The effects of variable fluid properties on thin film stability

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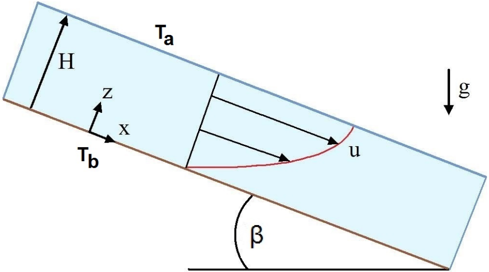
Problem description

The stability of two-dimensional laminar flow of a thin fluid layer down a heated inclined surface has been investigated



Previous work

- Isothermal case: Benjamin (1957), Yih (1963) and Benney (1966)
- Non-isothermal case: variable surface tension - Trevelyan *et al.* (2007) & D'Alessio *et al.* (2010)
  variable viscosity - Goussis & Kelly (1985) and Hwang & Weng (1988)
  variable surface tension & visocosity - Kabova & Kuznetsov (2002)
- Pascal *et al.* (2013) considered variable density, viscosity, surface tension, thermal conductivity and specific heat for small parameter variations

Governing equations

In the absence of viscous dissipation, the governing equations for a fluid possessing variable fluid properties are given by:

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = 0$$

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + g\rho\sin\beta + \frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - g\rho\cos\beta + \frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]$$

$$\rho\frac{D(c_p T)}{Dt} = \frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right) - p\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)$$

Governing equations

Allow the fluid properties to vary linearly with temperature as follows (in dimensionless form):

$$\frac{\rho}{\rho_0} = 1 - \alpha T \quad \text{denisity}$$
$$\frac{\mu}{\mu_0} = 1 - \lambda T \quad \text{viscosity}$$
$$\frac{c_p}{c_{p0}} = 1 + ST \quad \text{specific heat}$$
$$\frac{K}{K_0} = 1 + \Lambda T \quad \text{thermal conductivity}$$
$$\frac{\sigma}{\sigma_0} = 1 - \gamma T \quad \text{surface tension}$$

Here, α, γ, λ, Λ, S are positive dimensionless parameters measuring the rate of change with respect to temperature and ρ_0, μ_0, c_{p0}, K_0, σ_0 represent values at the reference temperature T_a (or $T = 0$ in dimensionless form).

Dimensionless equations

To cast the equations in dimensionless form the following scales are used. The length scale is the Nusselt thickness given by:

$$H = \left(\frac{3\mu_0 Q}{g \rho_0 \sin \beta} \right)^{1/3}$$

corresponding to a uniform steady isothermal flow.
Here, Q is the constant volume flux.
The pressure scale is $\rho_0 U^2$ with $U = Q/H$.
The time scale is H/U.
The temperature scale is $\Delta T = T_b - T_a$.

Dimensionless equations

Using the Boussinesq approximation and in dimensionless form the equations become:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$Re\frac{Du}{Dt} = -Re\frac{\partial p}{\partial x} + 3(1-\alpha T) + \frac{\partial}{\partial x}\left((1-\lambda T)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial z}\left((1-\lambda T)\frac{\partial u}{\partial z}\right) \\ -\lambda\frac{\partial T}{\partial x}\frac{\partial u}{\partial x} - \lambda\frac{\partial T}{\partial z}\frac{\partial w}{\partial x}$$

$$Re\frac{Dw}{Dt} = -Re\frac{\partial p}{\partial z} - 3\cot\beta(1-\alpha T) + \frac{\partial}{\partial x}\left((1-\lambda T)\frac{\partial w}{\partial x}\right) \\ +\frac{\partial}{\partial z}\left((1-\lambda T)\frac{\partial w}{\partial z}\right) - \lambda\frac{\partial T}{\partial x}\frac{\partial u}{\partial z} - \lambda\frac{\partial T}{\partial z}\frac{\partial w}{\partial z}$$

$$PrRe\frac{D}{Dt}\left[(1+S/\Delta T_r)T + ST^2\right] = \frac{\partial}{\partial x}\left[(1+\Lambda T)\frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial z}\left[(1+\Lambda T)\frac{\partial T}{\partial z}\right]$$

