Sheared Flow Past A Cylinder

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TALK OUTLINE

- Introduction
- Governing Equations
- Analytical Solution Procedure
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- Results & Comparisons
- Conclusions

COLLABORATORS

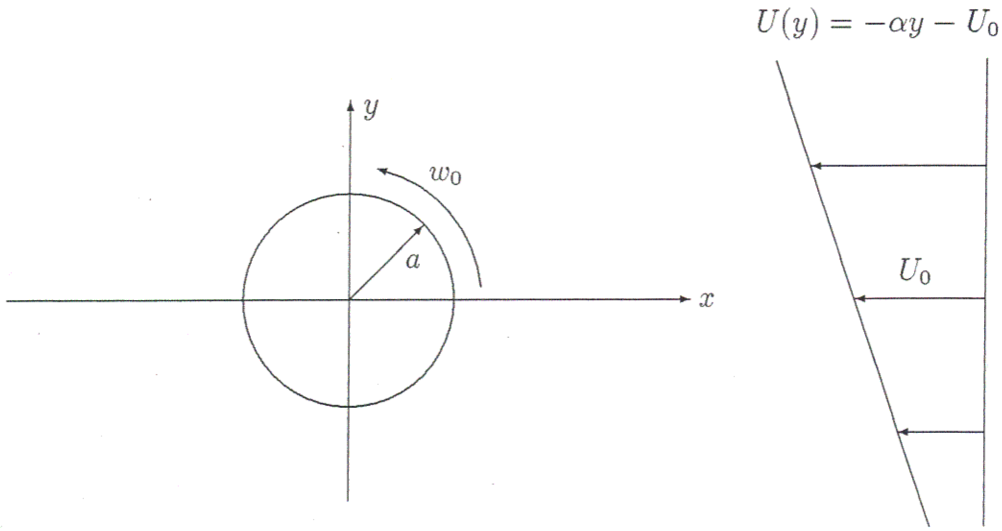
1) Prof. Kenzu Abdella (Trent U.): assisted in the numerical solution

2) Dr. Katrin Rohlf (U. of Toronto): generated the analytical solution

3) Mr. Alex Korobov (U. of Waterloo): produced the animated flow patterns

INTRODUCTION

We consider the unsteady two-dimensional problem of uniform shear flow of a viscous incompressible fluid past a rotating circular cylinder shown below:



The problem can be completely characterized by the following dimensionless parameters:

$R = 2aU_0/\nu$	Reynolds Number
$\Omega = a\omega_0/U_0$	Angular Velocity
$K = a\alpha/U_0$	Shear Parameter
$V = 0, \pm 1$	Centre-Line Velocity

MOTIVATION

The far-field flow approaching a cylindrical body is seldom exactly uniform but rather slightly sheared. In practice, structures are usually immersed within a boundary layer possessing shear.
This flow configuration can serve to model:

- offshore pipelines near the seabed,
- suspended cables in the atmosphere, and the
- effects of shear on hydrodynamic forces and vortex shedding.

PREVIOUS INVESTIGATIONS

Analytical (Stokes flow):
Bretherton *JFM* (1962),
Robertson & Acrivos *JFM* (1970)

Numerical:
Kossack & Acrivos *JFM* (1974),
Chew, Luo & Cheng *J Wind Eng Ind Aerodyn* (1997),
Lei, Cheng & Kavanagh *Ocean Eng* (2000),
Xu & Dalton *J Fluids Struct* (2001),
Mukhopadhyay, Venugopal & Vanka *Comp Fluids* (2002)

PRESENT STUDY

The main contributions of the present research are two fold:

1) An analytical solution valid for large R and small t is derived. Currently, only Stokes flow solutions (small R) exist. A multiple series expansion method is implemented to determine an approximate solution in the large R, small t limit.

2) An efficient numerical technique for solving the Navier-Stokes equations is proposed. Previous numerical studies are either of finite difference or hybrid vortex type. The method used in this study is a spectral - finite difference scheme and is designed to capture the early development of the flow.

A detailed comparison between the two methods of solution will also be carried out.

GOVERNING EQUATIONS

In terms of modified polar coordinates (ξ, θ) where $\xi = \ln(r/a)$ the dimensionless Navier-Stokes equations expressed in a stream function - vorticity formulation become:

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \theta^2} = e^{2\xi} \zeta$$

$$e^{2\xi} \frac{\partial \zeta}{\partial t} = \frac{\partial \psi}{\partial \theta} \frac{\partial \zeta}{\partial \xi} - \frac{\partial \psi}{\partial \xi} \frac{\partial \zeta}{\partial \theta} + \frac{2}{R} \left(\frac{\partial^2 \zeta}{\partial \xi^2} + \frac{\partial^2 \zeta}{\partial \theta^2} \right)$$

BOUNDARY CONDITIONS

These include the no-slip and impermeability conditions:

$$\psi = 0 \ , \ \frac{\partial \psi}{\partial \xi} = \Omega \text{ on } \xi = 0$$

and the far-field conditions:

$$\psi \to \frac{K}{4} e^{2\xi} + V e^{\xi} \sin\theta - \frac{K}{4} e^{2\xi} \cos 2\theta \ , \ \zeta \to K \text{ as } \xi \to \infty$$

INTEGRAL CONDITIONS

A straight-forward application of Green's second identity yields:

$$\frac{1}{\pi} \int_0^{\infty} \int_0^{2\pi} e^{(2-n)\xi} \zeta \sin(n\theta) d\theta d\xi = 2V \delta_{1,n} \ , \ n = 1, 2, \cdots$$

$$\frac{1}{\pi} \int_0^{\infty} \int_0^{2\pi} e^{(2-n)\xi} \zeta \cos(n\theta) d\theta d\xi = -K \delta_{2,n} \ , \ n = 1, 2, \cdots$$

$$\frac{1}{\pi} \int_0^{\xi_\infty} \int_0^{2\pi} e^{2\xi} \zeta d\theta d\xi = K e^{2\xi_\infty} - 2\Omega$$

