# DISCOVERING AN ALGEBRA OF CLASSES IN THE ALGEBRA OF NUMBERS—FROM GEORGE BOOLE TO THE PRESENT

STANLEY BURRIS

Many people mistakenly believe Boole's 1854 classic [1], *The Laws of Thought*, must be about his creation of the subject called Boolean Algebra. Actually he used the symbols and equational reasoning of the ordinary number system, augmented by requiring that variables be idempotent, to develop his Algebra of Logic (AoL). Trying to understand his AoL as either Boolean Algebra or Boolean Rings is a mistake—nonetheless some expressions in Boole's AoL can easily be read as belonging to these two subjects.

A down-side of Boole's Algebra of Numbers approach was, after choosing multiplication to be the totally defined operation of intersection, his operations of addition and subtraction on classes were *forced* to be partially defined—yet he carried out algebraic manipulations as though they were totally defined. His principle that one could derive meaningful results about classes by methods employing uninterpretable terms was not well-received—others soon replaced Boole's AoL with what would eventually become Boolean Algebra.<sup>1</sup> Although his AoL has more in common with Boolean Rings than Boolean Algebra,<sup>2</sup> it was only much later that Boolean Rings appeared—taking the symmetric difference as a fundamental operation was not an obvious choice.

Section 1 is a summary of Boole's AoL. Section 2 is Whitney's 1933 encoding of the modern Boolean Algebra of classes into the Algebra of Numbers using characteristic functions. Whitney did not realize how deeply connected his work was with that of Boole—such would have to wait till Hailperin's book of 1976/1986 with its focus on signed multisets (see Section 3). Section 4 explains Boole's Rule of 0 and 1 and the recent role of Horn sentences in proving his results. Section 5 looks at Brown's version of Boole's AoL using an algebra of proto-Boolean polynomials.

To achieve a smooth narrative the various original notations of the authors mentioned will frequently be set aside.

<sup>2</sup>All the quasi-equations that hold in the Algebra of Numbers hold in Boole's AoL. The equations also hold in Boolean Rings, but not all the quasi-equations—for example, Boolean Rings are not torsion-free.

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<sup>&</sup>lt;sup>1</sup>In 1864 Jevons [8] scrapped Boole's numerical superstructure and replaced *classes* with *properties*. Converting his properties back into classes one finds that, starting with the natural operations of union, intersection and complement, he launched the development of modern Boolean Algebra, followed by such luminaries as Peirce, Venn, Schröder, Huntington, Royce and Stone. This note, however, is concerned with results that continued along the lines of Boole's Algebra of Numbers approach.

## 1. George Boole's Algebra of Logic

In [1], p. 27, Boole asserted that only a few symbols were needed to develop an Algebra of Logic:

All the operations of Language, as an instrument of reasoning, may be conducted by a system of signs composed of the following elements, viz.:

1st. Literal symbols, as  $x, y, \mathcal{C}c.$ , representing things as subjects of our conceptions.

2nd. Signs of operation, as +, -,  $\times$ , standing for those operations of the mind by which the conceptions of things are combined or resolved so as to form new conceptions involving the same elements.

3rd. The sign of identity, =.

And these symbols of Logic are in their use subject to definite laws, partly agreeing with and partly differing from the laws of the corresponding symbols in the science of Algebra.

He would soon add the symbols 1 and 0 to the above. Boole did not have the following modern recursive definition of *terms* as strings of symbols for his AoL:

- Variables are terms, as are the symbols 0 and 1.
- Suppose s and t are terms. Then so are  $(s \cdot t)$ , (s + t), and (s t).

His notation followed that of the practice in the Algebra of Numbers, an informal mixture of terms and polynomials.

