17.	1	0	0	0	0	0/0	0/0	$z\cdot(1-v)\cdot(1-v')\cdot(1-w)\cdot(1-w')$
18.	1	0	0	0	1	0/0	0/0	$z\cdot(1-v)\cdot(1-v')\cdot(1-w)\cdot w'$
19.	1	0	0	1	0	1/0	∞	$z\cdot(1-v)\cdot(1-v')\cdot w\cdot(1-w')$
20.	1	0	0	1	1	0/0	0/0	$z\cdot(1-v)\cdot(1-v')\cdot w\cdot w'$
21.	1	0	1	0	0	0/0	0/0	$z\cdot(1-v)\cdot v'\cdot(1-w)\cdot(1-w')$
22.	1	0	1	0	1	0/0	0/0	$z\cdot(1-v)\cdot v'\cdot(1-w)\cdot w'$
23.	1	0	1	1	0	2/0	∞	$z\cdot(1-v)\cdot v'\cdot w\cdot(1-w')$
24.	1	0	1	1	1	1/0	∞	$z\cdot(1-v)\cdot v'\cdot w\cdot w'$
25.	1	1	0	0	0	1/1	1	$z\cdot v\cdot(1-v')\cdot(1-w)\cdot(1-w')$
26.	1	1	0	0	1	2/2	1	$z\cdot v\cdot(1-v')\cdot(1-w)\cdot w'$
27.	1	1	0	1	0	4/3	∞	$z\cdot v\cdot(1-v')\cdot w\cdot(1-w')$
28.	1	1	0	1	1	2/2	1	$z\cdot v\cdot(1-v')\cdot w\cdot w'$
29.	1	1	1	0	0	0/0	0/0	$z\cdot v\cdot v'\cdot(1-w)\cdot(1-w')$
30.	1	1	1	0	1	1/1	1	$z\cdot v\cdot v'\cdot(1-w)\cdot w'$
31.	1	1	1	1	0	2/0	∞	$z\cdot v\cdot v'\cdot w\cdot(1-w')$
32.	1	1	1	1	1	0/−1	0	$z\cdot v\cdot v'\cdot w\cdot w'$

Case I: Table for Equation (7)

Abbreviating the term $v'(1-w)+(1-v')w$ by $v'\triangle w$, the table gives

$$\begin{aligned}
\mathbf{1}-\boldsymbol{x} &= \Big[(25)+(26)+(28)+(30)+\frac{0}{0}\Big((17)+(18)+(20)+(21)+(22)+(29)\Big)\Big]\boldsymbol{z} \\
&+ \Big[(9)+(10)+(11)+(12)+(14)+(16) \\
&\qquad + \frac{0}{0}\Big((1)+(2)+(3)+(4)+(5)+(7)+(13)+(15)\Big)\Big](\mathbf{1}-\boldsymbol{z}) \\
&= \Big[\underbrace{v(1-v')(1-w)}_{(25)+(26)} + \underbrace{vw'(v'\triangle w)}_{(28)+(30)} \\
&\qquad + \frac{0}{0}\Big(\underbrace{(1-v)(1-w)}_{(17)+(18)+(21)+(22)}\ \underbrace{(1-v)(1-v')ww'}_{(20)} + \underbrace{vv'(1-w)(1-w')}_{(29)}\Big)\Big]\boldsymbol{z} \\
&+ \Big[\underbrace{v(1-v')}_{(9)+\cdots+(12)} + \underbrace{vv'w'}_{(14)+(16)} + \frac{0}{0}\Big(\underbrace{(1-v)}_{(1)+\cdots+(8)} + \underbrace{vv'(1-w')}_{(13)+(15)}\Big)\Big](\mathbf{1}-\boldsymbol{z}) \\
&= \Big[v(1-v')(1-w) + vw'(v'\triangle w) \\
&\qquad + \frac{0}{0}\Big((1-v)(1-w) + (1-v)(1-v')ww' + vv'(1-w)(1-w')\Big)\Big]\boldsymbol{z} \\
&+ \Big[\boldsymbol{v(1-v')} + v\,\boldsymbol{v'}w' + \frac{0}{0}\Big((1-v) + vv'(1-w')\Big)\Big](\mathbf{1}-\boldsymbol{z}). \qquad (8)
\end{aligned}$$

Formula (8) is Boole's (II.) on p. 233 of LT except for the red colored items: the term $v(1-v')$ is completely missing, and in the next term the v' in LT is given as $(1-w)$.