SWITCH TO BOUNDARY-LAYER VARIABLES

The early stages of the flow is best described by rescaling the coordinate ξ and flow variables ψ, ζ as follows:

$$\xi = \lambda z \ , \ \psi = \lambda \Psi \ , \ \zeta = \omega / \lambda \ , \ \lambda = \sqrt{\frac{8t}{R}}$$

TRANSFORMED EQUATIONS

$$\frac{\partial^2 \Psi}{\partial z^2} + \lambda^2 \frac{\partial^2 \Psi}{\partial \theta^2} = \mathrm{e}^{2\lambda z} \omega$$

$$\frac{\partial^2 \omega}{\partial z^2} + 2\mathrm{e}^{2\lambda z} \left(z \frac{\partial \omega}{\partial z} + \omega \right) = 4t\mathrm{e}^{2\lambda z} \frac{\partial \omega}{\partial t} - \lambda^2 \frac{\partial^2 \omega}{\partial \theta^2}$$
$$-4t \left(\frac{\partial \Psi}{\partial \theta} \frac{\partial \omega}{\partial z} - \frac{\partial \Psi}{\partial z} \frac{\partial \omega}{\partial \theta} \right)$$

ANALYTICAL SOLUTION PROCEDURE

For large R and small t, λ is also small, and it is possible to expand the variables in a double series in both λ and t as follows:

$$\Psi = \Psi_{00} + t\Psi_{01} + \lambda(\Psi_{10} + t\Psi_{11}) + \cdots$$

$$\omega = \omega_{00} + t\omega_{01} + \lambda(\omega_{10} + t\omega_{11}) + \cdots$$

This leads to a hierarchy of problems at various orders. For example, the unsteady boundary-layer equations emerge in the limit as $R \to \infty$ (or $\lambda = 0$):

$$\frac{\partial^2 \Psi_0}{\partial z^2} = \omega_0$$

$$\frac{\partial^2 \omega_0}{\partial z^2} + 2z\frac{\partial \omega}{\partial z} + 2\omega_0 = 4t\frac{\partial \omega_0}{\partial t} - 4t\left(\frac{\partial \Psi_0}{\partial \theta}\frac{\partial \omega_0}{\partial z} - \frac{\partial \Psi_0}{\partial z}\frac{\partial \omega_0}{\partial \theta}\right)$$

With the help of Maple all terms listed in the above series were found.

LEADING ORDER PROBLEM

The terms Ψ_{00} and ω_{00} satisfy:

$$\frac{\partial^2 \Psi_{00}}{\partial z^2} = \omega_{00}$$

$$\frac{\partial^2 \omega_{00}}{\partial z^2} + 2z\frac{\partial \omega_{00}}{\partial z} + 2\omega_{00} = 0$$

subject to:

$$\Psi_{00} = 0\ ,\ \frac{\partial \Psi_{00}}{\partial z} = \Omega \text{ on } z = 0 \text{ and } \omega_{00} \to 0 \text{ as } z \to \infty$$

$$\frac{1}{\pi}\int_0^\infty \int_0^{2\pi} \omega_{00} \sin(n\theta) d\theta dz = 2V\delta_{1,n}\ ,\ n = 1, 2, \cdots$$

$$\frac{1}{\pi}\int_0^\infty \int_0^{2\pi} \omega_{00} \cos(n\theta) d\theta dz = -K\delta_{2,n}\ ,\ n = 1, 2, \cdots$$

$$\frac{1}{\pi}\int_0^\infty \int_0^{2\pi} \omega_{00} d\theta dz = K - 2\Omega$$

The solution represents the initial condition and is given by:

$$\omega_{00} = \frac{2}{\sqrt{\pi}} \mathrm{e}^{-z^2} \left(2V\sin\theta - K\cos 2\theta + \frac{K - 2\Omega}{2}\right)$$

$$\Psi_{00} = \Omega z + \left[z\,\mathrm{erf}(z) + \frac{1}{\sqrt{\pi}}\left(\mathrm{e}^{-z^2} - 1\right)\right]\left(2V\sin\theta - K\cos 2\theta + \frac{K - 2\Omega}{2}\right)$$

Solutions to higher-order terms soon become very complicated.

For example:

$$\Psi_{01} = a_1(z)\cos\theta + a_2(z)\sin 2\theta + a_3(z)\cos 3\theta + a_4(z)\sin 4\theta$$

where

$$a_1(z) = -\frac{1}{9}\mathrm{e}^{-2z^2}V(8e^{z^2} + 12\pi\mathrm{e}^{z^2} - 12\mathrm{e}^{2z^2}\pi + 8z^2\mathrm{e}^{z^2} + 6z\sqrt{\pi}$$

$$-33\mathrm{erf}z\pi\mathrm{e}^{z^2} - 8z^3\mathrm{e}^{2z^2}\sqrt{\pi} - 12\mathrm{erf}z\pi\mathrm{e}^{2z^2} - 12z\sqrt{\pi}\mathrm{e}^{2z^2} - 6z^2\pi\mathrm{e}^{z^2}$$

$$+9z\mathrm{erf}z\mathrm{e}^{2z^2}\pi^{3/2} - 6z^3\mathrm{erf}ze^{2z^2}\pi^{3/2} + 12z\mathrm{erf}z\mathrm{e}^{2z^2}\sqrt{\pi}$$

$$-9z(\mathrm{erf}z)^2\pi^{3/2}\mathrm{e}^{2z^2} + 24\sqrt{2}\mathrm{erf}(\sqrt{2}z)\pi\mathrm{e}^{2z^2} + 6z^3(\mathrm{erf}z)^2\pi^{3/2}\mathrm{e}^{2z^2}$$

$$+8z^3\mathrm{erf}z\mathrm{e}^{2z^2}\sqrt{\pi} + 12z^2\mathrm{erf}z\pi\mathrm{e}^{z^2} - 8\mathrm{e}^{2z^2})\Omega/\pi^{3/2} - \frac{1}{9}\mathrm{e}^{-2z^2}V($$

