# THERMALLY DRIVEN GRAVITY CURRENTS

## PART 1: Formulation and Analytical Results

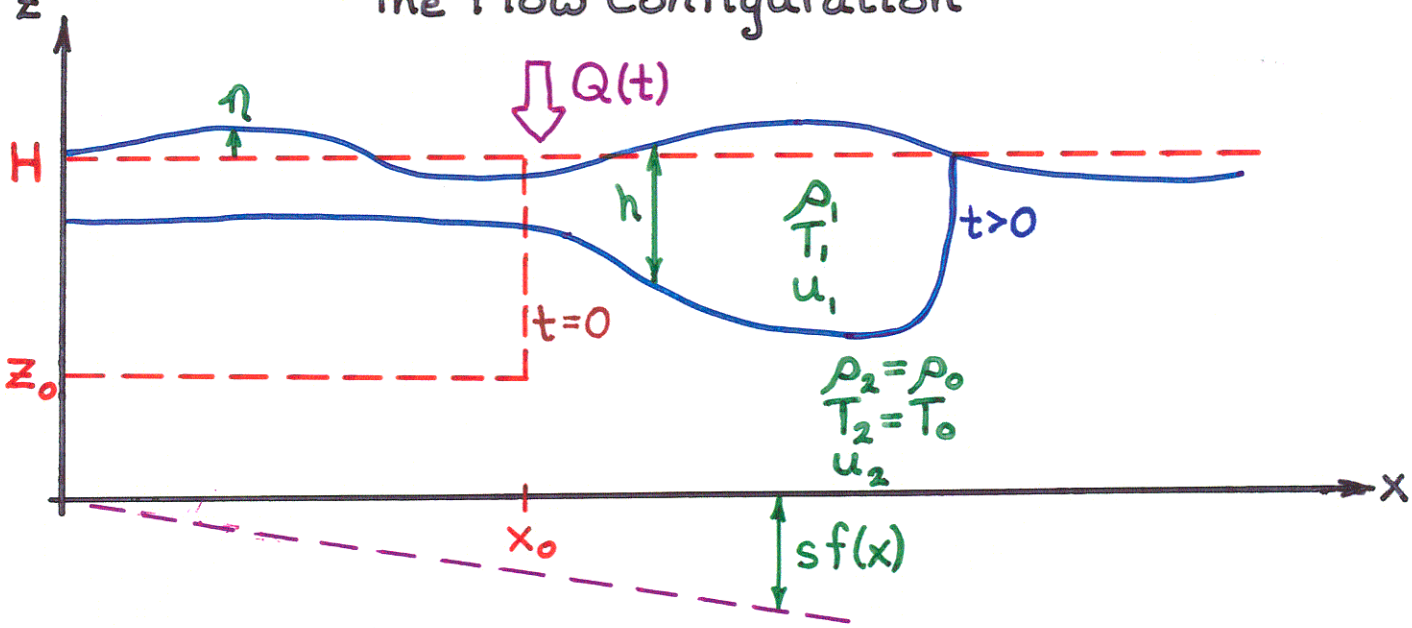
By: Serge D'Alessio

University of Waterloo

Faculty of Mathematics
University of Waterloo
Waterloo, Ontario

With: T.B. Moodie
University of Alberta
J.P. Pascal
Ryerson University

# The Flow Configuration




## Model Assumptions and Approximations:

- inviscid fluid and incompressible & immiscible
- small aspect ratio, $\delta = H/L$, $0 < \delta \ll 1$
- pressure is hydrostatic to $O(\delta^2)$
- Boussinesq approximation
- temperature remains uniform throughout surface layer
- ignore heat transfer between the fluid layers
- equation of state given by:

$$\rho_1(T) = \rho_0[1 - \alpha(T - T_0)^n], \quad n > 0$$

- ignore effects of surface tension

# Planar Shallow Water Equations

In dimensionless form the governing equations become:

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + (1 - \frac{g'}{g}\theta^n)\frac{\partial \eta}{\partial x} = 0$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(u_1 h) = 0$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + \frac{\partial \eta}{\partial x} - \theta^n \frac{\partial h}{\partial x} = 0$$

$$\frac{\partial}{\partial t}(h - \frac{g'}{g}\eta) + \frac{\partial}{\partial x}[u_2(h - H - \frac{g'}{g}\eta - sf(x))] = 0$$

$$\frac{d\theta}{dt} = Q(t) \; , \; \theta = \frac{T - T_0}{T_* - T_0}$$

An important dimensionless parameter is given by:

$$\frac{g'}{g} = \frac{\rho_0 - \rho_*}{\rho_0} = \alpha(T_* - T_0)^n$$

where $g'$ is the reduced gravity.