2.4. **Solving for $\boldsymbol{vx}$.** Multiplying (6) by v gives

$$\boldsymbol{vx} = \Big(vv'ww' + \frac{0}{0}\,vv'(1-w)(1-w')\Big)\boldsymbol{z} + \frac{0}{0}\,v\,\boldsymbol{v'}(1-w')(\mathbf{1}-\boldsymbol{z}). \qquad (9)$$

Formula (9) is Boole's (III.) on p. 233 of LT except that Boole omitted the $\boldsymbol{v'}$ indicated in red.

2.5. **Summary of Solutions for Case I.**

$$\boldsymbol{x} \;=\; \Big[vv'ww' \;+\; \frac{0}{0}\Big((1-v)(1-v')ww' \;+\; (1-v)(1-w) \;+\; vv'(1-w)(1-w')\Big)\Big]\boldsymbol{z}$$
$$+\; \frac{0}{0}\Big((1-v) \;+\; vv'(1-w')\Big)\boldsymbol{(1-z)} \tag{10}$$

$$\boldsymbol{1-x} \;=\; \Big[v(1-v')(1-w) \;+\; vw'(v'\triangle w)$$
$$+\; \frac{0}{0}\Big((1-v)(1-w) + \;(1-v)(1-v')ww' \;+\; vv'(1-w)(1-w')\Big)\Big]\boldsymbol{z}$$
$$+\; \Big[\boldsymbol{v(1-v')} \;+\; v\,\boldsymbol{v'}w' \;+\; \frac{0}{0}\Big((1-v) + vv'(1-w')\Big)\Big]\boldsymbol{(1-z)}. \tag{11}$$

$$\boldsymbol{vx} \;=\; \Big(vv'ww' \;+\; \frac{0}{0}\Big(vv'(1-w)(1-w')\Big)\Big)\boldsymbol{z} \;+\; \frac{0}{0}\Big(v\,\boldsymbol{v'}(1-w')\Big)\boldsymbol{(1-z)}. \tag{12}$$

We consider each of the equations (10)–(12) above in turn—the errors in LT noted in red type do not affect the conclusions Boole derived regarding valid syllogisms.

- For (10) the coefficient of z cannot be made to vanish using permissible substitutions of 1 for v, v', w, w'. To make the coefficient of $1-z$ vanish one needs to assign $v = w' = 1$. Then (10) reduces to $x = v'wz$.

- For (11) the only possibility is $v' = w = 1$ since one cannot force the coefficient of $1 - z$ to be 0 with a permissible substitution. Then (11) reduces to
$$1 - x = \Big(vw' \;+\; \frac{0}{0}(1 - vw')\Big)(1 - z).$$

- For (12) one must set $w' = 1$, and then (12) reduces $vx = vv'wz$. One can additionally set $v' = 1$, reducing (12) to $vx = vwz$.

2.6. **Details for Case II.** Given the two equations

$$vx = v'y \tag{13}$$

$$wz = w'(1-y) \tag{14}$$

reduce the system to a single equation $R(x, y, z, v, v', w, w') = 0$ where

$$R(x, y, z, v, v', w, w') := (vx - v'y)^2 + \Big(wz - w'(1-y)\Big)^2. \tag{15}$$

Eliminating y from $R = 0$ yields $E(x, z, v, v', w, w') = 0$ where

$$\begin{aligned} E(x, z, v, v', w, w') &:= R(x, 1, z, v, v', w, w')R(x, 0, z, v, v', w, w') \\ &= \Big((vx - v')^2 \; + \; wz\Big)\Big(vx + (wz - w')^2\Big). \end{aligned} \tag{16}$$

Now we want to solve $E = 0$ for x, for $1 - x$ and for vx.

