Thermally Driven Gravity Currents
Part 2: Numerical Results & Comparisons

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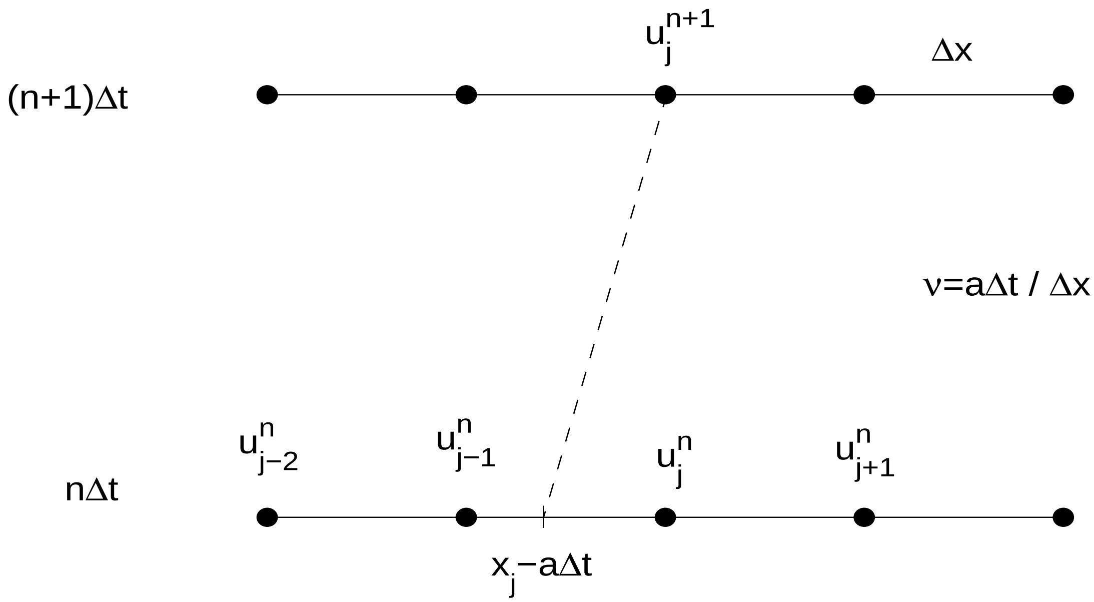
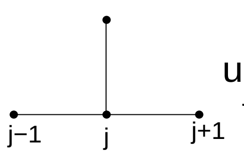
Finite-Difference Schemes

Consider

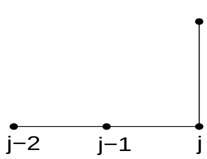
$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0\,, \quad u,\ a \in \mathbb{R}, \ \ a \text{ pos. constant}$$

Linear scalar equation.