Given a universe of discourse, his model was to let 1 denote the universe, 0 the empty class, and to define the three binary operations for classes x and y contained in the universe as follows:<sup>3</sup>

• 
$$x \times y := x \cap y$$
  
•  $x + y := \begin{cases} x \cup y & \text{if } x \cap y = 0 \\ \text{undefined otherwise} \end{cases}$   
•  $x - y := \begin{cases} x \setminus y & \text{if } y \subseteq x \\ \text{undefined otherwise.} \end{cases}$ 

Instead of 'undefined' Boole said 'uninterpretable'. In the following the product of x and y will be written either  $x \cdot y$ , or as Boole often wrote, simply xy (just as one does in the algebra of numbers). Clearly 1 - x is the complement of the class x. The *idempotent* law  $x^2 = x$  for classes followed from his definition of multiplication. A term  $t(x_1, \ldots, x_n)$  that is

<sup>&</sup>lt;sup>3</sup>Modern notation is used on the right sides of the equal signs, where ":=" means equals by definition.

interpretable for all choices of classes for the  $x_i$  is *totally* interpretable.<sup>4</sup> An equation s = t is totally interpretable if both s and t are totally interpretable.

Boole expressed the standard operations on classes by totally interpretable terms (thanks to "—" being binary subtraction) as well as by polynomials in his Algebra of Logic:<sup>5</sup>

$$x \cap y = xy$$
  

$$x \cup y = x + (1 - x)y = x + y - xy$$
  

$$x \triangle y = x(1 - y) + (1 - x)y = x + y - 2xy$$
  

$$x' = 1 - x$$
  

$$U = 1$$
  

$$\emptyset = 0.$$

He used this to express propositions about classes as equations. One can use Boole's AoL to derive the laws of Boolean algebra (but Boole did not do this), for example, to prove the absorption law  $x \cap (x \cup y) = x$  one has

$$x \cap (x \cup y) = x(x + y - xy)$$
$$= x^{2} + xy - x^{2}y$$
$$= x + xy - xy = x$$

The Boolean Algebra and Boolean Ring of classes exist just below the surface of Boole's AoL—any totally interpretable term is equivalent to the Boolean Algebra term obtained by replacing addition by union, multiplication by intersection, and each subterm s - t by  $s \cap t'$ . But this is not how Boolean Algebra came into existence—see the earlier footnote about Jevons. Any totally interpretable term can also be viewed as a Boolean Ring term. The Boolean Algebra and Boolean Ring of classes are embedded in Boole's AoL, but the converse is false.

1.1. The Rule of 0 and 1. Boole introduced a powerful method to determine the valid equations and equational arguments in his algebra of logic on p. 37 of [1], namely it suffices to check them in the algebra of numbers when the variables are restricted to 0 and 1:

Let us conceive, then, of an Algebra in which the symbols x, y, z, etc. admit indifferently of the values 0 and 1, and of these values alone. The laws, the axioms, and the processes, of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of

<sup>&</sup>lt;sup>4</sup>One can easily give a recursive definition of the collection of totally interpretable terms, and one can show that the domain of interpretability of a term  $t(x_1, \ldots, x_n)$  is given by the solutions of some equation  $d(x_1, \ldots, x_n) = 0$  where d is totally interpretable.

<sup>&</sup>lt;sup>5</sup>Modern notation is used on the left sides of the equal signs.

interpretation will alone divide them. Upon this principle the method of the following work is established.

The strength of this principle was not understood for a century and a half—it will be discussed in Section 4.

1.2. Constituents. Boole based the development of his algebra on the use of *constituents*. The constituents of x are simply x and 1 - x; for x, y they are xy, x(1 - y), (1 - x)y and (1 - x)(1 - y). In general for variables  $\vec{x} := x_1, \ldots, x_m$  the constituents are the  $C_{\sigma}(\vec{x})$  given by

$$C_{\sigma}(\vec{x}) := \prod_{i=1}^{m} C_{\sigma_i}(x_i)$$

where  $\sigma$  is a string of 0s and 1s of length m, and

$$C_{\sigma_i}(x_i) := \begin{cases} x_i & \text{if } \sigma_i = 1\\ 1 - x_i & \text{if } \sigma_i = 0 \end{cases}$$

Boole noted that the  $C_{\sigma}(\vec{x})$  are idempotent, pairwise disjoint and their sum is 1.