2.7. **Solving for x.** To solve $E = 0$ for x, first express E in the form $E_1 x + E_0(1-x)$ by expanding it about x:

$$\begin{aligned} & E(x, z, v, v', w, w') \\ = \; & E(1, z, v, v', w, w')x + E(0, z, v, v', w, w')(1-x) \\ = \; & \Big((v - v')^2 + wz\Big)\Big(v + (wz - w')^2\Big)x \; + \; (v' + wz)(wz - w')^2(1-x). \end{aligned}$$

From $E_1 x + E_0(1-x) = 0$ one has $(E_0 - E_1)x = E_0$, thus

$$\begin{aligned} x &= \frac{E(0, z, v, v', w, w')}{E(0, z, v, v', w, w') - E(1, z, v, v', w, w')} \\ &= \frac{(v' + wz)(wz - w')^2}{(v' + wz)(wz - w')^2 - \Big((v - v')^2 + wz\Big)\Big(v + (wz - w')^2\Big)}. \end{aligned} \tag{17}$$

Constructing the table for the right side of (17) gives:

	z	v	v'	w	w'	$Coeff$	$Value$	$Constituent$
1.	0	0	0	0	0	$0/0$	$0/0$	$(1-z)(1-v)(1-v')(1-w)(1-w')$
2.	0	0	0	0	1	$0/0$	$0/0$	$(1-z)(1-v)(1-v')(1-w)w'$
3.	0	0	0	1	0	$0/0$	$0/0$	$(1-z)(1-v)(1-v')w(1-w')$
4.	0	0	0	1	1	$0/0$	$0/0$	$(1-z)(1-v)(1-v')ww'$
5.	0	0	1	0	0	$0/0$	$0/0$	$(1-z)(1-v)v'(1-w)(1-w')$
6.	0	0	1	0	1	$-1/0$	∞	$(1-z)(1-v)v'(1-w)w'$
7.	0	0	1	1	0	$0/0$	$0/0$	$(1-z)(1-v)v'w(1-w')$
8.	0	0	1	1	1	$-1/0$	∞	$(1-z)(1-v)v'ww'$
9.	0	1	0	0	0	$0/1$	0	$(1-z)v(1-v')(1-w)(1-w')$
10.	0	1	0	0	1	$0/2$	0	$(1-z)v(1-v')(1-w)w'$
11.	0	1	0	1	0	$0/1$	0	$(1-z)v(1-v')w(1-w')$
12.	0	1	0	1	1	$0/2$	0	$(1-z)v(1-v')ww'$
13.	0	1	1	0	0	$0/0$	$0/0$	$(1-z)vv'(1-w)(1-w')$
14.	0	1	1	0	1	$-1/-1$	1	$(1-z)vv'(1-w)w'$
15.	0	1	1	1	0	$0/0$	$0/0$	$(1-z)vv'w(1-w')$
16.	0	1	1	1	1	$-1/-1$	1	$(1-z)vv'ww'$
17.	1	0	0	0	0	$0/0$	$0/0$	$z(1-v)(1-v')(1-w)(1-w')$
18.	1	0	0	0	1	$0/0$	$0/0$	$z(1-v)(1-v')(1-w)w'$
19.	1	0	0	1	0	$-1/0$	∞	$z(1-v)(1-v')w(1-w')$
20.	1	0	0	1	1	$0/0$	$0/0$	$z(1-v)(1-v')ww'$
21.	1	0	1	0	0	$0/0$	$0/0$	$z(1-v)v'(1-w)(1-w')$
22.	1	0	1	0	1	$-1/0$	∞	$z(1-v)v'(1-w)w'$
23.	1	0	1	1	0	$-2/0$	∞	$z(1-v)v'w(1-w')$
24.	1	0	1	1	1	$0/0$	$0/0$	$z(1-v)v'ww'$
25.	1	1	0	0	0	$0/1$	0	$zv(1-v')(1-w)(1-w')$
26.	1	1	0	0	1	$0/2$	0	$zv(1-v')(1-w)w'$
27.	1	1	0	1	0	$-1/3$	∞	$zv(1-v')w(1-w')$
28.	1	1	0	1	1	$0/2$	0	$zv(1-v')ww'$
29.	1	1	1	0	0	$0/0$	$0/0$	$zvv'(1-w)(1-w')$
30.	1	1	1	0	1	$-1/-1$	1	$zvv'(1-w)w'$
31.	1	1	1	1	0	$-2/0$	∞	$zvv'w(1-w')$
32.	1	1	1	1	1	$0/1$	0	$zvv'ww'.$

