Simulating Flows Over Wavy Inclined Surfaces

By: Serge D'Alessio[1]
With: J.P. Pascal[2]

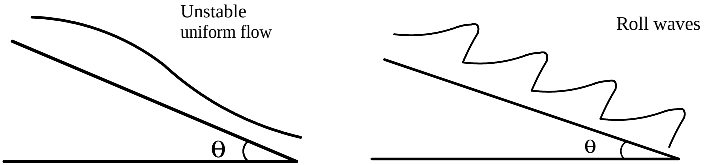
[1]Department of Applied Mathematics
University of Waterloo, Waterloo, Canada

[2]Department of Mathematics
Ryerson University, Toronto, Canada

CSC 2009

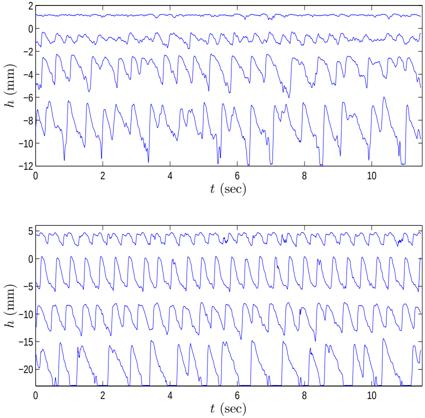
Outline

Unstable flow down an incline



- Critical conditions for the onset of Instability.
- Structure of Roll Waves
- Investigate the effect of bottom topography

The spillway from the Llyn Brianne Dam in Wales



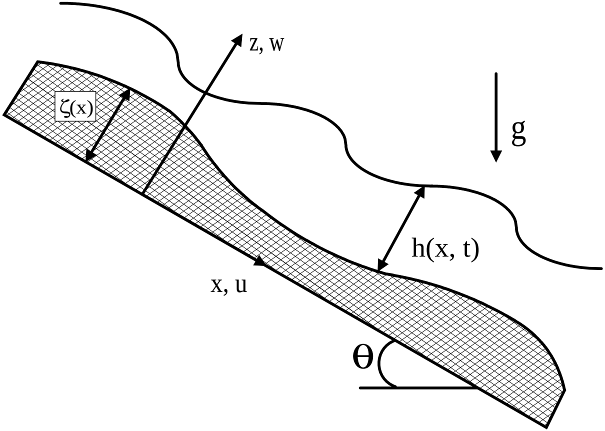
Experiment taken from Balmforth and Mandre (JFM, 2004)

Previous studies

Flow down an inclined plane has received numerous studies:

- pioneering experiments by Kapitza & Kapitza, 1949, further experiments by Liu *et al.* (Phys. Fluids, 1995)
- first analytical studies predicting onset of instability can be traced back to Benjamin (JFM, 1957), Yih (Phys. Fluids, 1963) and Benney (J Math. Phys., 1966)
- numerical investigation by Ramaswamy *et al.* (JFM, 1996)
- sinusoidal bottom topography was carried out by Balmforth & Mandre (JFM, 2004)
- effects of weak surface tension was considered by Wierschem *et al.* (Acta Mech., 2005)

Coordinate system



Equations of motion

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + g\rho\sin\theta + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} + g\cos\theta - \frac{\mu}{\rho}\frac{\partial^2 w}{\partial z^2} = 0$$

If $Re \sim O(1)$, then above equations represent a second-order approximation to the Navier-Stokes equations with respect to shallowness parameter $\delta = H/L$

Interface conditions

Free surface conditions:

$$\left.\begin{aligned} p &= 2\mu\frac{\partial w}{\partial z} - T\left(\frac{\partial^2 h}{\partial x^2} + \zeta''\right) \\ \frac{\partial u}{\partial z} &= 4\left(\frac{\partial h}{\partial x} + \zeta'\right)\frac{\partial u}{\partial x} - \frac{\partial w}{\partial x} \\ w &= \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + u\zeta'(x) \end{aligned}\right\} \text{ at } z = \zeta(x) + h(x,t)$$

Bottom boundary conditions:

$$u + \zeta'(x)w = 0 \text{ and } \zeta'(x)u - w = 0 \text{ at } z = \zeta(x)$$

$$\Rightarrow u = w = 0 \text{ at } z = \zeta(x)$$

Weighted residual method (Ruyer-Quil *et al.*, 2002)

