Film Flow Over Heated Wavy Inclined Surfaces

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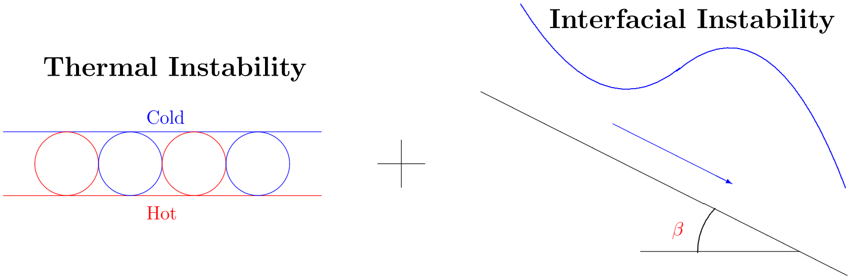
We consider the two-dimensional problem of gravity-driven laminar flow of a thin layer of fluid down a heated wavy inclined surface; this problem combines:

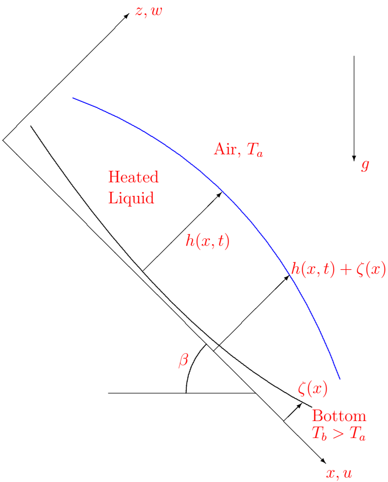


Previous studies

Flat bottom, variable temperature

- Thermocapillary effects were studied by the group: Demekhin, Kalliadasis, Kiyashko, Ruyer-Quil, Scheid, Trevelyan, Velarde, Zeytounian in a series of papers (JFM 2003, 2005, 2007)
  Mainly concerned with mathematical formulation and linear stability; also extended the weighted residual method to accommodate variable temperature
  The key finding is that heating destabilizes the flow, $Re_{crit} < 5\cot\beta/6$

Coordinate system



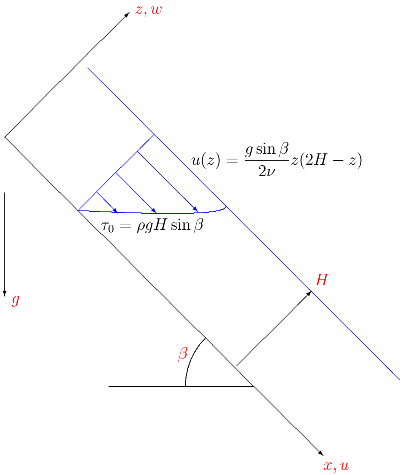
Variable surface tension:

$\sigma(T) = \sigma_0 - \gamma(T - T_a)$

Bottom profile:

$\zeta(x) = A_b \cos\left(\frac{2\pi x}{\lambda_b}\right)$

Scaling



Length scales:

H, λ_b

Velocity scale:

$U = Q/H$

$Q = \frac{g \sin\beta H^3}{3\nu}$

Temperature scale:

$\Delta T = T_b - T_a$

Pressure scale:

ρU^2

Time scale:

λ_b/U

Dimensionless parameters

$Re = \frac{Q}{\nu}$	Reynolds number
$We = \frac{\sigma_0 H}{\rho Q^2}$	Weber number
$Ma = \frac{\gamma H \Delta T}{\rho Q^2}$	Marangoni number
$Bi = \frac{\alpha_g H}{\rho c_p \kappa}$	Biot number
$Pr = \frac{\nu}{\kappa}$	Prandtl number
$\delta = \frac{H}{\lambda_b}$	Shallowness parameter
$a_b = \frac{A_b}{H}$	Bottom amplitude

Dimensionless equations

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\delta Re \frac{Du}{Dt} = -\delta Re \frac{\partial p}{\partial x} + 3 + \delta^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\delta^2 Re \frac{Dw}{Dt} = -Re \frac{\partial p}{\partial z} - 3 \cot\beta + \delta \frac{\partial^2 w}{\partial z^2}$$

$$\delta RePr \frac{DT}{Dt} = \delta^2 \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}$$

If $Re \sim O(1)$, then above equations represent a second-order approximation to the Navier-Stokes and energy equations with respect to δ