Case II: Table for Equation (17)

Thus

$$\begin{aligned}
\boldsymbol{x} &= \Big((30) \ + \ \frac{0}{0}\Big((17)+(18)+(20)+(21)+(24)+(29)\Big)\Big)\boldsymbol{z} \\
&+ \ \Big((14)+(16) \ + \ \frac{0}{0}\Big((1)+\cdots+(5)+(7)+(13)+(15)\Big)\Big)\mathbf{(1-z)} \\
&= \Big(\underbrace{vv'(1-w)w'}_{(30)} \ + \frac{0}{0}\Big(\underbrace{(1-v)(1-v')(1-w)}_{(17)+(18)} \ + \ \underbrace{(1-v)ww'}_{(20)+(24)} \ + \ \underbrace{v'(1-w)(1-w')}_{(21)+(29)}\Big)\Big)\boldsymbol{z} \\
&+ \ \Big(\underbrace{vv'w'}_{(14)+(16)} \ + \frac{0}{0}\Big(\underbrace{(1-v)(1-v')}_{(1)+\cdots+(4)} \ + \ \underbrace{v'(1-w')}_{(5)+(7)+(13)+(15)}\Big)\Big)\mathbf{(1-z)}
\end{aligned}$$

$$\begin{aligned}
&= \Big(vv'(1-w)w' \ + \ \frac{0}{0}\Big((1-v)(1-v')(1-w) \ + \ (1-v)ww' \ + \ v'(1-w)(1-w')\Big)\Big)\boldsymbol{z} \\
&+ \ \Big(vv'w' \ + \frac{0}{0}\Big((1-v)(1-v') \ + \ v'(1-w')\Big)\Big)\mathbf{(1-z).} \qquad (18)
\end{aligned}$$

Formula (18) is Boole's (IV.) on p. 234 of LT.

2.8. **Solving for $1-x$.** From $(E_1-E_0)(1-x)=E_1$ one has

$$\begin{aligned}
1-x &= \frac{E(1,z,v,v',w,w')}{E(1,z,v,v',w,w')-E(0,z,v,v',w,w')} \\
&= \frac{\Big((v-v')^2+wz\Big)\Big(v+(wz-w')^2\Big)}{\Big((v-v')^2+wz\Big)\Big(v+(wz-w')^2\Big)-(v'+wz)(wz-w')^2} \qquad (19)
\end{aligned}$$

so construct the table for (19):