Boundary conditions

The dynamic conditions along the free surface, $z = h(x,t)$, are:

$$p = \frac{2(1-\lambda T)}{ReF}\left(\left[\frac{\partial h}{\partial x}\right]^2 \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} - \frac{\partial h}{\partial x}\frac{\partial u}{\partial z} - \frac{\partial h}{\partial x}\frac{\partial w}{\partial x}\right) - \frac{(We-MaT)}{F^{3/2}}\frac{\partial^2 h}{\partial x^2}$$

$$-MaRe\sqrt{F}\left(\frac{\partial T}{\partial x} + \frac{\partial h}{\partial x}\frac{\partial T}{\partial z}\right) = (1-\lambda T)\left[G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) - 4\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}\right]$$

$$\text{where } F = 1 + \left[\frac{\partial h}{\partial x}\right]^2, \; G = 1 - \left[\frac{\partial h}{\partial x}\right]^2$$

Based on Newton's Law of Cooling, the heat transfer across the free surface can be expressed as:

$$-Bi\sqrt{F}T = (1+\Lambda T)\left(\frac{\partial T}{\partial z} - \frac{\partial h}{\partial x}\frac{\partial T}{\partial x}\right)$$

Boundary conditions

The kinematic condition along the free surface is given by:

$$w = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}$$

Lastly, the bottom temperature, no-slip and impermeability conditions are:

$$T = 1 \text{ at } z = 0$$

$$u = w = 0 \text{ at } z = 0$$

Dimensionless parameters

$Re = \dfrac{\rho_0 UH}{\mu_0}$	Reynolds number
$We = \dfrac{\sigma_0}{\rho_0 U^2 H}$	Weber number
$Ma = \dfrac{\gamma \Delta T}{\rho_0 U^2 H}$	Marangoni number
$Pr = \dfrac{\mu_0 c_{p0}}{K_0}$	Prandtl number
$Bi = \dfrac{\alpha_g H}{K_0}$	Biot number
$\Delta T_r = \dfrac{T_b - T_a}{T_a}$	Relative Temperature Difference

Also, $\alpha, \lambda, \Lambda, S$ represent dimensionless rates of change of density, viscosity, thermal conductivity and specific heat with respect to temperature.

Steady-state equations

Steady uniform flow in the streamwise direction is given by $h \equiv 1$, $w \equiv 0$, $u = u_s(z)$, $p = p_s(z)$, $T = T_s(z)$ and satisfies the following boundary-value problems ($D \equiv d/dz$):

$$D[(1+\Lambda T_s)DT_s] = 0\ ,\ (1+\Lambda T_s)DT_s + BiT_s = 0 \ \text{ at }\ z = 1\ ,\ T_s(0) = 1$$

$$D[(1-\lambda T_s)Du_s] + 3(1-\alpha T_s) = 0\ ,\ Du_s = 0 \ \text{ at }\ z = 1\ ,\ u_s(0) = 0$$

$$ReDp_s = -3\cot\beta(1-\alpha T_s)\ ,\ p_s(1) = 0$$

Steady-state solutions

The steady-state solutions are given by:

$$T_s(z) = \sqrt{a - bz} - \frac{1}{\Lambda}$$

$$u_s(z) = a_0 \ln\left(\frac{A - \lambda\sqrt{a - bz}}{A - \lambda\sqrt{a}}\right) + a_1 z - \frac{\alpha}{\lambda} z^2 + a_2(\sqrt{a - bz} - \sqrt{a})$$

$$+ a_3[(a - bz)^{3/2} - a^{3/2}]$$

$$p_s(z) = \frac{3\cot\beta}{Re}\left(1 + \frac{\alpha}{\Lambda}\right)(1 - z) + \frac{2\alpha\cot\beta}{bRe}[(a - b)^{3/2} - (a - bz)^{3/2}]$$

where the constants a, b, a_0, a_1, a_2, a_3 and A are related to the parameters Λ, Bi, λ and α.

Steady-state solutions

Some special cases:

$$\Lambda = 0 \ , \ \ T_s(z) = 1 - \frac{Bi}{1+Bi} z$$

For $Bi = 0$, $T_s(z) = 1$ while as $Bi \to \infty$, $T_s(z) \to 1 - z$

For $Bi = 0$, $u_s(z) = 3\left(\frac{1-\alpha}{1-\lambda}\right) z \left(1 - \frac{z}{2}\right)$

Perturbation equations

Next impose small disturbances on the steady-state flow:

$$u = u_s(z) + \tilde{u}(x, z, t)\ ,\ \ w = \tilde{w}(x, z, t)\ ,\ \ p = p_s(z) + \tilde{p}(x, z, t)$$

$$T = T_s(z) + \tilde{T}(x, z, t)\ ,\ \ h = 1 + \eta(x, t)$$

Substitute these into the governing equations, linearize and assume the disturbances have the form:

$$(\tilde{u}, \tilde{w}, \tilde{p}, \tilde{T}, \eta) = (\hat{u}(z), \hat{w}(z), \hat{p}(z), \hat{T}(z), \hat{\eta})e^{ik(x-ct)}$$

where k (real & positive) represents the wavenumber of the perturbation and c is a complex quantity with the real part denoting the phase speed of the perturbation while the imaginary part is related to the growth rate.