Interface conditions

Free surface conditions:

$$\left.\begin{aligned}
p &= \frac{2\delta}{Re}\frac{\partial w}{\partial z} - \delta^2(We - MaT)\frac{\partial^2 z_1}{\partial x^2} \\
\frac{\partial u}{\partial z} &= 4\delta^2\frac{\partial z_1}{\partial x}\frac{\partial u}{\partial x} - \delta^2\frac{\partial w}{\partial x} - MaRe\delta(\frac{\partial T}{\partial x} + \frac{\partial z_1}{\partial x}\frac{\partial T}{\partial z}) \\
\frac{\partial T}{\partial z} - \delta^2\frac{\partial z_1}{\partial x}\frac{\partial T}{\partial x} &= -BiT[1 + \frac{\delta^2}{2}(\frac{\partial z_1}{\partial x})^2] \\
w &= \frac{\partial h}{\partial t} + u\frac{\partial z_1}{\partial x}
\end{aligned}\right\} \text{at } z = z_1 \equiv \zeta(x) + h(x,t$$

Bottom boundary conditions:

$$u = w = 0\ ,\ \ T = 1\ \ \text{at}\ \ z = \zeta(x)$$

Weighted-residual method

First, solve for p from the z-momentum equation to obtain:

$$p = \frac{3\cot\beta}{Re}(z_1 - z) - \frac{\delta}{Re}\left.\frac{\partial u}{\partial x}\right|_{z=z_1} - \frac{\delta}{Re}\frac{\partial u}{\partial x} - \delta^2(We - MaT)\frac{\partial^2 z_1}{\partial x^2}$$

Then, substitute into x-momentum equation to eliminate pressure. Next, assume the following profiles for velocity and temperature:

$$u(x,z,t) = \frac{3q}{2h^3}W_1 + \frac{\delta MaRe}{4h}\frac{\partial\theta}{\partial x}W\ ,\ \ T = 1 + \frac{(\theta - 1)}{h}W_2$$

where $W_1 = (z-\zeta)(2h - z + \zeta)\ ,\ W_2 = (z-\zeta)\ ,$

$$W = (z-\zeta)(2h - 3z + 3\zeta)\ ,\ \theta = T(x, z = z_1, t)$$

Weighted-residual method

Now, multiply momentum equation by the weight function W_1 and the energy equation by the weight function W_2
Next, depth-integrate the equations to eliminate z-dependence and introduce new flow variables:

$$h(x,t)\ ,\ q(x,t)=\int_{\zeta}^{z_1} u dz\ ,\ \theta(x,t)=T(x,z=z_1,t)$$

The equations for h, q, θ then become:

Equations for *h*, *q*:

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0$$

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left[\frac{9}{7}\frac{q^2}{h} + \frac{5}{4}\frac{\cot\beta}{Re}h^2 + \frac{5}{4}Ma\theta\right] = \frac{q}{7h}\frac{\partial q}{\partial x} + \frac{5}{2\delta Re}\left(h - \frac{q}{h^2}\right)$$

$$+\frac{5}{6}\delta^2 Weh\left(\frac{\partial^3 h}{\partial x^3} + \zeta'''\right) - \frac{5\cot\beta}{2Re}\zeta' h + \frac{\delta}{Re}\left[\frac{9}{2}\frac{\partial^2 q}{\partial x^2} - \frac{9}{2h}\frac{\partial q}{\partial x}\frac{\partial h}{\partial x}\right.$$

$$\left. + \frac{4q}{h^2}\left(\frac{\partial h}{\partial x}\right)^2 - \frac{6q}{h}\frac{\partial^2 h}{\partial x^2} - \frac{5\zeta' q}{2h^2}\frac{\partial h}{\partial x} - \frac{15\zeta'' q}{4h} - \frac{5(\zeta')^2 q}{h^2}\right]$$

$$+\frac{\delta ReMa}{16}\left[\frac{h^2}{3}\frac{\partial^2\theta}{\partial x\partial t} + \frac{15hq}{14}\frac{\partial^2\theta}{\partial x^2} + \frac{19h}{21}\frac{\partial q}{\partial x}\frac{\partial\theta}{\partial x} + \frac{5q}{7}\frac{\partial h}{\partial x}\frac{\partial\theta}{\partial x}\right]$$