	z	v	v'	w	w'	$Coeff$	$Value$	$Constituent$
1.	0	0	0	0	0	$0/0$	$0/0$	$(1-z)(1-v)(1-v')(1-w)(1-w)$
2.	0	0	0	0	1	$0/0$	$0/0$	$(1-z)(1-v)(1-v')(1-w)w$
3.	0	0	0	1	0	$0/0$	$0/0$	$(1-z)(1-v)(1-v')w(1-w)$
4.	0	0	0	1	1	$0/0$	$0/0$	$(1-z)(1-v)(1-v')ww$
5.	0	0	1	0	0	$0/0$	$0/0$	$(1-z)(1-v)v'(1-w)(1-w)$
6.	0	0	1	0	1	$1/0$	∞	$(1-z)(1-v)v'(1-w)w$
7.	0	0	1	1	0	$0/0$	$0/0$	$(1-z)(1-v)v'w(1-w)$
8.	0	0	1	1	1	$1/0$	∞	$(1-z)(1-v)v'ww$
9.	0	1	0	0	0	$1/1$	1	$(1-z)v(1-v')(1-w)(1-w)$
10.	0	1	0	0	1	$2/2$	1	$(1-z)v(1-v')(1-w)w$
11.	0	1	0	1	0	$1/1$	1	$(1-z)v(1-v')w(1-w)$
12.	0	1	0	1	1	$2/2$	1	$(1-z)v(1-v')ww$
13.	0	1	1	0	0	$0/0$	$0/0$	$(1-z)vv'(1-w)(1-w)$
14.	0	1	1	0	1	$0/-1$	0	$(1-z)vv'(1-w)w$
15.	0	1	1	1	0	$0/0$	$0/0$	$(1-z)vv'w(1-w)$
16.	0	1	1	1	1	$0/-1$	0	$(1-z)vv'ww$
17.	1	0	0	0	0	$0/0$	$0/0$	$z(1-v)(1-v')(1-w)(1-w')$
18.	1	0	0	0	1	$0/0$	$0/0$	$z(1-v)(1-v')(1-w)w'$
19.	1	0	0	1	0	$1/0$	∞	$z(1-v)(1-v')w(1-w')$
20.	1	0	0	1	1	$0/0$	$0/0$	$z(1-v)(1-v')ww'$
21.	1	0	1	0	0	$0/0$	$0/0$	$z(1-v)v'(1-w)(1-w')$
22.	1	0	1	0	1	$1/0$	∞	$z(1-v)v'(1-w)w'$
23.	1	0	1	1	0	$2/0$	∞	$z(1-v)v'w(1-w')$
24.	1	0	1	1	1	$0/0$	$0/0$	$z(1-v)v'ww'$
25.	1	1	0	0	0	$1/1$	1	$zv(1-v')(1-w)(1-w')$
26.	1	1	0	0	1	$2/2$	1	$zv(1-v')(1-w)w'$
27.	1	1	0	1	0	$4/3$	∞	$zv(1-v')w(1-w')$
28.	1	1	0	1	1	$2/2$	1	$zv(1-v')ww'$
29.	1	1	1	0	0	$0/0$	$0/0$	$zvv'(1-w)(1-w')$
30.	1	1	1	0	1	$0/-1$	0	$zvv'(1-w)w'$
31.	1	1	1	1	0	$2/0$	∞	$zvv'w(1-w')$
32.	1	1	1	1	1	$1/1$	1	$zvv'ww'$

Case II: Table for Equation (19)

The table gives

$$\begin{aligned}
\mathbf{1-x} \;&=\; \Big(\;(25)+(26)+(28)+(32)\;+\;\frac{0}{0}\Big((17)+(18)+(20)+(21)+(24)+(29)\Big)\Big)\boldsymbol{z}\\
&+\;\Big((9)+\cdots+(12)\;+\;\frac{0}{0}\Big((1)+\cdots+(5)+(7)+(13)+(15)\Big)\Big)\mathbf{(1-z)}\\
&=\;\Big(\underbrace{v(1-v')(1-w)}_{(25)+(26)}\;+\;\underbrace{vww'}_{(28)+(32)}\\
&\qquad+\frac{0}{0}\Big(\underbrace{(1-v)(1-v')(1-w)}_{(17)+(18)}\;+\;\underbrace{(1-v)ww'}_{(20)+(24)}\;+\;\underbrace{v'(1-w)(1-w')}_{(21)+(29)}\Big)\Big)\boldsymbol{z}\\
&+\;\Big(\underbrace{v(1-v')}_{(9)+\cdots+(12)}\;+\frac{0}{0}\Big(\underbrace{(1-v)(1-v')}_{(1)+\cdots+(4)}\;+\;\underbrace{v'(1-w')}_{(5)+(7)+(13)+(15)}\Big)\Big)\mathbf{(1-z)}\\
&=\;\Big(v(1-v')(1-w)\;+\;vww'\\
&\qquad+\frac{0}{0}\Big((1-v)(1-v')(1-w)\;+\;(1-v)ww'\;+\;v'(1-w)(1-w')\Big)\Big)\boldsymbol{z}\\
&+\;\Big(v(1-v')\;+\frac{0}{0}\Big((1-v)(1-v')\;+\;v'(1-w')\Big)\Big)\mathbf{(1-z)}. \qquad (20)
\end{aligned}$$

Formula (20) is Boole's (V.) on p. 235 of LT.

2.9. **Solving for vx.** Multiplying (18) by v gives

$$\begin{aligned}
\boldsymbol{vx} \;&=\; \Big(vv'(1-w)w'\;+\;\frac{0}{0}vv'(1-w)(1-w')\Big)\boldsymbol{z}\\
&+\;\Big(vv'w'\;+\frac{0}{0}vv'(1-w')\Big)\mathbf{(1-z)}. \qquad (21)
\end{aligned}$$

Formula (21) is Boole's (VI.) on p. 235 of LT.