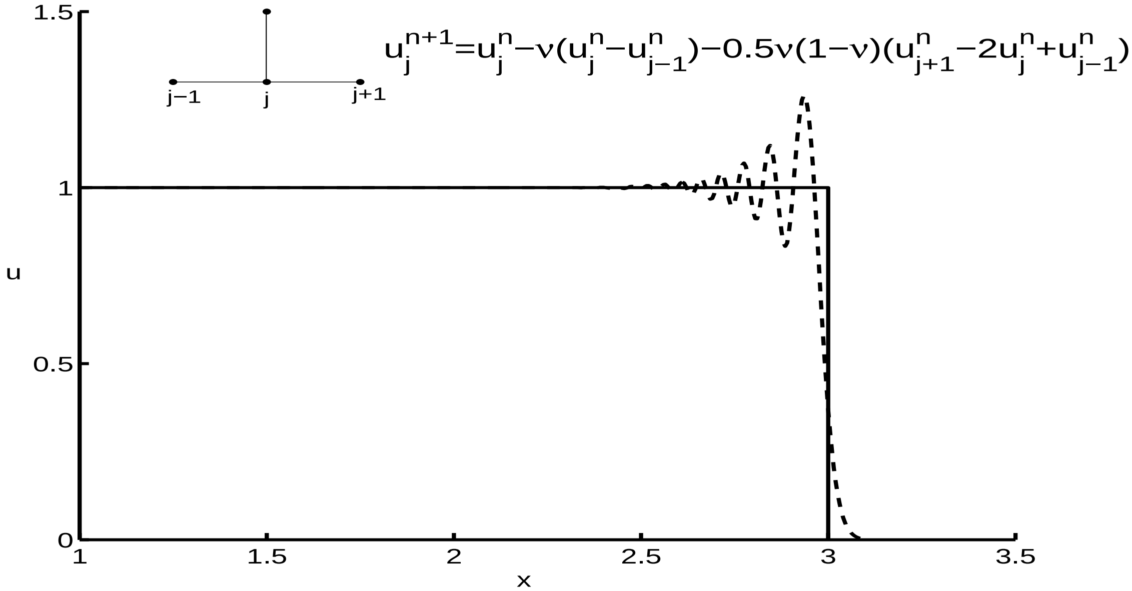




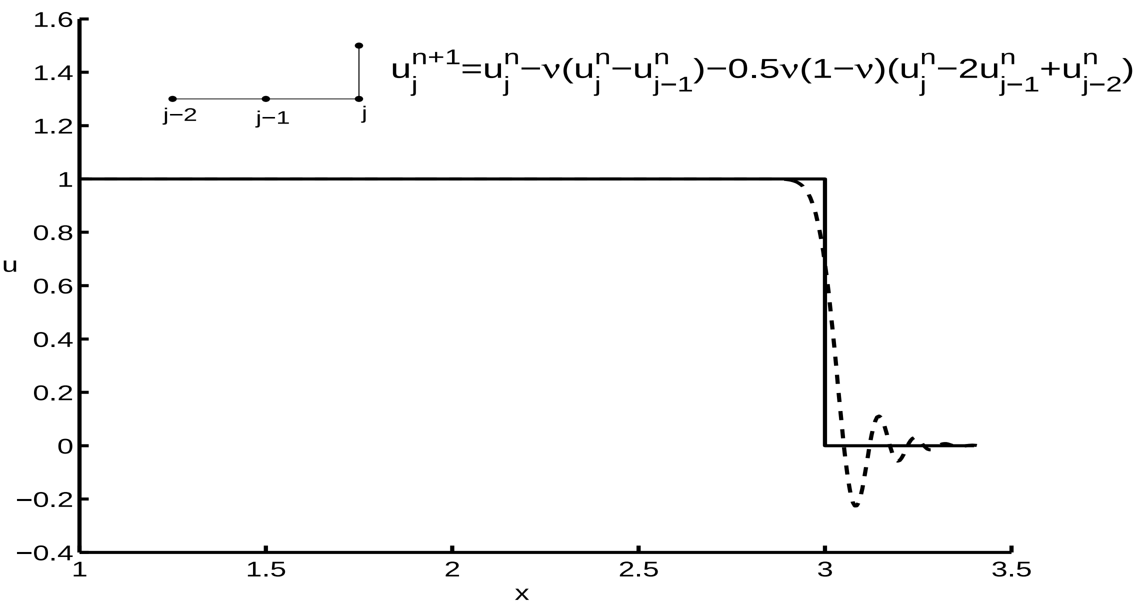
$$u_j^{n+1} = u_j^n - \nu(u_j^n - u_{j-1}^n) - 0.5\nu(1-\nu)(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$



$$u_j^{n+1} = u_j^n - \nu(u_j^n - u_{j-1}^n) - 0.5\nu(1-\nu)(u_j^n - 2u_{j-1}^n + u_{j-2}^n)$$



$u_j^{n+1}=u_j^n-\nu(u_j^n-u_{j-1}^n)-0.5\nu(1-\nu)(u_{j+1}^n-2u_j^n+u_{j-1}^n)$
j−1
j
j+1
u
x
1.5
1
0.5
0
1
1.5
2
2.5
3
3.5



$u_j^{n+1}=u_j^n-\nu(u_j^n-u_{j-1}^n)-0.5\nu(1-\nu)(u_j^n-2u_{j-1}^n+u_{j-2}^n)$
j−2
j−1
j
u
x
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
1
1.5
2
2.5
3
3.5

Total Variation

$$TV(u^n) = \sum_j \left|u_{j+1}^n - u_j^n\right|$$

Total Variation Diminishing Schemes (TVD) $\Rightarrow \quad TV(u^{n+1}) \leq TV(u^n)$

$\Rightarrow$ Oscillation-free solutions

$$u_j^{n+1} = u_j^n - (u_j^n - u_{j-1}^n)\left[\nu + \frac{\nu}{2}(1-\nu)\left(\frac{\phi(r_{j+1/2})}{r_{j+1/2}} - \phi(r_{j-1/2})\right)\right]$$

$$\text{slope ratio:} \quad r_{j+1/2} = \frac{u_j^n - u_{j-1}^n}{u_{j+1}^n - u_j^n}$$

$\phi(r) = 1 \quad \Rightarrow \quad$ Central Scheme

$\phi(r) = r \quad \Rightarrow \quad$ Upwind Scheme

Roe's symmetric limiter:

$$\phi(r) = \begin{cases} r\,, & |r| < 1 \\ 1\,, & \text{otherwise} \end{cases}$$

Systems of Nonlinear Equations

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial}{\partial x}\mathbf{F}(x, t, \mathbf{U}) = 0\,, \quad \mathbf{U},\ \mathbf{F} \in \mathbb{R}^4$$

Second-order accurate component-wise TVD scheme

(Yu and Liu, Journal of Computational Physics, 2001)

Consider the kth equation: Let u and f be the kth components of $\mathbf{U}$ and $\mathbf{F}$

For right moving waves,

$$u_j^* = u_j^n - \frac{\Delta t}{\Delta x}(f_j^n - f_{j-1}^n)$$

$$u_j^{n+1} = u_j^* - \frac{\Delta t}{2\Delta x}\left[\phi(r_{j+1/2})(f_{j+1}^* - f_j^n) - \phi(r_{j-1/2})(f_j^* - f_{j-1}^n)\right]$$

where

$$r_{j+1/2} = \frac{f_j^* - f_{j-1}^n}{f_{j+1}^* - f_j^n}\,, \quad f_j^n = f(j\Delta x, n\Delta t, \mathbf{U}_j^n)\,, \quad f_j^* = f(j\Delta x, n\Delta t, \mathbf{U}_j^*)$$