Perturbation equations

The linearized perturbation equations become:

$$D\hat{w} + ik\hat{u} = 0$$

$$Re[ik(u_s - c)\hat{u} + \hat{w}Du_s] = -ikRe\hat{p} + k^2(\lambda T_s - 1)\hat{u}$$
$$+D[(1-\lambda T_s)D\hat{u}] - \lambda \hat{T} D^2 u_s - \lambda Du_s D\hat{T} - ik\lambda \hat{w} DT_s - 3\alpha \hat{T}$$

$$ikRe(u_s - c)\hat{w} = -ReD\hat{p} + 3\alpha \cot\beta \, \hat{T} - k^2(1-\lambda T_s)\hat{w}$$
$$+D[(1-\lambda T_s)D\hat{w}] - ik\lambda \hat{T} Du_s - \lambda DT_s D\hat{w}$$

$$PrRe(1 + S/\Delta T_r + 2ST_s)[ik(u_s - c)\hat{T} + \hat{w}DT_s]$$
$$= -k^2(1+\Lambda T_s)\hat{T} + D^2[(1+\Lambda T_s)\hat{T}]$$

Boundary conditions

Along the free surface $(z = 1)$ the perturbations satisfy:

$$\hat{p} = -\hat{\eta} D p_s + \frac{2}{Re}(1 - \lambda T_s) D\hat{w} + k^2(We - Ma T_s)\hat{\eta}$$

$$(1 - \lambda T_s)(\hat{\eta} D^2 u_s + D\hat{u} + ik\hat{w}) = -ikMaRe(\hat{T} + \hat{\eta} D T_s)$$

$$D[(1 + \Lambda T_s)\hat{T}] + \hat{\eta} D[(1 + \Lambda T_s) D T_s + Bi T_s] + Bi\hat{T} = 0$$

$$\hat{w} = ik(u_s - c)\hat{\eta}$$

while along the bottom the conditions are:

$$\hat{u}(0) = \hat{w}(0) = \hat{T}(0) = 0$$

Small wavenumber expansion

Recall that for isothermal flow small wavenumber perturbations are the most unstable. Assume this is also true for non-isothermal flow and expand the perturbations in the following series:

$$\hat{u} = u_0(z) + ku_1(z) + O(k^2)$$

$$\hat{w} = w_0(z) + kw_1(z) + O(k^2)$$

$$\hat{p} = p_0(z) + kp_1(z) + O(k^2)$$

$$\hat{T} = T_0(z) + kT_1(z) + O(k^2)$$

$$\hat{\eta} = \eta_0 + k\eta_1 + O(k^2)$$

$$c = c_0 + kc_1 + O(k^2)$$

This leads to a hierarchy of problems at various orders of k.

Neutral stability

With the help of the Maple Computer Algebra System an exact, but lenghty, expression for the critical Reynolds number, Re_{crit}, has been found having the functional form

$$Re^*_{crit} = f(\alpha, \lambda, \Lambda, \Delta T_r, S, Pr, Ma, Bi) \text{ where } Re^*_{crit} = \frac{Re_{crit}}{\cot\beta}$$

which predicts the onset of instability. With no heating the isothermal result, $Re^*_{crit} = 5/6$, is recovered. Re^*_{crit} does not depend on We (as is the case with isothermal flow).