First eliminate pressure using

$$p = \cos\theta \; \rho g(z_1 - z) - \mu \left.\frac{\partial u}{\partial x}\right|_{z=z_1} - \mu \frac{\partial u}{\partial x} - T\frac{\partial^2 z_1}{\partial x^2} \; , \; z_1 = h + \zeta$$

Next, multiply momentum equation by weight function

$$W(x,z,t) = 2[h(x,t) + \zeta(x)]z - z^2 - [2h(x,t) + \zeta(x)]\zeta(x)$$

Depth-integrate and introduce: $h(x,t) \; , \; q(x,t) = \int_{\zeta}^{\zeta+h} u dz$

To convert terms like: $\int_{\zeta}^{\zeta+h} W u^2 dz \quad , \quad \frac{\mu}{\rho}\left.\frac{\partial u}{\partial z}\right|_{z=\zeta}$

assume the parabolic velocity profile: $u(x,z,t) = \frac{3q}{2h^3} W$

Dimensionless equations

In terms of h, q the dimensionless equations become

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0$$

$$\frac{\partial q}{\partial t} + \frac{9}{7}\frac{\partial}{\partial x}\left(\frac{q^2}{h}\right) = \frac{q}{7h}\frac{\partial q}{\partial x} + \frac{5}{2}\frac{\cot\theta}{Re}\left(h - \zeta' h - h\frac{\partial h}{\partial x} - \frac{q}{h^2}\right)$$
$$+\frac{5}{6}\frac{We}{\cot^2\theta}h\left(\frac{\partial^3 h}{\partial x^3} + \zeta'''\right) + \frac{1}{Re\cot\theta}\left[\frac{9}{2}\frac{\partial^2 q}{\partial x^2} - \frac{9}{2h}\frac{\partial q}{\partial x}\frac{\partial h}{\partial x}\right.$$
$$\left. + \frac{4q}{h^2}\left(\frac{\partial h}{\partial x}\right)^2 - \frac{6q}{h}\frac{\partial^2 h}{\partial x^2} - \frac{5\zeta' q}{2h^2}\frac{\partial h}{\partial x} - \frac{15\zeta'' q}{4h} - \frac{5(\zeta')^2 q}{h^2}\right]$$

where $We = \dfrac{TH}{\rho Q^2}$, $Re = \dfrac{\rho Q}{\mu}$ and $\zeta(x) = a_b\cos(k_b x)$

Begin by expressing equations in the form

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0$$

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{9}{7}\frac{q^2}{h} + \frac{5\cot\theta}{4Re}h^2\right) = \Psi + \chi$$

where $\Psi = \Psi(h, q)$

and $\chi = \chi\left(x, h, q, \frac{\partial h}{\partial x}, \frac{\partial q}{\partial x}, \frac{\partial^2 h}{\partial x^2}, \frac{\partial^2 q}{\partial x^2}, \frac{\partial^3 h}{\partial x^3}\right)$

Fractional-step method (LeVeque, 2002)

Decouple the advective and diffusive components, first solve

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0$$

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{9}{7}\frac{q^2}{h} + \frac{5\cot\theta}{4Re}h^2\right) = \Psi(h, q)$$

over a time step Δt, and then solve

$$\frac{\partial q}{\partial t} = \chi\left(x, h, q, \frac{\partial h}{\partial x}, \frac{\partial q}{\partial x}, \frac{\partial^2 h}{\partial x^2}, \frac{\partial^2 q}{\partial x^2}, \frac{\partial^3 h}{\partial x^3}\right)$$

using the solution obtained from the first step as an initial condition for the second step; the second step returns the solution for q at the new time $t + \Delta t$

First step

This involves solving a nonlinear system of hyperbolic conservation laws; express in vector form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \mathbf{b}(\mathbf{U})$$

where $\mathbf{U} = \begin{bmatrix} h \\ q \end{bmatrix}$, $\mathbf{F}(\mathbf{U}) = \begin{bmatrix} q \\ \frac{9}{7}\frac{q^2}{h} + \frac{5\cot\theta}{4Re}h^2 \end{bmatrix}$, $\mathbf{b}(\mathbf{U}) = \begin{bmatrix} 0 \\ \Psi \end{bmatrix}$

Utilize MacCormack's method to solve this system; this is a conservative second-order accurate finite difference scheme which correctly captures discontinuities and converges to the physical weak solution of the problem