$$-6\pi\mathrm{e}^{z^2} - 6z\sqrt{\pi} + 6\mathrm{e}^{2z^2}\pi - 8\mathrm{e}^{z^2} + 8\mathrm{e}^{2z^2} - 8z^3\mathrm{erf}z\mathrm{e}^{2z^2}\sqrt{\pi} - 8z^2\mathrm{e}^{z^2}$$

$$-12z\mathrm{erf}z\mathrm{e}^{2z^2}\sqrt{\pi} - 6z^3(\mathrm{erf}z)^2\pi^{3/2}\mathrm{e}^{2z^2} + 12z^3\mathrm{e}^{2z^2}\pi^{3/2}$$

$$-9z\mathrm{erf}z\mathrm{e}^{2z^2}\pi^{3/2} + 12z\sqrt{\pi}\mathrm{e}^{2z^2} + 9z(\mathrm{erf}z)^2\pi^{3/2}\mathrm{e}^{2z^2}$$

$$-24\sqrt{2}\mathrm{erf}(\sqrt{2}z)\pi\mathrm{e}^{2z^2} + 33\mathrm{erf}z\pi\mathrm{e}^{z^2} - 6z^2\pi\mathrm{e}^{z^2} - 12z^2\mathrm{erf}z\pi\mathrm{e}^{z^2}$$

$$+8z^3\mathrm{e}^{2z^2}\sqrt{\pi} + 12\mathrm{erf}\ z\pi\mathrm{e}^{2z^2} - 6z^3\mathrm{erf}z\mathrm{e}^{2z^2}\pi^{3/2})K/\pi^{3/2}$$

JUSTIFICATION OF ANALYTICAL SOLUTION

The inviscid solution can be obtained by setting $\zeta = K$ and satisfies:

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \theta^2} = K\mathrm{e}^{2\xi}$$

subject to:

$$\frac{\partial \psi}{\partial \theta} = 0 \text{ on } \xi = 0$$

$$\psi \to \frac{K}{4}\mathrm{e}^{2\xi} + V\mathrm{e}^{\xi}\sin\theta - \frac{K}{4}\mathrm{e}^{2\xi}\cos 2\theta \text{ as } \xi \to \infty$$

The solution is easily found to be:

$$\psi(\xi, \theta) = \frac{K}{4}\mathrm{e}^{2\xi} + 2V\sinh\xi\sin\theta - \frac{K}{2}\sinh 2\xi\cos 2\theta$$

The transverse velocity on the surface is then

$$v(0, \theta) = \frac{K}{2} + 2V\sin\theta - K\cos 2\theta$$

The above can also be determined from the analytical solution:

$$v = \lim_{z \to \infty}\left(\frac{\partial \Psi_{00}}{\partial z}\right) = \frac{K}{2} + 2V\sin\theta - K\cos 2\theta$$

which is in full agreement with the inviscid solution.

NUMERICAL SOLUTION PROCEDURE

The scheme implemented is a spectral - finite difference method similar to that used by Badr & Dennis (JFM 1985). The flow variables are first expanded in a truncated Fourier series:

$$\Psi(z,\theta,t) = \frac{F_0(z,t)}{2} + \sum_{n=1}^{N} \left(F_n(z,t)\cos(n\theta) + f_n(z,t)\sin(n\theta)\right)$$

$$\omega(z,\theta,t) = \frac{G_0(z,t)}{2} + \sum_{n=1}^{N} \left(G_n(z,t)\cos(n\theta) + g_n(z,t)\sin(n\theta)\right)$$

The Fourier coefficients satisfy the following equations:

$$\frac{\partial^2 F_0}{\partial z^2} = \mathrm{e}^{2\lambda z} G_0(z,t)$$

$$\frac{\partial^2 F_n}{\partial z^2} - n^2\lambda^2 F_n = \mathrm{e}^{2\lambda z} G_n(z,t)$$

$$\frac{\partial^2 f_n}{\partial z^2} - n^2\lambda^2 f_n = \mathrm{e}^{2\lambda z} g_n(z,t)$$

$$\mathrm{e}^{-2\lambda z}\frac{\partial^2 G_0}{\partial z^2} + 2z\frac{\partial G_0}{\partial z} + 2G_0 = 4t\frac{\partial G_0}{\partial t} - 4t\mathrm{e}^{-2\lambda z} S_0$$

$$\mathrm{e}^{-2\lambda z}\frac{\partial^2 G_n}{\partial z^2} + \left(2z + 4nt\mathrm{e}^{-2\lambda z} f_{2n}\right)\frac{\partial G_n}{\partial z} + \left(2 - n^2\lambda^2\mathrm{e}^{-2\lambda z} + 2nt\mathrm{e}^{-2\lambda z}\frac{\partial f_{2n}}{\partial z}\right) G_n$$
$$= 4t\frac{\partial G_n}{\partial t} - 2nt\mathrm{e}^{-2\lambda z}\left(f_n\frac{\partial G_0}{\partial z} - g_n\frac{\partial F_0}{\partial z} - g_n\frac{\partial F_{2n}}{\partial z} - 2F_{2n}\frac{\partial g_n}{\partial z}\right) - 2t\mathrm{e}^{-2\lambda z} S_n$$

$$\mathrm{e}^{-2\lambda z}\frac{\partial^2 g_n}{\partial z^2} + \left(2z - 4nt\mathrm{e}^{-2\lambda z} f_{2n}\right)\frac{\partial g_n}{\partial z} + \left(2 - n^2\lambda^2\mathrm{e}^{-2\lambda z} - 2nt\mathrm{e}^{-2\lambda z}\frac{\partial f_{2n}}{\partial z}\right) g_n$$
$$= 4t\frac{\partial g_n}{\partial t} + 2nt\mathrm{e}^{-2\lambda z}\left(F_n\frac{\partial G_0}{\partial z} - G_n\frac{\partial F_0}{\partial z} + G_n\frac{\partial F_{2n}}{\partial z} + 2F_{2n}\frac{\partial G_n}{\partial z}\right) - 2t\mathrm{e}^{-2\lambda z} T_n$$

for $n = 1, 2, \cdots$.