1.3. Development Theorem. Perhaps the single most important result for his algebra of logic was that for any term  $t(\vec{x}, \vec{y})$ , where  $\vec{y} := y_1, \ldots, y_n$ ,

$$t(\vec{x}, \vec{y}) = \sum_{\sigma} t(\sigma, \vec{y}) \cdot C_{\sigma}(\vec{x}).$$

When there are no  $y_i$  then one has the *complete* development

$$t(\vec{x}) = \sum_{\sigma} t(\sigma) \cdot C_{\sigma}(\vec{x})$$

with the coefficients  $t(\sigma)$  being integers. The right side of this equation is totally interpretable iff each  $t(\sigma)$  is idempotent, that is, it is either 0 or 1. This is precisely the condition for  $t(\vec{x})$  to be idempotent, that is,  $t(\vec{x})^2 = t(\vec{x})$ . The *idempotence* of a term is Boole's *condition* of *interpretability*—a term is equivalent to a totally interpretable term iff it is idempotent. Note that x + y does not satisfy this condition as 1 + 1 is neither 0 nor 1; x + y is only interpretable under the condition that xy = 0.

Boole stated that two terms  $s(\vec{x}), t(\vec{x})$  were equal in his algebra iff they had the same complete developments, that is,  $s(\sigma) = t(\sigma)$  for all  $\sigma$ . In Boole's Algebra every equation  $t_1(\vec{x}) = t_2(\vec{x})$  is equivalent to an equation  $t(\vec{x}) = 0$ , namely put  $t(\vec{x}) := t_1(\vec{x}) - t_2(\vec{x})$ . The Development Theorem, the properties of constituents, and the torsion-free property show that an equation  $t(\vec{x}) = 0$  is equivalent to the collection of constituent equations  $C_{\sigma}(\vec{x}) = 0$ where  $t(\sigma) \neq 0$ . Thus any equation  $t(\vec{x}) = 0$  is equivalent to an equation  $t^*(\vec{x}) = 0$  that is totally interpretable, namely let

$$t^{\star}(\vec{x}) := \sum_{4} \{ C_{\sigma}(\vec{x}) : t(\sigma) \neq 0 \}.$$

1.4. Reduction Theorem. This says that any collection of equations

$$t_1(\vec{x}) = 0, \dots, t_k(\vec{x}) = 0$$

can be reduced to the single equation

$$t_1(\vec{x})^2 + \dots + t_k(\vec{x})^2 = 0$$

This follows from the torsion-free property and properties of constituents—they imply that the collection of equations is equivalent to the collection of constituent equations  $C_{\sigma}(\vec{x}) = 0$ for which some  $t_i(\sigma) \neq 0$ . Clearly  $t_i(\sigma) \neq 0$  for some *i* iff  $t_1(\sigma)^2 + \cdots + t_k(\sigma)^2 \neq 0$ .

1.5. Elimination Theorem. This says that the complete result of eliminating the variables  $\vec{x}$  from an equation  $t(\vec{x}, \vec{y}) = 0$  is the equation

$$\prod_{\sigma} t(\sigma, \vec{y}) = 0.$$

Thus eliminating y from  $t(\vec{x}, y) = 0$  gives  $t(\vec{x}, 0) \cdot t(\vec{x}, 1) = 0$ . Boole did not define what he meant by the *complete result*.

1.6. Solution Theorem. Boole claimed that the solution for y in an equation  $t(\vec{x}, y) = 0$  consisted of a pair of equations

(1) 
$$t(\vec{x}, 0) \cdot t(\vec{x}, 1) = 0$$

(2) 
$$y = t^{*}(\vec{x}, 0) + v \cdot (1 - t^{*}(\vec{x}, 0)) \cdot (1 - t^{*}(\vec{x}, 1)), \text{ for some } v.$$

(This is a tidied-up version of Boole's solution.) Boole frequently used this result in his examples in [1], and for the application of his AoL to Probability Theory.

# 2. Hassler Whitney encodes Modern Boolean Algebra in the Algebra of Numbers using Characteristic Functions

In 1933 Whitney [11] showed that one did not need to be familiar with Boolean Algebra<sup>6</sup> to verify equations and equational arguments in this algebra—one could convert them into equations and equational arguments in the Algebra of Numbers. His main tool was *charac*-teristic functions  $\chi_A$  mapping a universe U into  $\{0, 1\}$ , taking the value 1 on A, the value 0 off A.<sup>7</sup>

<sup>&</sup>lt;sup>6</sup>Like many, Whitney still used the traditional name Algebra of Logic instead of Boolean Algebra, the latter a name which had been promoted by the philosopher Josiah Royce of Harvard and his students.