First step

LeVeque & Yee (JCP, 1990) extended MacCormack's method to include source terms; this explicit predictor-corrector scheme takes the form

$$\mathbf{U}_j^* = \mathbf{U}_j^n - \frac{\Delta t}{\Delta x}\left[\mathbf{F}(\mathbf{U}_{j+1}^n) - \mathbf{F}(\mathbf{U}_j^n)\right] + \Delta t\,\mathbf{b}(\mathbf{U}_j^n)$$

$$\mathbf{U}_j^{n+1} = \frac{1}{2}\left(\mathbf{U}_j^n + \mathbf{U}_j^*\right) - \frac{\Delta t}{2\Delta x}\left[\mathbf{F}(\mathbf{U}_j^*) - \mathbf{F}(\mathbf{U}_{j-1}^*)\right] + \frac{\Delta t}{2}\mathbf{b}(\mathbf{U}_j^*)$$

where the notation $\mathbf{U}_j^n \equiv \mathbf{U}(x_j, t_n)$ was adopted, Δx is the grid spacing and Δt is the time step; second-order accuracy is achieved by first forward differencing and then backward differencing

Second step

This reduces to solving the generalized one-dimensional nonlinear diffusion equation of the form:

$$\frac{\partial q}{\partial t} = \frac{9}{2Re\cot\theta}\frac{\partial^2 q}{\partial x^2} + \frac{q}{7h}\frac{\partial q}{\partial x} + S_1\frac{\partial q}{\partial x} + S_0 q + S$$

Since h is known from the first step and remains constant during the second step, the functions S, S_0, S_1 are known
Discretizing the above equation using the Crank-Nicolson scheme, imposing periodicity conditions, and using the output from the first step as an initial condition, leads to a nonlinear system of algebraic equations which was solved iteratively using a robust algorithm which takes advantage of the structure and sparseness of the resulting linearized system

Computational parameters

The problem is completely specified by Re, $\cot\theta$, We, a_b, k_b
Typical computational parameters used were:
Computational Domain: $0 \leq x \leq L$
with $\lambda_b \leq L \leq 20\lambda_b$, $\lambda_b = \dfrac{2\pi}{k_b}$
Grid Spacing: $\Delta x = .01$
Time Step: $\Delta t = .002$ for $We = 0$
(smaller Δt required for $We \neq 0$)

Linear stability results for $a_b = 0$

The steady-state flow is: $q_s = h_s = 1$

For all values of the wavenumber k and We, the flow is stable if $Re \leq \frac{5}{6}\cot\theta$, while for $Re > \frac{5}{6}\cot\theta$ instability occurs

The predicted onset of instability is in exact agreement with previous analytical predictions (Benjamin, Benney & Yih) and experimentally verified by Liu *et al.* (Phys. Fluids, 1995)

Linear stability: $a_b \neq 0$ case

The steady state solution is $q_s = 1$ and $h_s(x)$ satisfies

$$\frac{5We}{6\cot^2\theta} h_s^3 h_s''' - \frac{2}{Re\cot\theta}[3h_s h_s'' - 2(h_s')^2]$$

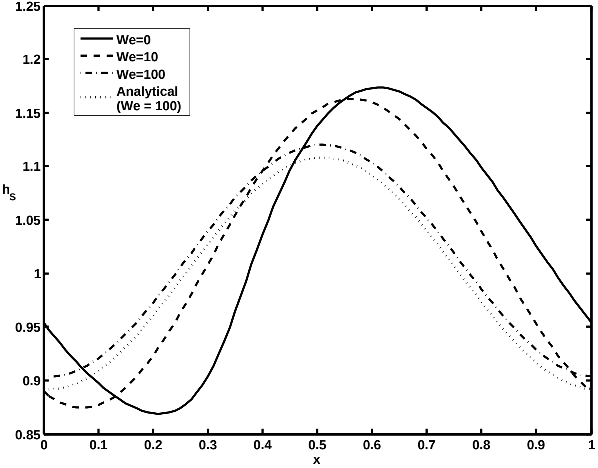
$$-\left(\frac{5\cot\theta}{2Re} h_s^3 + \frac{5}{2Re\cot\theta}\zeta' - \frac{9}{7}\right) h_s' - \frac{15}{4Re\cot\theta}\zeta'' h_s$$

$$+\left(\frac{5\cot\theta}{2Re}(1-\zeta') + \frac{5We}{6\cot^2\theta}\zeta'''\right) h_s^3 = \frac{5\cot\theta}{2Re} + \frac{5}{Re\cot\theta}(\zeta')^2$$