The solution procedure is illustrated using the equation for G_0:

$$\mathrm{e}^{-2\lambda z}\frac{\partial^2 G_0}{\partial z^2} + 2z\frac{\partial G_0}{\partial z} + 2G_0 = 4t\frac{\partial G_0}{\partial t} - 4t\mathrm{e}^{-2\lambda z}S_0$$

subject to:

$$G_0 \longrightarrow 2K\lambda \quad \text{as} \quad z \longrightarrow \infty$$

$$\int_0^{z_\infty} \mathrm{e}^{2\lambda z} G_0 dz = K\mathrm{e}^{2\lambda z_\infty} - 2\Omega$$

First, define $\hat{G}_0 = G_0 - 2K\lambda$, then $\hat{G}_0 \longrightarrow 0$ as $z \longrightarrow \infty$.
Note that the equation for $\hat{G}_0$ will be identical to that of G_0.
In terms of $\hat{G}_0$ the integral condition transforms to

$$\int_0^{\infty} \mathrm{e}^{2\lambda z} \hat{G}_0 dz = K - 2\Omega$$

The equation for $\hat{G}_0$ may be rewritten in the form:

$$4t\frac{\partial \hat{G}_0}{\partial t} = Q(z,t)$$

Now advance the solution from time t to $t + \Delta t$:

$$4\tau\hat{G}_0\Big|_t^{t+\Delta t} - 4\int_t^{t+\Delta t} \hat{G}_0 d\tau = \int_t^{t+\Delta t} Q d\tau$$

Next, approximate the integrals using the general formula:

$$\int_t^{t+\Delta t} \chi d\tau \approx \Delta t[\beta\chi(z, t+\Delta t) + (1-\beta)\chi(z,t)]$$

where $\beta = 1/2 \Rightarrow$ Crank-Nicolson scheme
while $\beta = 1 \Rightarrow$ fully implicit scheme.

The equation for $\hat{G}_0$ then becomes:

$$4[t + (1-\beta)\Delta t](\hat{G}_0(z, t+\Delta t) - \hat{G}_0(z,t)) =$$

$$\Delta t[\beta Q(z, t+\Delta t) + (1-\beta)Q(z,t)]$$

Substituting the expression for $Q(z, t+\Delta t)$ and replacing all derivatives by central differences leads to

$$A(z, t+\Delta t)\hat{G}_0(z - \Delta z, t+\Delta t) + B(z, t+\Delta t)\hat{G}_0(z, t+\Delta t)$$

$$+C(z, t+\Delta t)\hat{G}_0(z + \Delta z, t+\Delta t) = D(z, t+\Delta t) + E(z,t)$$

In matrix form this becomes a tri-diagonal system. To satisfy the integral condition express

$$\hat{G}_0(z, t+\Delta t) = \gamma \hat{G}_0^h(z, t+\Delta t) + \hat{G}_0^p(z, t+\Delta t)$$

with $\hat{G}_0^h = \hat{G}_0^p = 1$ at $z = 0$ and $\hat{G}_0^h = \hat{G}_0^p = 0$ at $z = z_\infty$. The constant γ is chosen to satisfy the integral condition:

$$\gamma = \frac{K - 2\Omega - \int_0^{z\infty} e^{2\lambda z}\hat{G}_0^p dz}{\int_0^{z\infty} e^{2\lambda z}\hat{G}_0^h dz}$$