<sup>&</sup>lt;sup>7</sup>Evidently using characteristic functions was considered novel as Whitney gave a reference to a 1916 work of de la Vallée Poussin for the definition.

The basic idea was that the characteristic function of a Boolean combination of  $A_i$  could be expressed as a polynomial in the characteristic functions of the  $A_i$ , a polynomial with integer coefficients, since

$$\chi_{A\cap B} = \chi_A \cdot \chi_B$$
  

$$\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$$
  

$$\chi_{A\triangle B} = \chi_A + \chi_B - 2\chi_A \cdot \chi_B$$
  

$$\chi_{A'} = 1 - \chi_A$$
  

$$\chi_U = 1, \quad \chi_{\emptyset} = 0.$$

A polynomial in characteristic functions defines a function in the ring  $\mathbf{Z}^{U}$ , but Whitney did not mention this ring. He did not see the connection with Boole's AoL, namely if one restricted the ring operations to the characteristic functions then one had a partial algebra isomorphic to Boole's partial algebra.<sup>8</sup> Although he had Boole's Development Theorem (which he called his First Normal Form) and the reduction of a system of equations to constituent equations, there was no Elimination Theorem or Solution Theorem.

Whitney briefly considered arbitrary functions from the universe U to the integers, generalizing the polynomials in the characteristic functions, but did not make explicit that these were precisely the elements of the ring  $\mathbf{Z}^{U}$ .

## 3. Theodore Hailperin's Axioms and Signed Multisets

Hailperin [6] noted that by identifying subsets of the universe U with their characteristic functions, the ring  $\mathbf{Z}^U$  formed an extension of Boole's partial algebra that made all terms in Boole's AoL interpretable.  $\mathbf{Z}^U$  is a non-trivial torsion-free<sup>9</sup> commutative ring with unity— Hailperin called members of this ring *signed multi-sets*. He axiomatized Boole's AoL by the first-order axioms of non-trivial torsion-free commutative rings with unity,<sup>10</sup> and proceeded to prove each of Boole's main theorems—expressed as a first-order sentence  $\varphi$ —by proving  $\varphi_E$ from these axioms, where  $\varphi_E$  is the result of restricting the quantifiers of  $\varphi$  to idempotent elements. Finally Boole's work on logic had a firm foundation. Note that Boole's rather awkward idempotent law, which applied only to variables, was dropped from the axioms

 $<sup>^{8}</sup>$ The restriction needs to use the binary subtraction operation, not the unary minus operation.

<sup>&</sup>lt;sup>9</sup>Torsion-free means that nx = 0 implies x = 0, for n = 1, 2, ...

<sup>&</sup>lt;sup>10</sup>Perhaps Hailperin was the first to recognize the critical role that the torsion-free property played in the construction of Boole's AoL. Actually Boole only used it to show that  $nC_{\sigma}(\vec{x}) = 0$  implies  $C_{\sigma}(\vec{x}) = 0$ . By the Development Theorem this leads to  $nt(\vec{x}) = 0$  implies  $t(\vec{x}) = 0$  for any term  $t(\vec{x})$ . In an unpublished note Boole used the torsion-free property to prove that  $(x+y)^2 = x+y$  implies xy = 0, forcing him to define addition only for disjoint classes. Hailpern also included the unnecessary axiom  $x^2 = 0$  implies x = 0, i.e., there are no nilpotent elements.

by Hailperin—it was implicitly replaced by the restriction of the quantifiers to idempotent elements.