Equation for θ:

$$h\frac{\partial\theta}{\partial t}+\frac{27q}{20}\frac{\partial\theta}{\partial x}-\frac{7\theta}{40}(1-\theta)\frac{\partial q}{\partial x}=\frac{3}{\delta RePrh}\left[1-\theta(1+Bih)\right]$$

$$+\frac{\delta}{RePr}\left[(1-\theta)\frac{\partial^2 h}{\partial x^2}+h\frac{\partial^2\theta}{\partial x^2}+\frac{\partial h}{\partial x}\frac{\partial\theta}{\partial x}-\left(\frac{3Bi\theta}{2}+\frac{2(1-\theta)}{h}\right)\left(\frac{\partial h}{\partial x}\right)^2\right.$$

$$\left.-3\zeta'\left(\frac{(1-\theta)}{h}+Bi\theta\right)\frac{\partial h}{\partial x}+\frac{3\zeta''}{2}(1-\theta)-\frac{3Bi(\zeta')^2\theta}{2}\right]$$

$$+\frac{3\delta ReMa}{80}\left[2h^2\left(\frac{\partial\theta}{\partial x}\right)^2-h^2(1-\theta)\frac{\partial^2\theta}{\partial x^2}-2h(1-\theta)\frac{\partial h}{\partial x}\frac{\partial\theta}{\partial x}\right]$$

Linear stability with $a_b = 0$

The instability threshold can be determined from the Benney equation describing the evolution of the free surface

Proceed by expanding u, w, p and T in powers of δ:

$$u = u_0 + \delta u_1 + \cdots \;, \; w = w_0 + \delta w_1 + \cdots$$

$$p = p_0 + \delta p_1 + \cdots \;, \; T = T_0 + \delta T_1 + \cdots$$

Substituting these into the governing equations then leads to a hierarchy of problems at various orders

At each order n the quantities u_n, w_n, p_n and T_n can be found by applying the boundary conditions (also expanded in δ)

Evaluating these expressions at $z = h + \zeta$ and inserting them into the kinematic condition yields to first-order

Linear stability with $a_b = 0$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(h^3\right) + \delta\frac{\partial}{\partial x}\left[\frac{6Re}{5}h^6\frac{\partial h}{\partial x} + \frac{ReMaBi}{2}\frac{h^2}{(1+Bih)^2}\frac{\partial h}{\partial x}\right.$$

$$\left. -\cot\beta h^3\left(\frac{\partial h}{\partial x}+\zeta'\right) + \frac{\delta^2 WeRe}{3}h^3\left(\frac{\partial^3 h}{\partial x^3}+\zeta'''\right)\right] = 0$$

Next, linearize using $h = 1 + \hat{h}$, set $\zeta = 0$ for a flat bottom, and introduce the perturbation $\hat{h} = h_0 e^{ikx} e^{\sigma t}$ to obtain:

$$Re_{crit}^{even} = \frac{10(1+Bi)^2\cot\beta}{5MaBi + 12(1+Bi)^2}$$

Linear stability: $a_b \neq 0$ case

The steady-state solutions are: $q_s = 1, h_s(x), \theta_s(x)$ where $h_s(x), \theta_s(x)$ satisfy

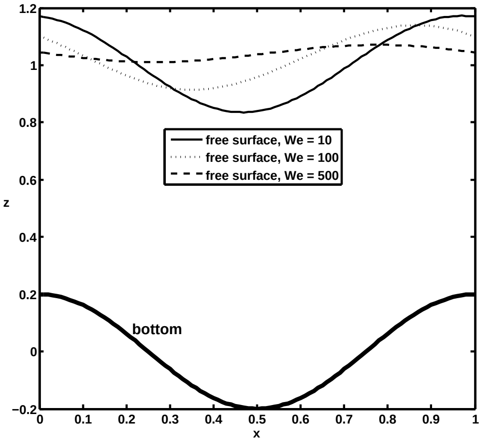
$$\frac{5\delta^2 We}{6} h_s^3 h_s''' - \frac{6\delta}{Re} h_s h_s'' + \frac{4\delta}{Re}(h_s')^2 - \frac{15\delta}{4Re}\zeta'' h_s - \frac{5Ma}{4}\theta_s' h_s^2$$
$$- \left[\frac{5\cot\beta}{2Re} h_s^3 + \frac{5\delta}{2Re}\zeta' - \frac{5\delta ReMa}{112} h_s^2 \theta_s' - \frac{9}{7}\right] h_s'$$
$$+ \left[\frac{15\delta ReMa}{224}\theta_s'' + \frac{5}{2\delta Re} - \frac{5\cot\beta}{2Re}\zeta' + \frac{5\delta^2 We}{6}\zeta'''\right] h_s^3$$
$$= \frac{5}{2\delta Re} + \frac{5\delta}{Re}(\zeta')^2$$