$\phi(r) = 1 \quad \Rightarrow \quad$ MacCormack's Scheme

$\phi(r) = r \quad \Rightarrow \quad$ Beam-Warming Scheme

For left moving waves,

$$u_j^* = u_j^n - \frac{\Delta t}{\Delta x}(f_{j+1}^n - f_j^n)$$

$$u_j^{n+1} = u_j^* - \frac{\Delta t}{2\Delta x}\left[\phi(r_{j-1/2})(f_j^n - f_{j-1}^*) - \phi(r_{j+1/2})(f_{j+1}^n - f_j^*)\right]$$

where

$$r_{j+1/2} = \frac{f_{j+2}^n - f_{j+1}^*}{f_{j+1}^n - f_j^*}$$

Flux Splitting

$$\mathbf{F} = \mathbf{F}^+ + \mathbf{F}^-$$

such that the eigenvalues of $\mathbf{F}_\mathbf{U}^+$ are positive and of $\mathbf{F}_\mathbf{U}^-$ are negative.

$$\mathbf{F}^\pm = \frac{1}{2}(\mathbf{F} \pm \alpha\mathbf{U})$$

where α is an upper bound on the absolute value of the eigenvalues of $\mathbf{F}_\mathbf{U}$

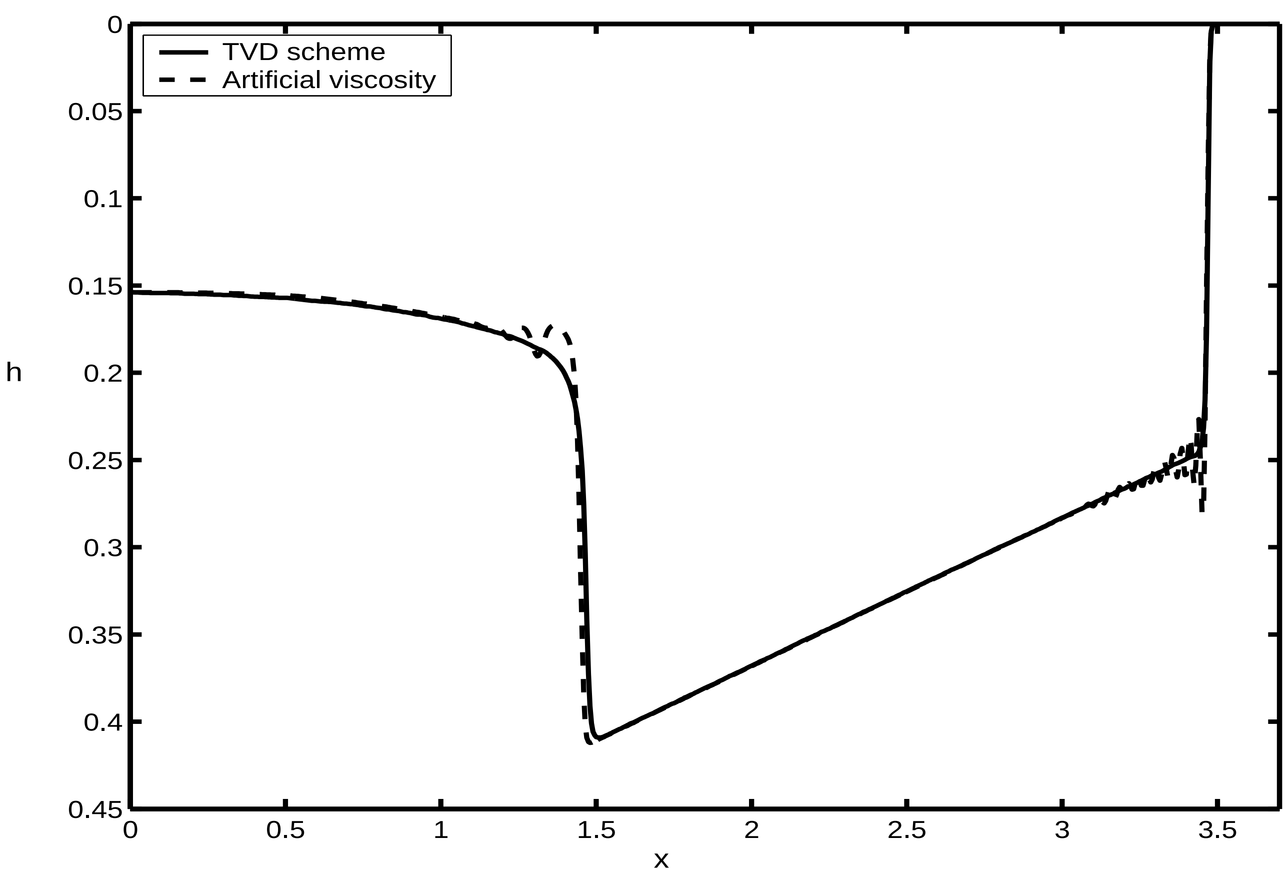


Fig. 2

Relative density difference: $\dfrac{\rho_0 - \rho_1}{\rho_0} = 0.01(t+1)$

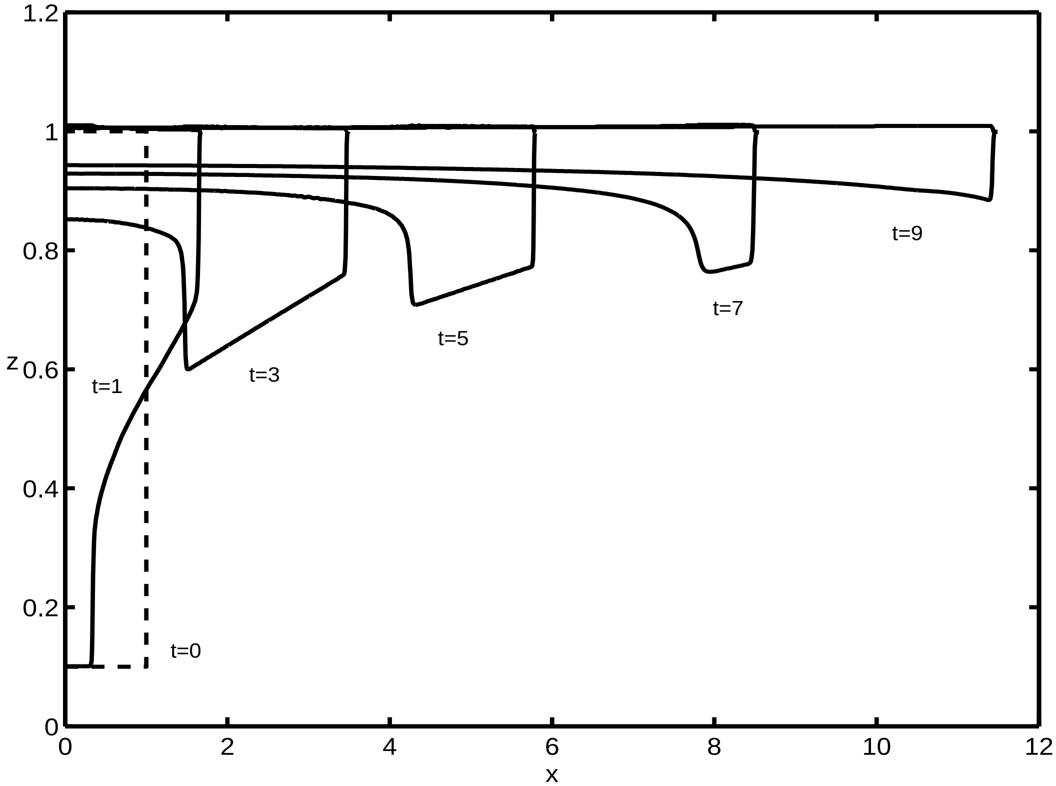


Fig. 3a

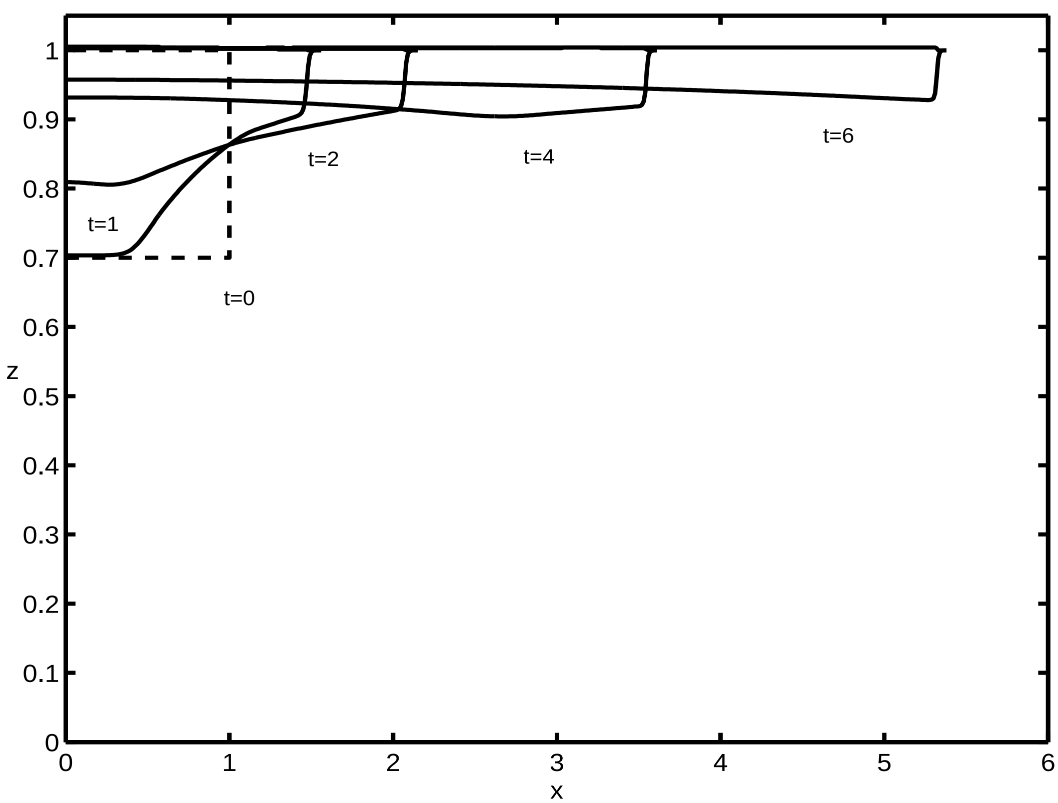


Fig. 3b

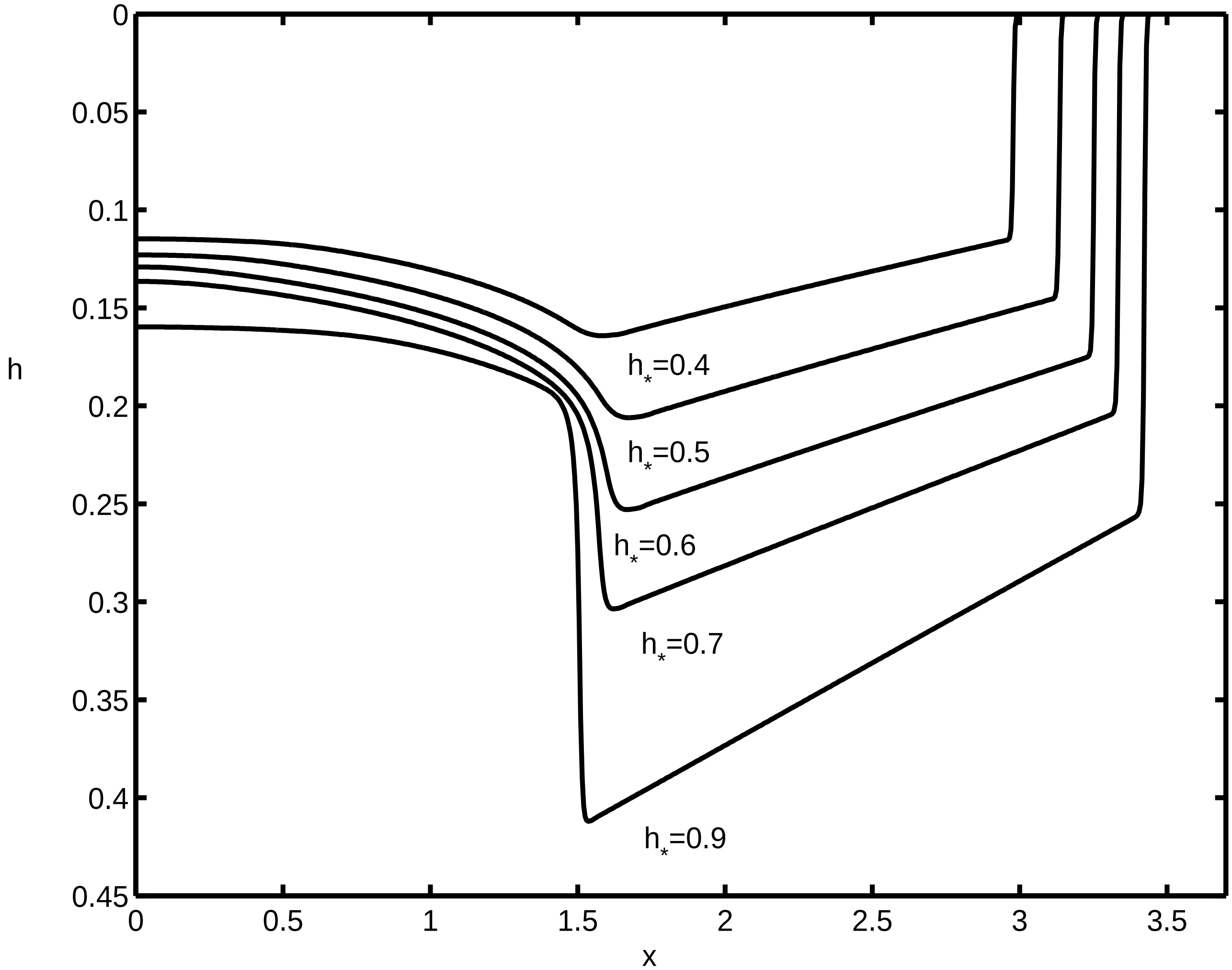

0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
h
h*=0.4
h*=0.5
h*=0.6
h*=0.7
h*=0.9
0
0.5
1
1.5
2
2.5
3
3.5
x