#### 4. R01 and Horn sentences

Boole's Rule of 0 and 1, abbreviated as R01, was finally given its proper formulation in 2013 in [5]. Following Hailperin's lead, a first-order sentence  $\varphi$  will be said to hold in Boole's AoL if  $\varphi_E$  follows from the above-mentioned axioms of Hailperin.<sup>11</sup> By the completeness theorem for first-order logic this is equivalent to saying that  $\varphi_E$  holds in all rings satisfying Hailperin's axioms. Let **R** be such a ring, and let **R**<sub>E</sub> be the subring generated by the idempotents of **R**.<sup>12</sup> Clearly **R**  $\models \varphi_E$  iff **R**<sub>E</sub>  $\models \varphi_E$ . It is known that **R**<sub>E</sub> is isomorphic to a bounded Boolean power **Z**[**B**]<sup>\*</sup> of **Z** (see [9], [10]). From [4] we know that Horn sentences true in **Z** also hold in **Z**[**B**]<sup>\*</sup>, giving the following result.

## **Theorem 4.1.** A Horn sentence $\varphi$ holds in Boole's Algebra of Logic iff $\mathbf{Z} \models \varphi_{\mathrm{E}}$ .

Note that  $\mathbf{Z} \models \varphi_{\rm E}$  simply says that  $\varphi$  holds in the ring of integers when the variables are restricted to the values 0 and 1.<sup>13</sup> Since each of Boole's main theorems can be expressed as a Horn sentence  $\varphi$ , Theorem 4.1 readily proves that they hold in Boole's AoL.

## 5. FRANK BROWN'S PROTO-BOOLEAN ALGEBRAS

In his 2009 paper [2] Brown said that Boole did not base his AoL on signed multisets and rings but on the algebra of polynomials, and he set out to give an updated presentation of the latter. Brown worked with polynomials in which variables can only occur to the first power. He called these *proto-Boolean polynomials*, abbreviating the name to *p*-polynomials. The sum and difference of two *p*-polynomials is again a *p*-polynomial, but not in general their product. A new multiplication  $q \odot r$  is introduced for *p*-polynomials that takes the ordinary product  $q \cdot r$  and reduces to 1 all exponents greater than 1. For example,  $xy \odot (x+y) = 2xy$ . The new product of *p*-polynomials is again a *p*-polynomial.

Let P(X) be the set of *p*-polynomials with variables from X.<sup>14</sup> With the usual plus and minus operations along with the new multiplication one has a ring  $\mathbf{P}(X)$  that satisfies

<sup>&</sup>lt;sup>11</sup>Boole surely viewed his models as being *atomic*, that is, each non-empty class contains an atom (a singleton). So to extend Boole's theorems to first-order properties  $\varphi$  not considered by him, one needs to add an axiom expressing the atomic property to Hailperin's axioms.

<sup>&</sup>lt;sup>12</sup>For the ring  $\mathbf{Z}^U$  the subring generated by the idempotents consists of all members of  $Z^U$  with a finite range.

<sup>&</sup>lt;sup>13</sup>Thus Boole's AoL satisfies all the equations and quasi-equations that hold for the integers. Some of the more exciting quasi-equations that hold in the integers become quite trivial in Boole's AoL, for example,  $x^n + y^n = z^n \rightarrow xyz = 0$  for  $n \ge 3$  becomes  $x + y = z \rightarrow xyz = 0$  in Boole's AoL, a quasi-equation that R01 immediately validates.

<sup>&</sup>lt;sup>14</sup>Brown only considered finite X.

Hailperin's axioms—indeed it is isomorphic to the quotient ring  $\mathbf{Z}[X]/\mathscr{I}$  where  $\mathscr{I}$  is the ideal of  $\mathbf{Z}[X]$  generated by  $\{x^2 - x : x \in X\}$ . The isomorphism is given by mapping each *p*-polynomial *q* to the coset  $q + \mathscr{I}$ . This ring in turn is isomorphic to  $\mathbf{Z}^{2^n}$  if |X| = n.

Brown proved Boole's results hold in these rings of *p*-polynomials by proving the appropriate  $\varphi_E$  where the variables range over the idempotent polynomials in P(X). The connection with theorems about classes is not as obvious as when working with Hailperin's models  $\mathbf{Z}^U$ and letting the variables range over the idempotent elements, namely over the characteristic functions. Brown's connection was simply to point out that, for |X| = n, the Boolean Algebra of idempotents in the ring  $\mathbf{P}(X)$  is isomorphic to a Boolean algebra of  $2^{2^n}$  sets freely generated by *n* sets.

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Email address: snburris@uwaterloo.ca