2.10. **Summary of Solutions for Case II.**

$$\begin{aligned}
\boldsymbol{x} \;&=\; \Big(vv'(1-w)w' \;+\; \frac{0}{0}\Big((1-v)(1-v')(1-w) \;+\; (1-v)ww' \;+\; v'(1-w)(1-w')\Big)\Big)\boldsymbol{z} \\
&+\; \Big(vv'w' \;+\frac{0}{0}\Big((1-v)(1-v') \;+\; v'(1-w')\Big)\Big)(\mathbf{1-\boldsymbol{z}}) \qquad (22)
\end{aligned}$$

$$\begin{aligned}
\mathbf{1}-\boldsymbol{x} \;&=\; \Big(v(1-v')(1-w) \;+\; vww' \\
&\qquad +\frac{0}{0}\Big((1-v)(1-v')(1-w) \;+\; (1-v)ww' \;+\; v'(1-w)(1-w')\Big)\Big)\boldsymbol{z} \\
&+\; \Big(v(1-v') \;+\frac{0}{0}\Big((1-v)(1-v') \;+\; v'(1-w')\Big)\Big)(\mathbf{1-\boldsymbol{z}}). \qquad (23)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{vx} \;&=\; \Big(vv'(1-w)w' \;+\; \frac{0}{0}vv'(1-w)(1-w')\Big)\boldsymbol{z} \\
&+\; \Big(vv'w' \;+\frac{0}{0}vv'(1-w')\Big)(\mathbf{1-\boldsymbol{z}}). \qquad (24)
\end{aligned}$$

We consider each of the equations (22)–(24) above in turn.

- For (22) the coefficient of $1-z$ cannot be made to vanish using a permissible substitution. To make the coefficient of z vanish one needs to assign $v = w = 1$. Then (22) reduces to $x = \Big(v'w' + \frac{0}{0}v'(1-w')\Big)(1-z)$.

- For (23) the coefficient of z cannot be made to vanish using a permissible substitution. To make the coefficient of $1-z$ vanish one needs to assign $v' = w' = 1$. Then (23) reduces to $1-x = \Big(vw + \frac{0}{0}(1-v)w\Big)z$.

- For (24) the coefficient of $1-z$ cannot be made to vanish using a permissible substitution. To make the coefficient of z vanish one needs to assign $w = 1$. Then (24) reduces to $vx = \Big(vv'w' + \frac{0}{0}vv'(1-w')\Big)(1-z)$. One can additionally set $v' = 1$, giving the reduction to $vx = \Big(vw' + \frac{0}{0}v(1-w')\Big)(1-z)$.

3. Relevant Equational Arguments

The above justifies the following equational arguments:

For Case I:

(**a**)
$$\begin{aligned} x &= vy \\ wz &= y \\ \therefore x &= vwz \end{aligned}$$

(**b**)
$$\begin{aligned} vx &= y \\ z &= wy \\ \therefore 1-x &= \big(vw \ + \ \tfrac{0}{0}(1-vw)\big)(1-z) \end{aligned}$$

(**c**)
$$\begin{aligned} vx &= vy \\ wz &= y \\ \therefore vx &= vwz \end{aligned}$$

(**d**)
$$\begin{aligned} vx &= y \\ wz &= y \\ \therefore vx &= vwz. \end{aligned}$$

From Case I Boole claimed for categorical premises with like middle terms (p. 234 of LT):[2]

Condition of Inference.— One middle term, at least, universal.

Rule of Inference.— Equate the extremes.