Special case

For $Bi = 0$ the critical Reynolds number is given by

$$Re^*_{crit} = \frac{5}{6}\frac{(1-\lambda)^2}{(1-\alpha)}$$

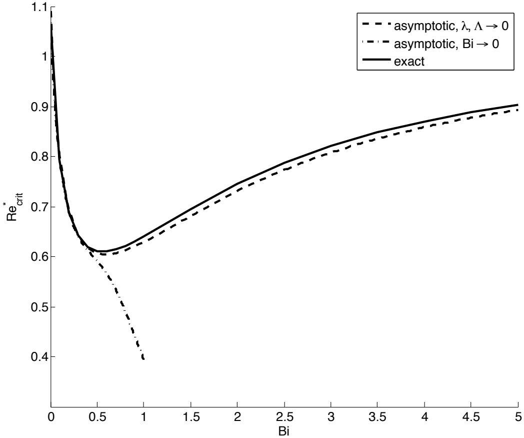
The dependence of Re^*_{crit} on α, λ can be explained by examining how the flow rate, Q, is influenced. Recall that

$$Q = \frac{\rho g \sin\beta H^3}{3\mu}$$

for steady uniform isothermal flow.

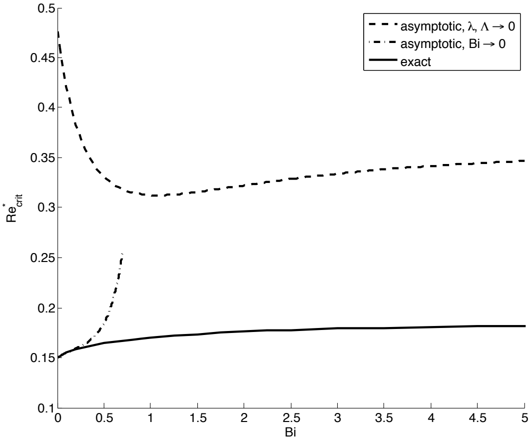
Comparisons with previous research

Comparison between our exact results and the asymptotic expansions of Pascal *et al.* (2013). Parameter values:
$\alpha = 0.5, \lambda = 0.2,$
$\Lambda = 0.25, S = 1,$
$Ma = \Delta T_r = 1,$
$Pr = 7$



Comparisons with previous research

Comparison between our exact results and the asymptotic expansions of Pascal *et al.* (2013). Parameter values: $\alpha = 0.5, \lambda = 0.7$, $\Lambda = 0.7, S = 1$, $Ma = \Delta T_r = 1$, $Pr = 7$



Comparisons with previous research

With $\lambda = \Lambda = \alpha = S = 0$ the expression for Re^*_{crit} becomes:

$$Re^*_{crit} = \frac{10(1+Bi)^2}{5Ma\,Bi + 12(1+Bi)^2}$$

which coincides with the result obtained by D'Alessio *et al.* (2010). Thus, thermocapillarity is destabilizing as Re^*_{crit} decreases with Ma. The scaled critical Reynolds number attains a minimum at $Bi = 1$, given by $Re^*_{crit,min} = 40/(48 + 5Ma)$. The limit as Bi tends to infinity is equal to the value at $Bi = 0$ which is given by $Re^*_{crit} = 5/6$ and corresponds to the isothermal case.

Variation in density

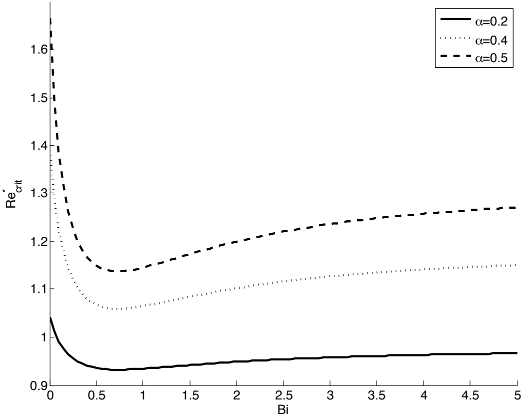
For $0 < Bi < \infty$ a negative temperature gradient within the fluid layer ensues which establishes a gravitationally unstable top-heavy density gradient within the fluid layer. Increasing α triggers competing stability mechanisms which is reflected in the Rayleigh number, Ra:

$$Ra = \frac{\hat{\alpha} g \cos\beta H^3 \Delta T}{\left(\frac{\mu}{\rho}\right)\left(\frac{K}{\rho c_p}\right)} = 3\rho_0 \cot\beta \left(\frac{\alpha \rho c_p Q}{K}\right)$$

Increasing α does not necessarily increase Ra since both ρ and Q decrease while holding c_p, K constant.