RESULTS & COMPARISONS

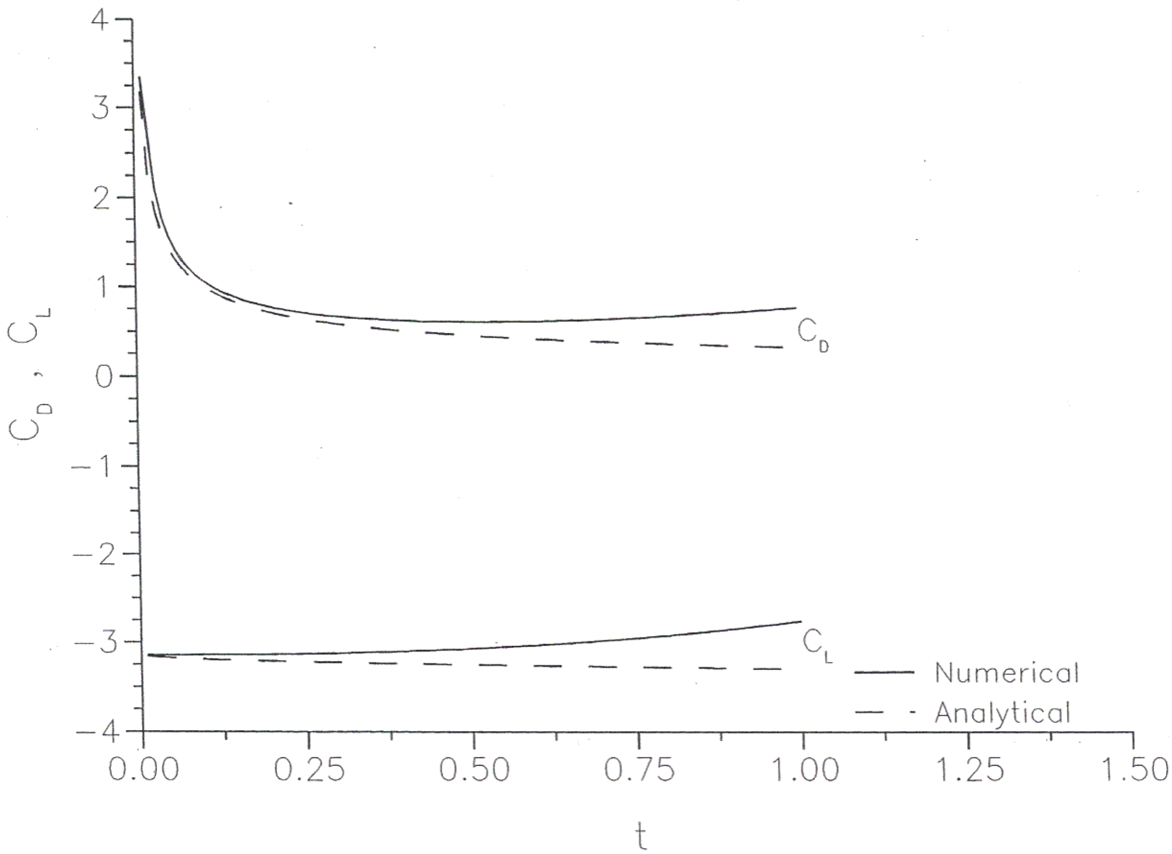
Computational Parameters

Outer Boundary: $z_\infty = 8$
Grid Spacing Used: $\Delta z = .05$
Time Steps Used: $\Delta t = .0001$ till $t = .001$
then $\Delta t = .001$ till $t = .01$
then $\Delta t = .01$ till $t = 1$
then $\Delta t = .05$ for $t > 1$
Differencing Parameter Used: $\beta = 1$ (Fully Implicit Scheme)
Convergence Criterion Adopted: $|\hat{G}_0^{(k+1)}(z,t) - \hat{G}_0^{(k)}(z,t)| < \varepsilon$
Tolerance: $\varepsilon = 10^{-5}$
Number of Terms in Fourier Series: $N = 25$

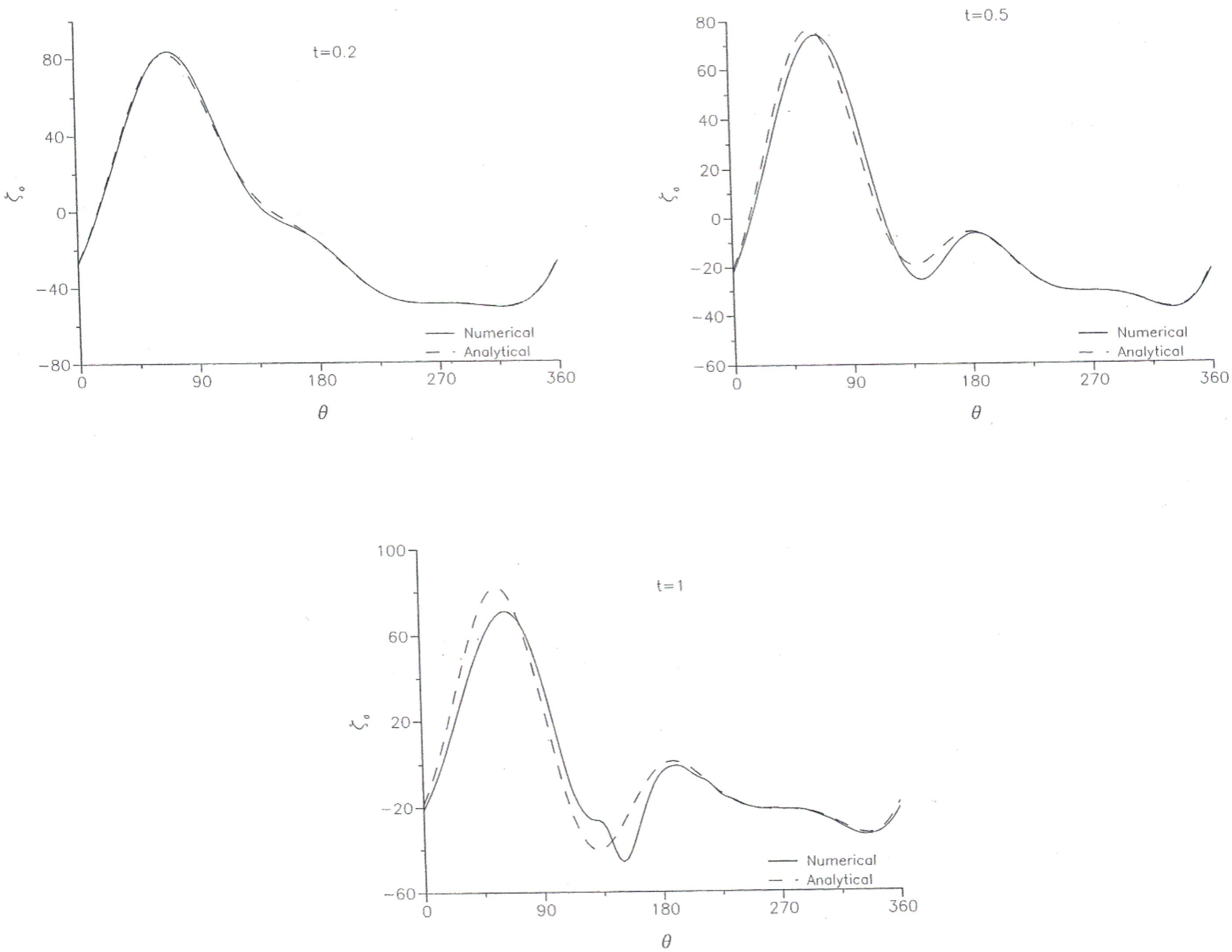
Numerical Validation

The numerics can be verified against the analytical results found. The following quantities are compared for large R and small t:

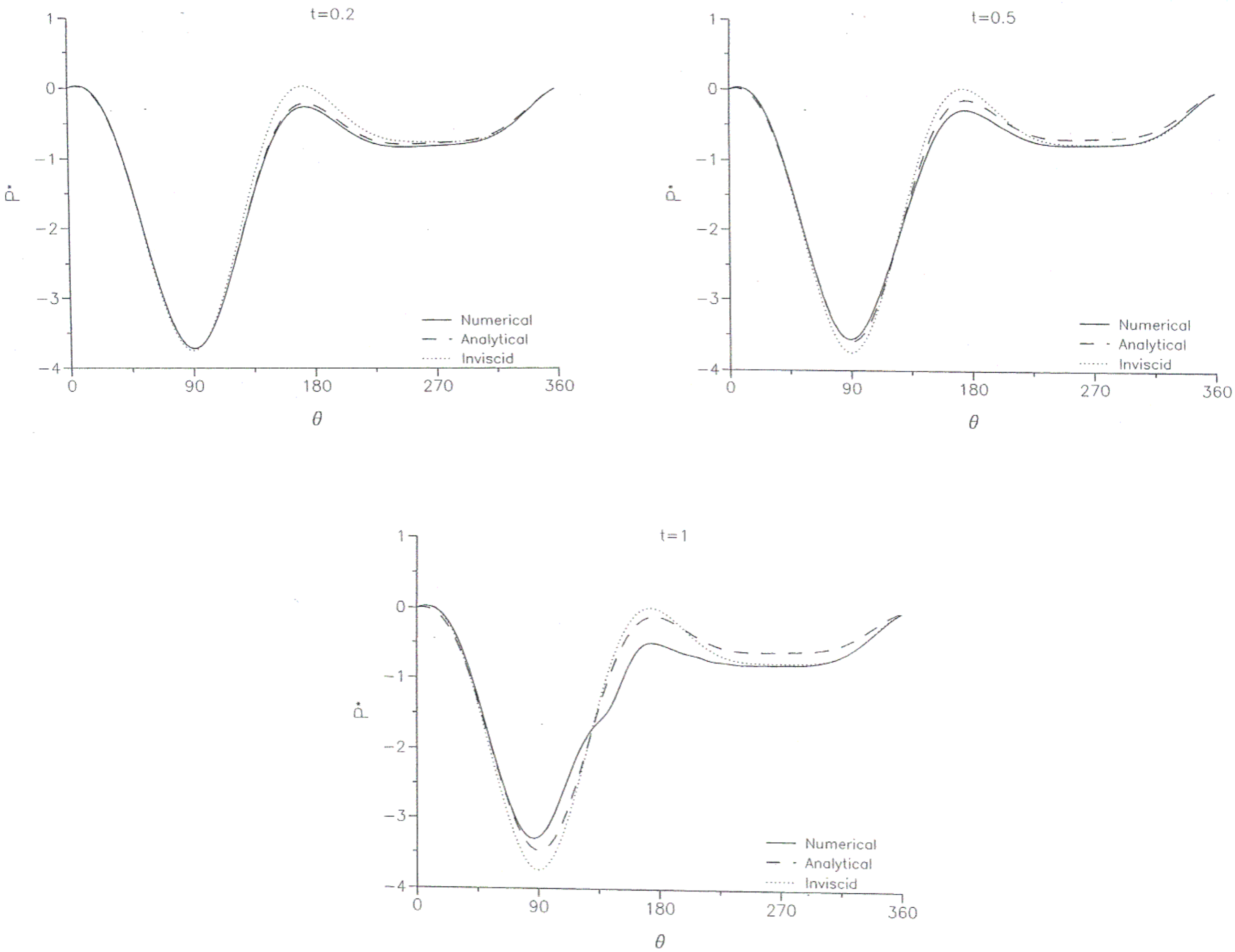
$$\zeta_0(\theta, t)\ ,\ C_D(t)\ ,\ C_L(t)\ ,\ P^*(\theta, t)$$



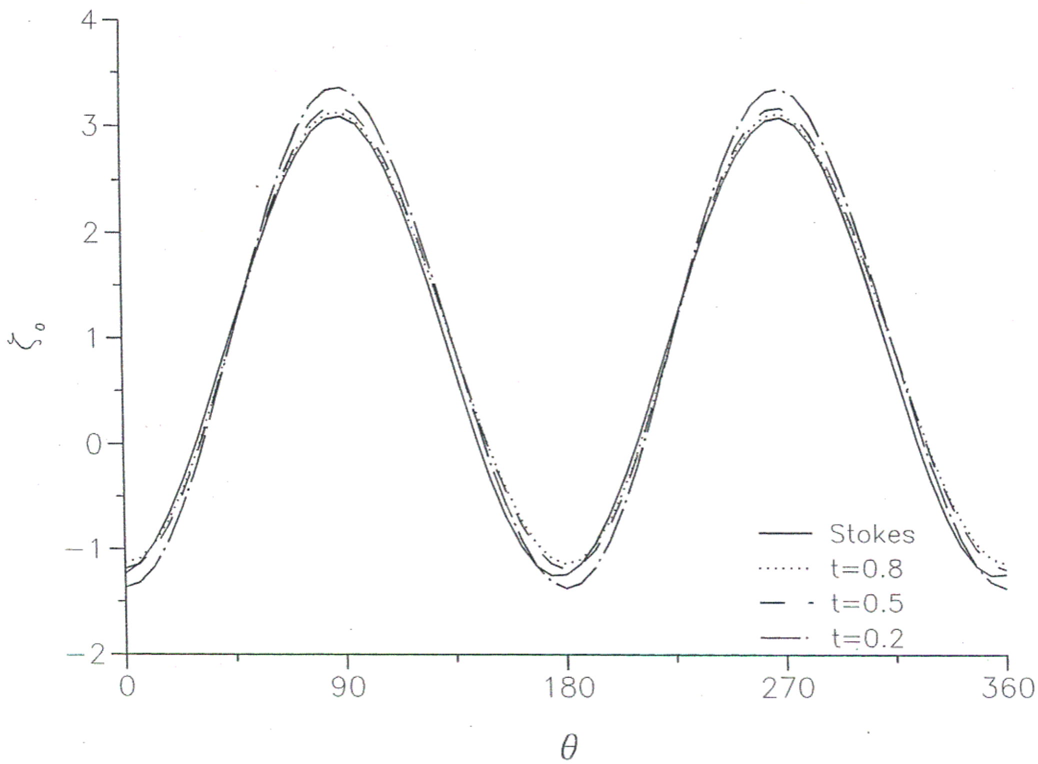
Comparison of C_D and C_L for the case:
$R = 1000, \Omega = 1/2, K = 1/2$ and $V = -1$.



Comparison of ζ_0 at times $t = 0.2, 0.5, 1$ for the case: $R = 1000, \Omega = 1/2, K = 1/2$ and $V = -1$.