For Case II:

(**a**)
$$\begin{aligned} x &= vy \\ z &= w(1-y) \\ \therefore x &= \big(vw + \tfrac{0}{0}v(1-w)\big)(1-z) \end{aligned}$$

(**b**)
$$\begin{aligned} vx &= y \\ wz &= 1-y \\ \therefore 1-x &= \big(vw + \tfrac{0}{0}(1-v)w\big)z \end{aligned}$$

(**c**)
$$\begin{aligned} vx &= vy \\ z &= w(1-y) \\ \therefore vx &= \big(vw + \tfrac{0}{0}v(1-w)\big)(1-z) \end{aligned}$$

(**d**)
$$\begin{aligned} vx &= y \\ z &= w(1-y) \\ \therefore vx &= \big(vw + \tfrac{0}{0}v(1-w)\big)(1-z). \end{aligned}$$

From Case II Boole claimed (pp. 235, 236 of LT) for premises with unlike middle terms:[3]

First Condition of Inference.— *At least one universal extreme.*

Rule of Inference.— *Change the quantity and quality of that extreme, and equate the result to the other extreme.*

Second Condition of Inference.— *Two universal middle terms.*

Rule of Inference.— *Change the quantity and quality of either extreme, and equate the result to the other extreme unchanged.*

[2]These three rules from Cases I and II, for determining valid syllogisms, were originally announced as the main contribution of Boole's 1848 paper [2] *The Calculus of Logic*. Boole was quite pleased that he had abolished the need for the Aristotelian concepts of *figure* and *mood* after redefining the notions of *quality* and *quantity* to apply to terms instead of to propositions.

[3]Boole mistakenly said the first of these two rules applied when the middle terms had *like* quality.

4. The Advantage of Using Negated Equations

Note that all of Boole's categorical propositions $\Phi(X, Y)$ can be expressed in either the form $\alpha\beta = 0$ or $\alpha\beta \neq 0$, where α is either x or $1 - x$ and β is either y or $1 - y$. For example, All not-X is Y is expressed by $(1 - x)(1 - y) = 0$, and Some not-X is Y is expressed by $(1 - x)y \neq 0$.

By using negated equations as well as equations the mathematical description of valid syllogisms can be condensed into four cases where $\overline{\alpha}$ changes x to $1 - x$ and vice-versa, etc.

$$\begin{array}{rcl} \alpha\gamma & = & 0 \\ \beta\gamma & = & 0 \\ \hline \therefore\ \overline{\alpha}\overline{\beta} & \neq & 0 \end{array} \qquad \begin{array}{rcl} \alpha\gamma & = & 0 \\ \beta\overline{\gamma} & = & 0 \\ \hline \therefore\ \alpha\beta & = & 0 \end{array} \qquad \begin{array}{rcl} \alpha\gamma & = & 0 \\ \beta\gamma & \neq & 0 \\ \hline \therefore\ \overline{\alpha}\beta & \neq & 0 \end{array} \qquad \begin{array}{rcl} \alpha\gamma & = & 0 \\ \beta\overline{\gamma} & \neq & 0 \\ \hline \therefore\ \alpha\beta & \neq & 0 \end{array}$$

These arguments are easily confirmed using Venn diagrams, just as the two forms of premises

$$\begin{array}{rcl} \alpha\gamma & \neq & 0 \\ \beta\gamma & \neq & 0 \end{array} \qquad \begin{array}{rcl} \alpha\gamma & \neq & 0 \\ \beta\overline{\gamma} & \neq & 0 \end{array}$$

can be shown to have no valid conclusion. (A formal proof theory for dealing with $\neq$, however, requires more work.)

Boole limited his general theory to universal propositions for a good reason—there was no general elimination theorem if one included particular propositions. For example consider the premises

Some Y is X

Some not-Y is X.

The result of eliminating Y is that X has at least two elements. However this conclusion cannot be expressed in his algebra of logic.

References

(1) George Boole, *The Mathematical Analysis of Logic, Being an Essay Towards a Calculus of Deductive Reasoning*, Originally published in Cambridge by Macmillan, Barclay, & Macmillan, 1847. Reprinted in Oxford by Basil Blackwell, 1951.

(2) ______, *The Calculus of Logic*, The Cambridge and Dublin Mathematical Journal, **3** (1848), 183–198.

(3) ______, *An Investigation of The Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities*. Originally published by Macmillan, London, 1854. Reprint by Dover, 1958.

(4) Stanley Burris, *George Boole*. The online Stanford Encyclopedia of Philosophy at `http://plato.stanford.edu/entries/boole/`.

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