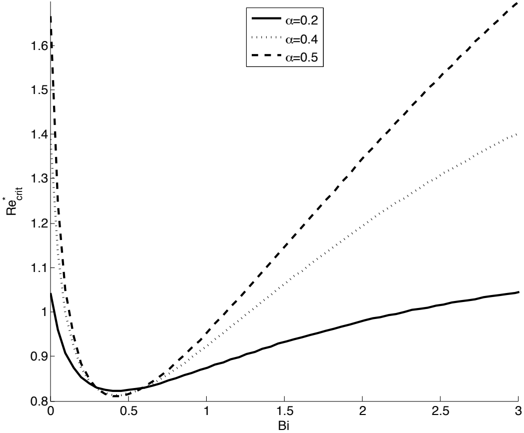
Variation in density

Parameter values:
$\lambda = \Lambda = S = Ma = 0$,
$\Delta T_r = 1, Pr = 7$



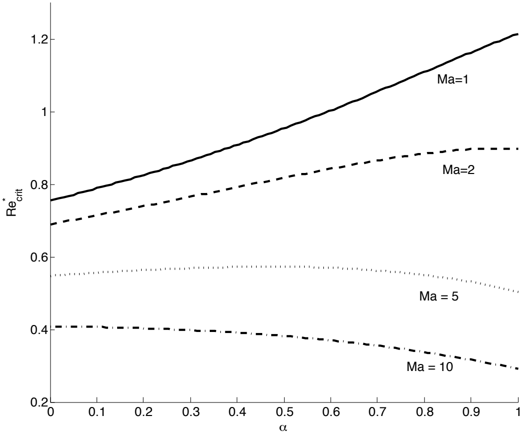
Variation in density

Parameter values:
$\lambda = \Lambda = S = Ma = 0$,
$\Delta T_r = 1, Pr = 30$



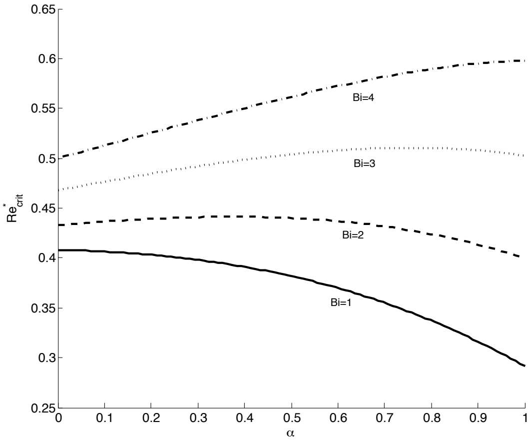
Marangoni effect

Parameter values:
$\lambda = \Lambda = S = 0$,
$Bi = 1, \Delta T_r = 1$,
$Pr = 7$



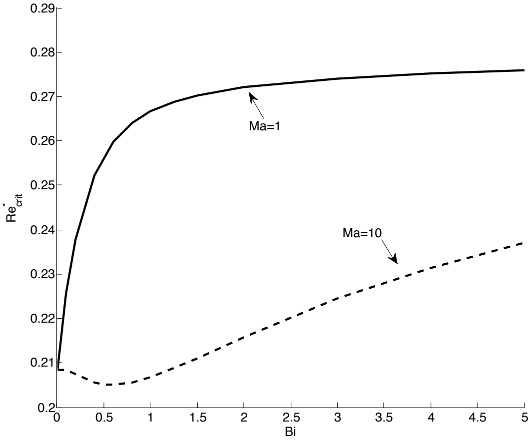
Marangoni effect

Parameter values:
$\lambda = \Lambda = S = 0$,
$Ma = 10, \Delta T_r = 1$,
$Pr = 7$



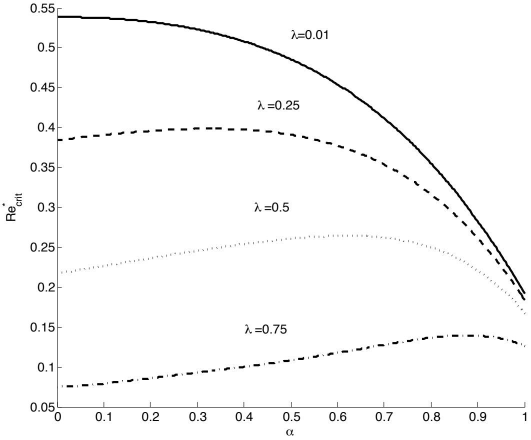
Variation in viscosity

Parameter values:
$\alpha = \Lambda = S = 0$,
$\lambda = 0.5, \Delta T_r = 1$,
$Pr = 7$