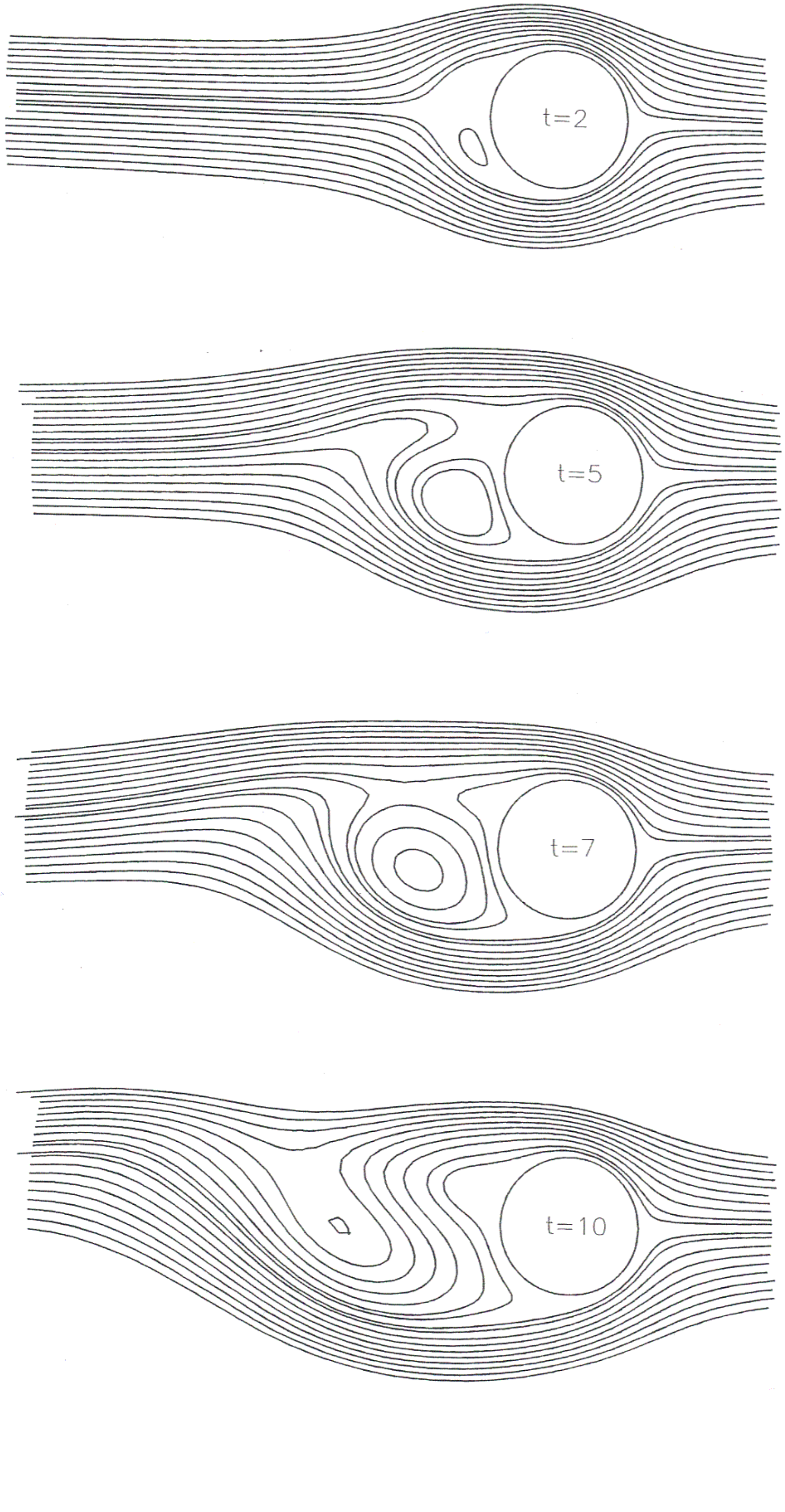


Comparison of P^* at times $t = 0.2, 0.5, 1$ for the case:
$R = 1000, \Omega = 1/2, K = 1/2$ and $V = -1$.