Linear stability: $a_b \neq 0$ case

$$
\begin{aligned}
&\left(\frac{\delta}{RePr} + \frac{3\delta ReMa}{80} h_s(\theta_s - 1)\right) h_s^2 \theta_s'' + \frac{3\delta ReMa}{40} h_s^3 (\theta_s')^2 \\
&+ \left[\frac{3\delta ReMa}{40} h_s h_s' (\theta_s - 1) + \frac{\delta}{RePr} h_s' - \frac{27}{20}\right] h_s \theta_s' \\
&+ \left[\frac{2\delta}{RePr}(h_s')^2 - \frac{3}{\delta RePr}(1 + Bih_s) + \frac{3\delta}{RePr}\zeta' h_s' \right.\\
&\left. - \frac{\delta}{RePr}\left(\frac{3}{2}\zeta'' + h_s''\right) h_s - \frac{3Bi\delta}{2RePr} h_s (\zeta' + h_s')^2 \right] \theta_s \\
&= -\frac{3}{\delta RePr} - \frac{\delta}{RePr} h_s \left(\frac{3}{2}\zeta'' + h_s''\right)
\end{aligned}
$$

The MATLAB routine bvp4c was used to solve these ODEs

Periodic steady-state solutions



Free surface with bottom profile for $Re = 0.5$, $\cot\beta = 0.5$, $a_b = 0.2$, $\delta = 0.1$, $Ma = 1$, $Bi = 1$ and $Pr = 7$ with $We = 10, 100, 500$

Linear stability: $a_b \neq 0$ case

To study how small disturbances will evolve, introduce perturbations $\hat{h}, \hat{q}, \hat{\theta}$ superimposed on the steady-state solutions and linearize equations using

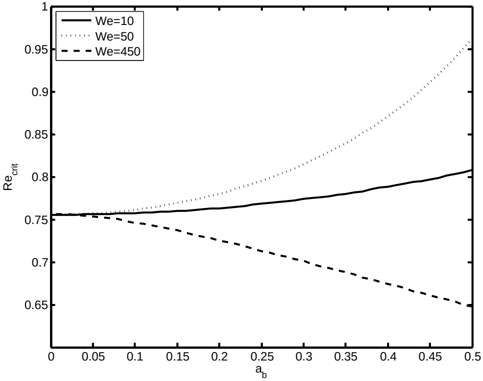
$$h = h_s(x) + \hat{h}\ , \ \ q = 1 + \hat{q}\ , \ \ \theta = \theta_s(x) + \hat{\theta}$$

For a wavy bottom, the coefficients in the linearized equations are periodic functions; hence apply Floquet theory and express the perturbations as follows

$$\hat{h} = e^{\sigma t} e^{iKx} \sum_{n=-\infty}^{\infty} \hat{h}_n e^{i2n\pi x}\ , \ \ \hat{q} = e^{\sigma t} e^{iKx} \sum_{n=-\infty}^{\infty} \hat{q}_n e^{i2n\pi x}\ ,$$

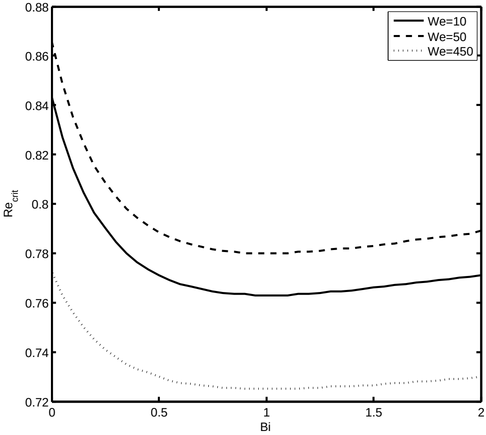
$$\hat{\theta} = e^{\sigma t} e^{iKx} \sum_{n=-\infty}^{\infty} \hat{\theta}_n e^{i2n\pi x}$$