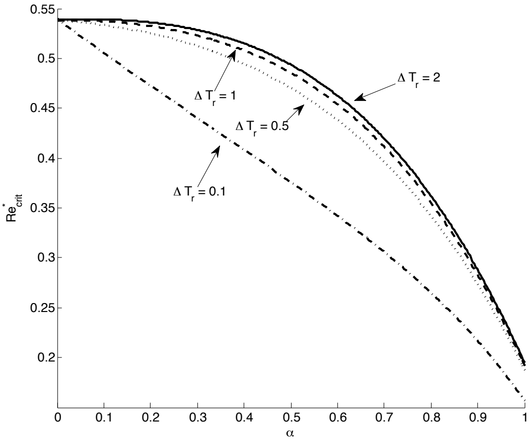
Variation in viscosity

Parameter values:
$\Lambda = 0.5, S = 1,$
$Bi = 0.25, Ma = 10,$
$\Delta T_r = 1, Pr = 7$



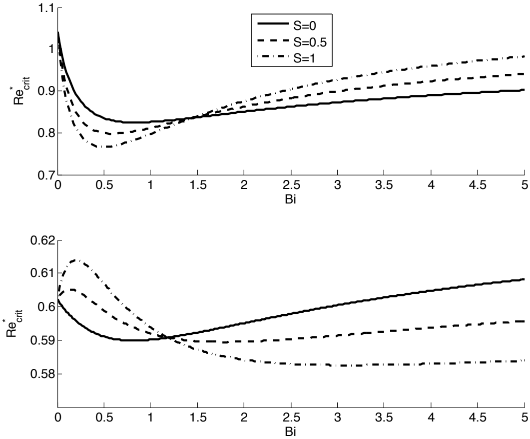
Variation in ΔT_r

Parameter values:
$\lambda = 0, \Lambda = 0.5, S = 1,$
$Ma = 10, Bi = 0.25,$
$Pr = 7$



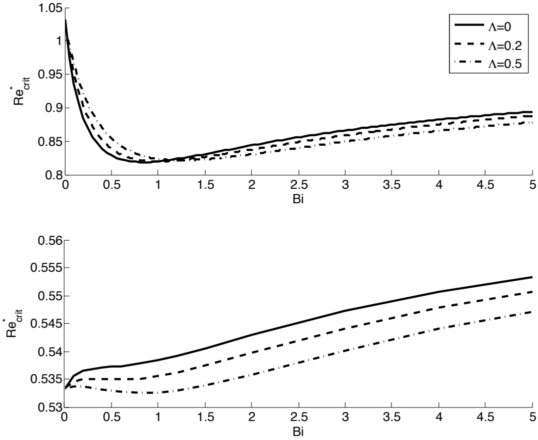
Variation in S

Parameter values:
$\Lambda = 0, Ma = 1,$
$\Delta T_r = 1, Pr = 7$
Top panel:
$\alpha = 0.2, \lambda = 0$
Bottom panel:
$\alpha = 0, \lambda = 0.15$



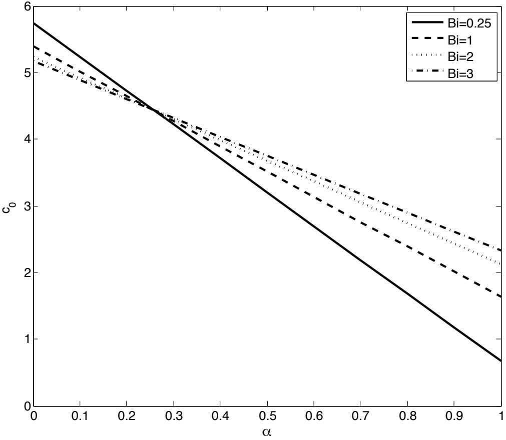
Variation in Λ

Parameter values:
$S = 0, Ma = 1,$
$\Delta T_r = 1, Pr = 7$
Top panel:
$\alpha = 0.2, \lambda = 0$
Bottom panel:
$\alpha = 0, \lambda = 0.2$



Variation in perturbation phase speed, c_0

Parameter values:
$\lambda = \Lambda = 0.5$
c_0 does not depend on $Ma, S, \Delta T_r, Pr$ and varies linearly with α



Variation in perturbation phase speed, c_0

Parameter values:
$\lambda = 0.5, \Lambda = 0.7$
For isothermal flow
$c_0 = 3$