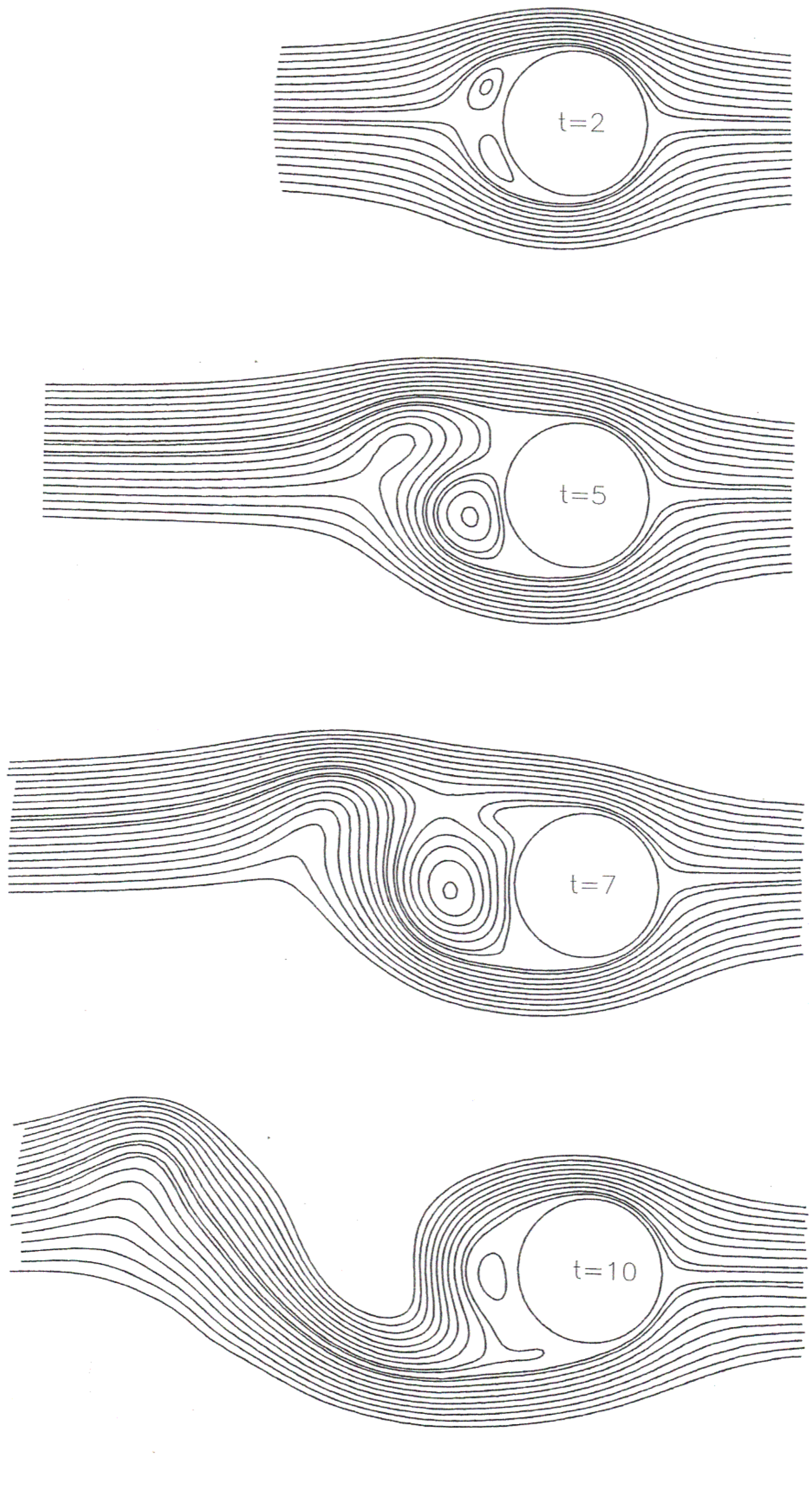


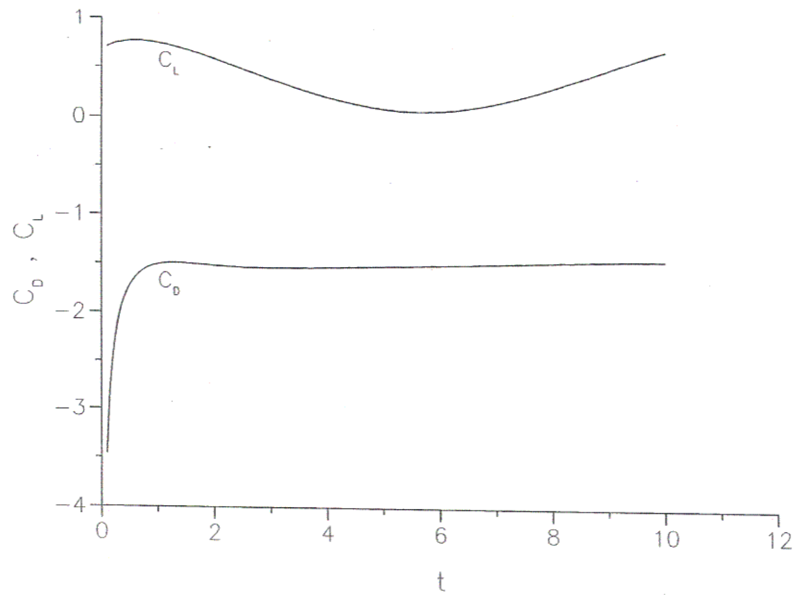
Comparison of ζ_0 with the Stokes solution of Robertson & Acrivos JFM (1970) for the case: $R = 1/2, \Omega = 1/2, K = 1$ and $V = 0$.

Streamline plots for $R = 100, \Omega = 0.25, K = 0.1$ and $V = 1$.

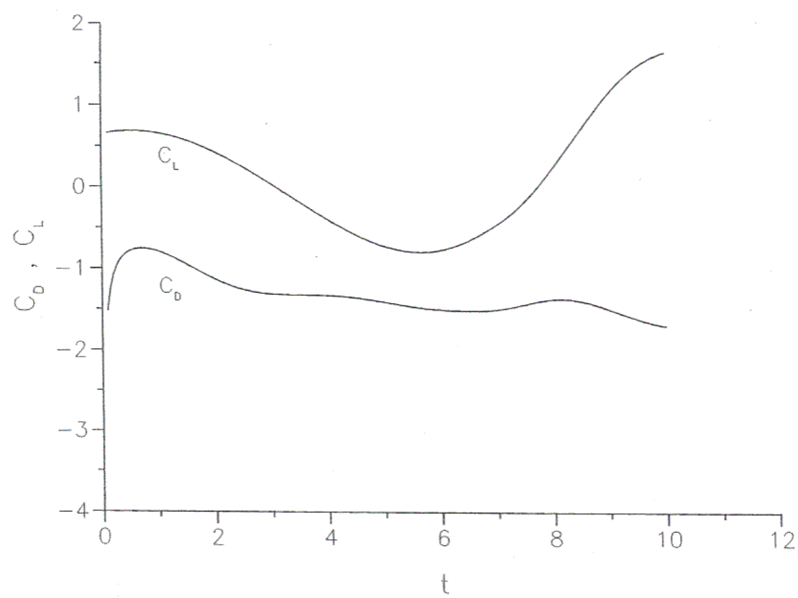


Streamline plots for $R = 500, \Omega = 0.25, K = 0.1$ and $V = 1$.





Time variation of C_D, C_L for
$R = 100, \Omega = 0.25, K = 0.1, V = 1.$



Time variation of C_D, C_L for
$R = 500, \Omega = 0.25, K = 0.1, V = 1.$

CONCLUSIONS

- An approximate analytical solution valid for large R and small t has been constructed.
- An alternate numerical method for solving the governing equations has been outlined.
- Good agreement between the two types of solutions was found. Comparisons with other studies also demonstrated good agreement.
- Possible extensions include different cylindrical cross sections and other forms of shear profiles.