Fig. 8

weakly stratified model: neglect terms of $O(g'/g)$,

where g'/g - initial relative density difference

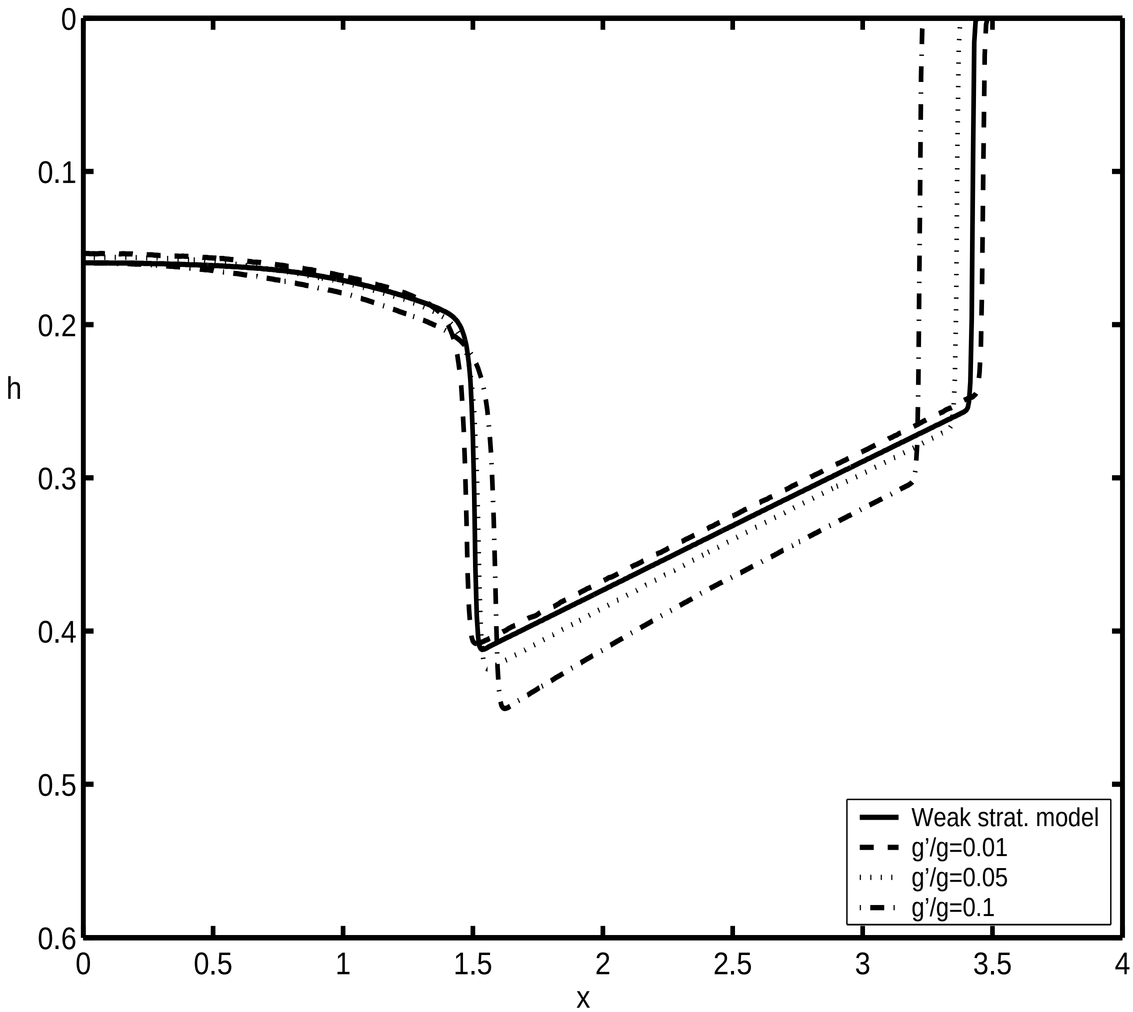


Fig. 4

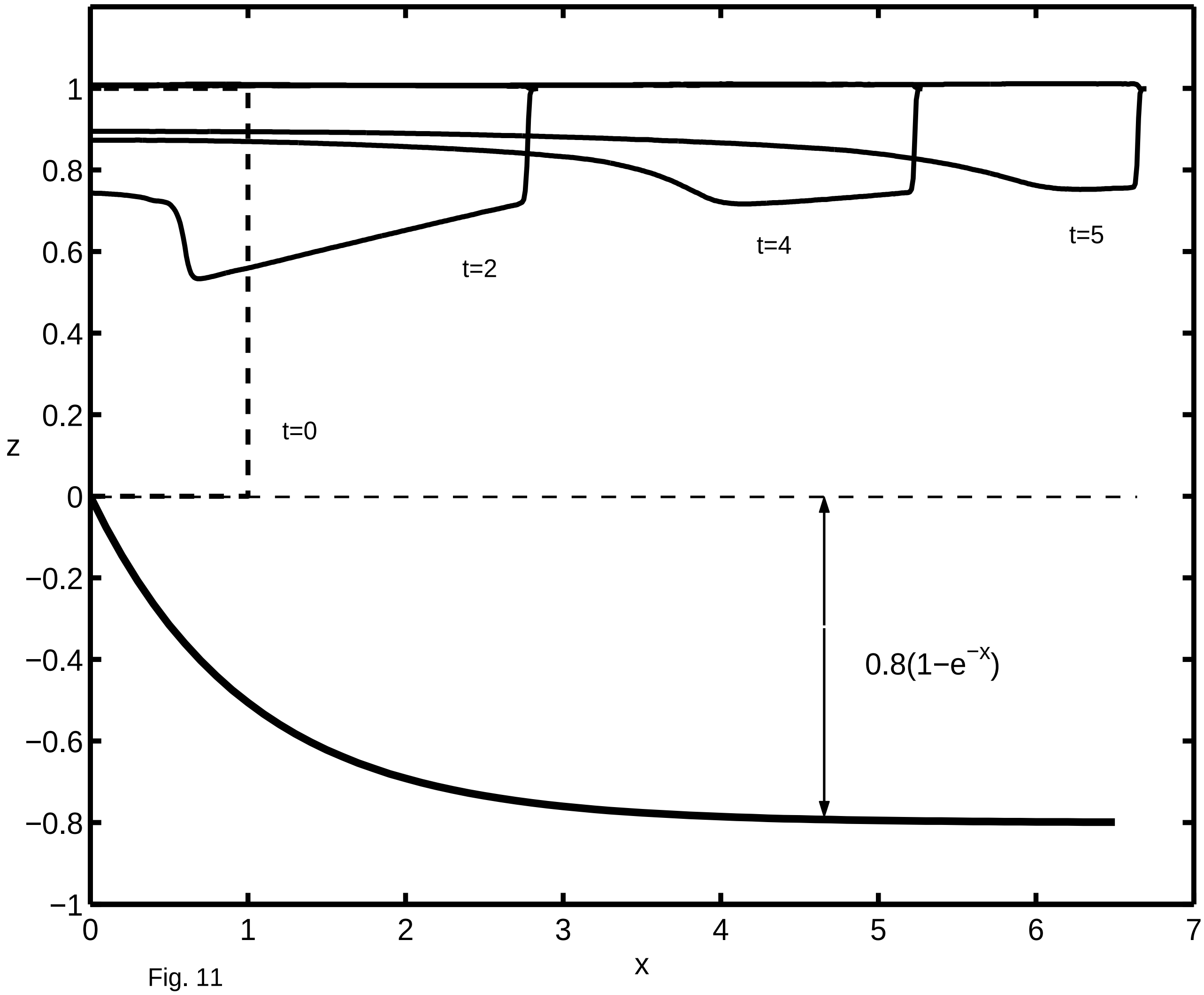


Fig. 11

$$\text{Equation of State:} \quad \rho_1 = \rho_0(1 - \alpha\theta^n)$$

where ρ_0 - ambient fluid density, θ - temperature difference,
α - thermal expansion coefficient