Numerical linear stability results for $a_b \neq 0$



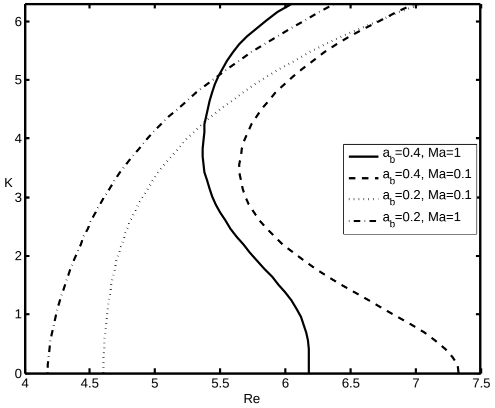
Critical Reynolds number as a function of bottom amplitude with $\cot\beta = 1$, $\delta = 0.05$, $Ma = 1$, $Bi = 1$ and $Pr = 7$ for $We = 10, 50, 450$

Numerical linear stability results for $a_b \neq 0$



Critical Reynolds number as a function of Biot number with $a_b = 0.2$, $\delta = 0.05$, $\cot\beta = 1$, $Ma = 1$ and $Pr = 7$

Numerical linear stability results for $a_b \neq 0$



Neutral stability curves for the case $\delta = 0.05$, $\cot\beta = 5$, $We = 5$, $Bi = 1$ and $Pr = 7$

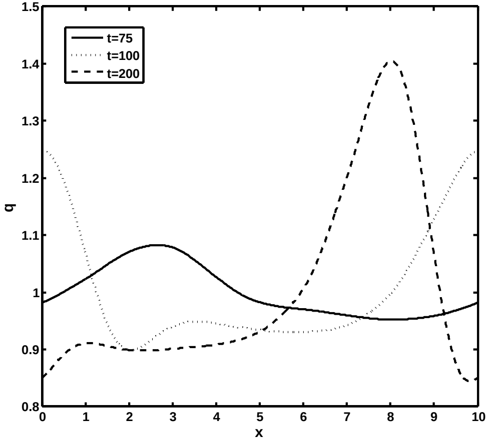
Comparisons

Experiments of Wierschem *et al.* (Acta Mech., 2005)
Contrasted below are Re_{crit} values with $Re_{crit}^{flat} = \frac{5}{6}\cot\beta$:

β	Re_{crit}^{flat}	Re_{crit}^{wavy} Experimental	Numerical	Theoretical
15°	3.3	5.1 ± 0.4	(5.5,5.6)	5.6
30°	1.4	2.2 ± 0.2	(1.8,1.9)	1.7
40.7°	0.97	1.3 ± 0.1	(1.1,1.2)	1.1

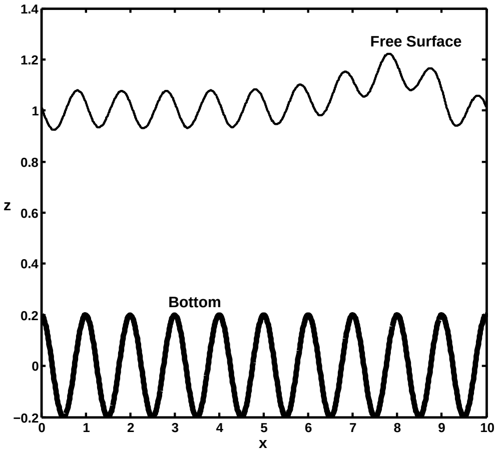
Comparison between experimental and numerical values of Re_{crit} for $a_b = 0.5, \delta = 0.1, We = 0$

Non-isothermal Simulation



Parameters: $Re = 1$, $a_b = 0.2, \delta = 0.1$, $\cot\beta = 0.5$, $Bi = 1$, $Ma = 1$, $Pr = 7$, $We = 100$ and $L = 10$

Non-isothermal Simulation



Parameters: $Re = 1$, $a_b = 0.2, \delta = 0.1$, $\cot\beta = 0.5$, $Bi = 1$, $Ma = 1$, $Pr = 7$, $We = 100$ and $L = 10$ at $t = 200$

Concluding remarks

- A mathematical model based on weighted residuals to simulate the flow over heated wavy inclined surfaces was presented
- Good agreement with experimental data was found
- The threshold for instability for a flat bottom is given by:

$$Re_{crit}^{even} = \frac{10(1+Bi)^2 \cot\beta}{5MaBi + 12(1+Bi)^2}$$

- Heating destabilizes the flow
- Bottom topography can either stabilize or destabilize the flow depending on surface tension