An approximate solution can be constructed in the form

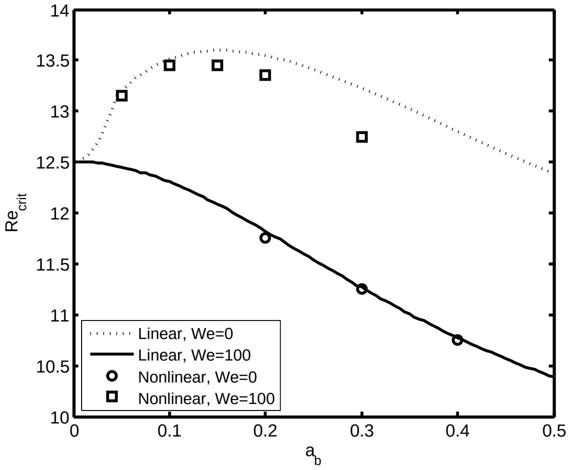
$$h_s(x) = 1 + (a_b k_b) h_s^{(1)}(x) + \cdots$$

Periodic steady state solution



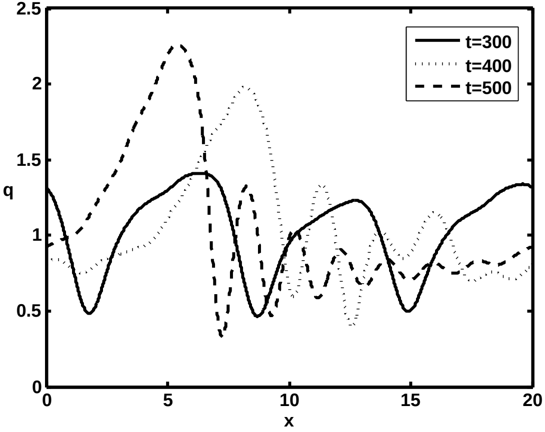
$Re = 20,$
$\cot\theta = 15,$
$a_b = 0.1,$
$k_b = 2\pi$

Linear versus nonlinear results



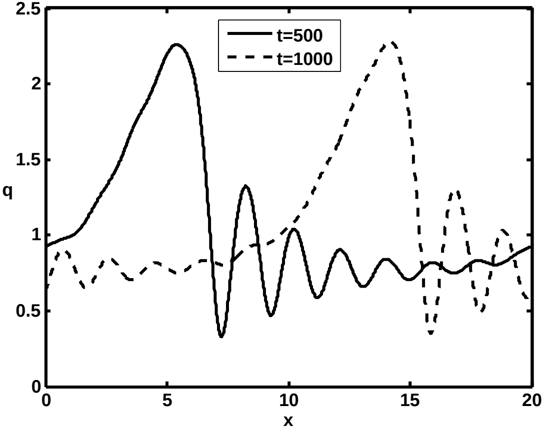
Critical Reynolds numbers for $\cot\theta = 15$ and $k_b = 2\pi$

Evolution of flow rate



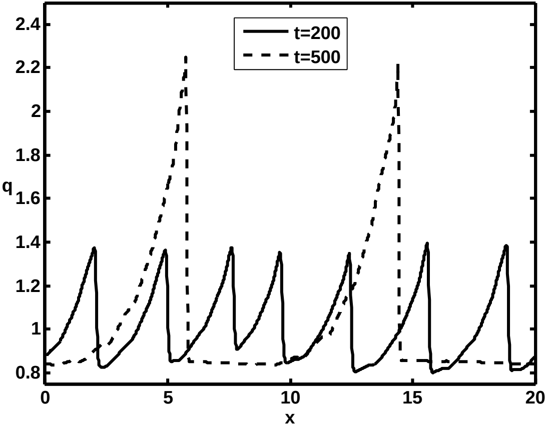
Parameters:
$a_b = 0.1, k_b = 2\pi,$
$Re = 20, \cot\theta = 15,$
$We = 100, L = 20$

Evolution of flow rate



Parameters:
$a_b = 0.1, k_b = 2\pi,$
$Re = 20, \cot\theta = 15,$
$We = 100, L = 20$

Evolution of flow rate



Parameters:

$a_b = 0.1, k_b = 2\pi,$

$Re = 20, \cot\theta = 15,$

$We = 0, L = 20$

Concluding remarks

- A mathematical model along with a numerical method to simulate the flow down a wavy incline was presented
- Numerically investigated the combined effect of bottom topography and surface tension on the stability of the flow
- For weak surface tension bottom topography acts to stabilize the flow, while for stronger surface tension bottom topography can destabilize the flow
- Future work includes repeating the analysis for the case of a porous wavy bottom and also to include thermocapillary effects