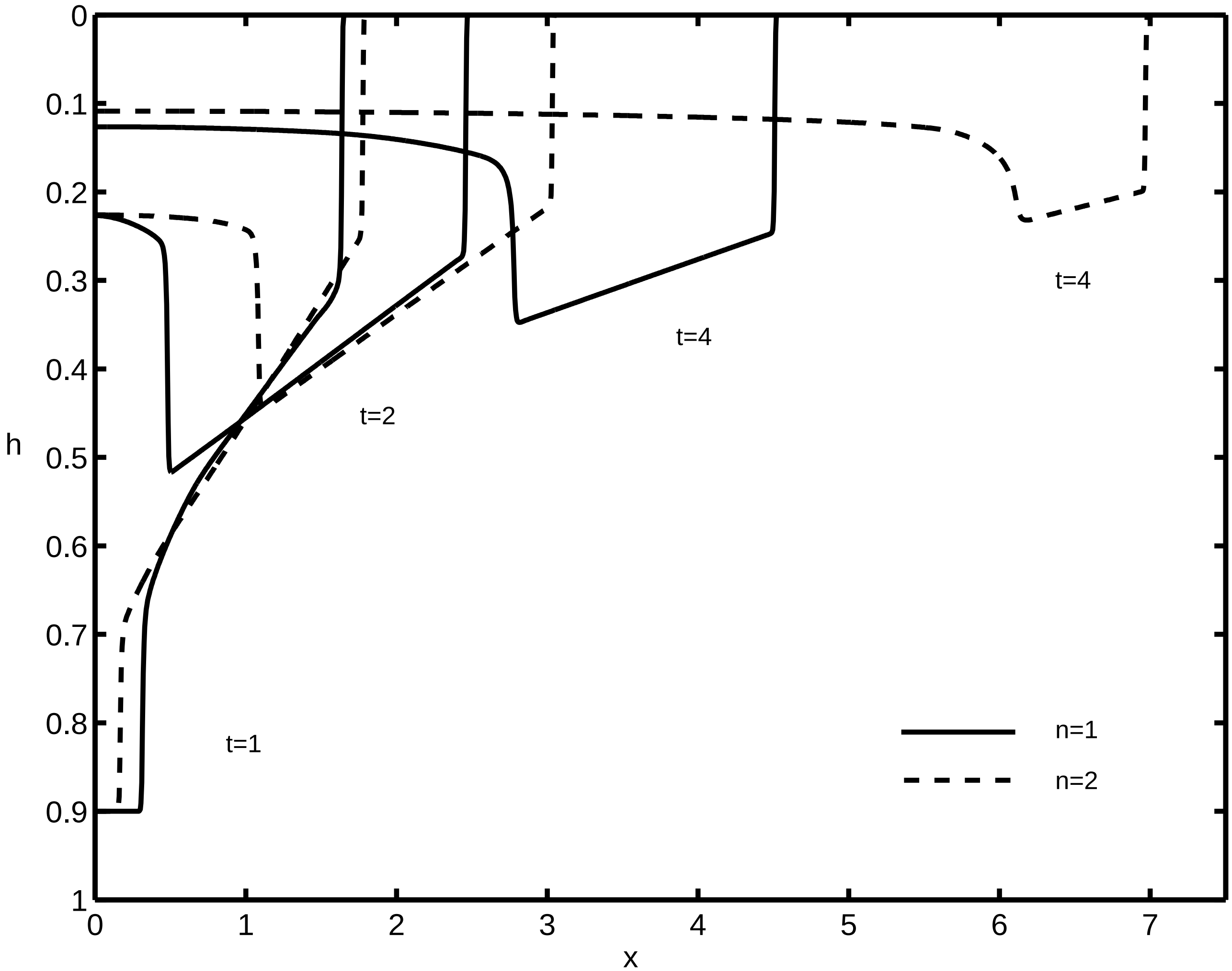


Fig. 9

$$\theta = e^{-0.1t}, \quad n = 1$$

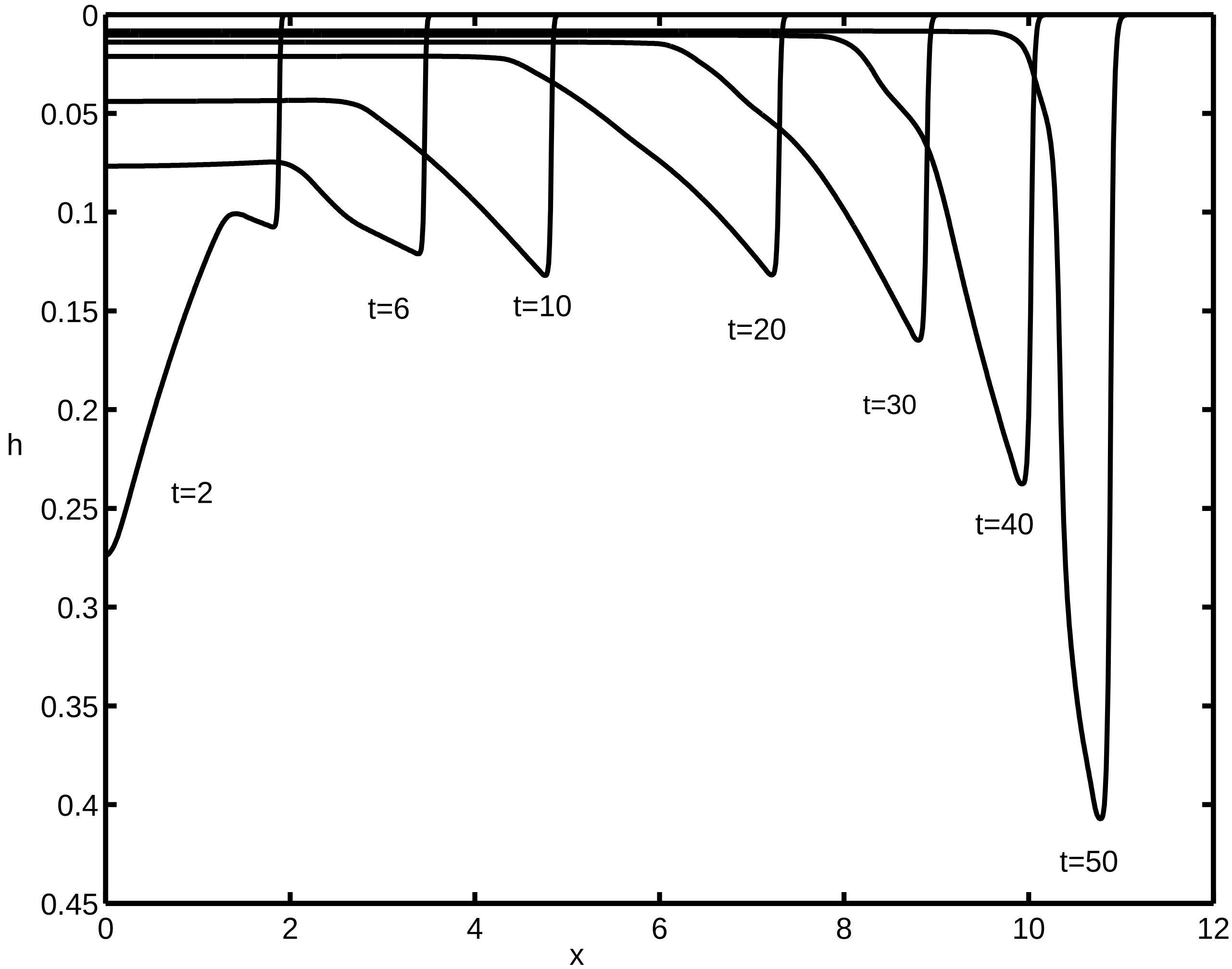


Fig. 7