## Initial and Boundary Conditions

The equations are posed as an initial value problem subject to:

$$u_1(x,0) = 0\ ,\ u_2(x,0) = 0\ ,\ \eta(x,0) = 0\ ,\ \theta(0) = 1$$

$$h(x,0) = \begin{cases} h_* & \text{for } x \le x_0 \\ 0 & \text{for } x > x_0 \end{cases}$$

$$u_1(0,t) = 0\ ,\ u_2(0,t) = 0\ ,\ t > 0$$

$$\frac{\partial h}{\partial x}(0,t) = 0\ ,\ \frac{\partial \eta}{\partial x}(0,t) = 0\ ,\ t > 0$$

$$h \to 0\ ,\ \eta \to 0 \text{ as } x \to \infty$$

To close the system we need to specify $Q(t)$ & $n$.
We assume that $\theta(t)$ is known once $Q(t)$ is given.

We next identify two limiting cases:

- Weak Stratification Limit
- Deep Ambient Layer Limit

## Weak Stratification Limit

If the initial density difference is small we can neglect terms of $O(g'/g)$. In this limit the governing equations can be simplified to (setting $H = 1$ and $s = 0$):

$$\frac{\partial u_1}{\partial t} + \left(\frac{1-3h}{1-h}\right) u_1 \frac{\partial u_1}{\partial x} + \left[\theta^n(1-h) - \frac{u_1^2}{(1-h)^2}\right] \frac{\partial h}{\partial x} = 0$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(u_1 h) = 0$$

$$\frac{d\theta}{dt} = Q(t)$$

$$\eta = -\frac{\theta^n}{2}(1-h)^2 - \frac{u_1^2 h}{1-h} \quad , \quad u_2 = -\frac{u_1 h}{1-h}$$

## Deep Ambient Layer Limit

If we let $H \to \infty$ with $h = O(1)$ and again ignore terms of $O(g'/g)$ and set $s = 0$ we obtain:

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \theta^n \frac{\partial h}{\partial x} = 0$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(u_1 h) = 0$$

$$\frac{d\theta}{dt} = Q(t)$$

$$\frac{\partial \eta}{\partial x} = \theta^n \frac{\partial h}{\partial x} \quad , \quad u_2 = 0$$

# Analysis of Deep Ambient Layer System

Rewrite the equations in vector form:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial}{\partial x}\mathbf{F}(\mathbf{U}, t) = 0$$

$$\frac{d\theta}{dt} = Q(t)$$

where

$$\mathbf{U} = \begin{pmatrix} u_1 \\ h \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \frac{1}{2}u_1^2 + \theta^n h \\ u_1 h \end{pmatrix}$$

The eigenvalues associated with the Jacobian matrix $\partial \mathbf{F}/\partial \mathbf{U}$ are:

$$\lambda^{\pm} = u_1 \pm \sqrt{\theta^n h}$$

Since $\lambda^{\pm}$ are real, the system is hyperbolic and can be expressed in characteristic form as:

$$\frac{d}{dt}\left\{u_1 \pm 2\sqrt{\theta^n h}\right\} = \pm\frac{n\sqrt{\theta^n h}Q(t)}{\theta} \quad \text{along} \quad \frac{dx}{dt} = u_1 \pm \sqrt{\theta^n h}$$

# Similarity Solutions

The deep ambient layer system admits similarity solutions for:

$$Q(t) = Q_0 \Rightarrow \theta(t) = 1 + Q_0 t \quad \text{and}$$

$$Q(t) = -k\theta \Rightarrow \theta(t) = e^{-kt}$$

For $Q(t) = Q_0$ we obtain:

$$u_1(x,t) = \left(t + \frac{1}{Q_0}\right)^{(n-1)/3} v(\xi)\ ,\ h(x,t) = \left(t + \frac{1}{Q_0}\right)^{-(n+2)/3} f(\xi)$$

$$v(\xi) = \frac{(n+2)}{3}\xi\ ,\ f(\xi) = \frac{(n+2)}{18Q_0^n}\left[(2n+1)\xi_f^2 - (n-1)\xi^2\right]$$

$$\xi = x\left(t + \frac{1}{Q_0}\right)^{-(n+2)/3}\ ,\ \xi_f = 3\left(\frac{2Q_0^n h_* x_0}{(n+2)(5n+4)}\right)^{1/3}$$

The shock front speed is given by: $\dot{x}_f = \xi_f \dfrac{(n+2)}{3}\left(t + \dfrac{1}{Q_0}\right)^{(n-1)/3}$

For $Q(t) = -k\theta$ we obtain:

$$u_1(x,t) = e^{-nkt/3} v(\xi)\ ,\ h(x,t) = e^{nkt/3} f(\xi)$$

$$v(\xi) = -\frac{nk}{3}\xi\ ,\ f(\xi) = \frac{n^2k^2}{18}\left[2\xi_f^2 - \xi^2\right]$$

$$\xi = x e^{nkt/3}\ ,\ \xi_f = 3\left(\frac{2h_* x_0}{5n^2k^2}\right)^{1/3}$$

The shock front speed is given by: $\dot{x}_f = -\xi_f \dfrac{nk}{3} e^{-nkt/3}$

# Analysis of Weak Stratification System

Rewrite the equations in vector form:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial}{\partial x}\mathbf{F}(\mathbf{U}, t) = 0$$

$$\frac{d\theta}{dt} = Q(t)$$

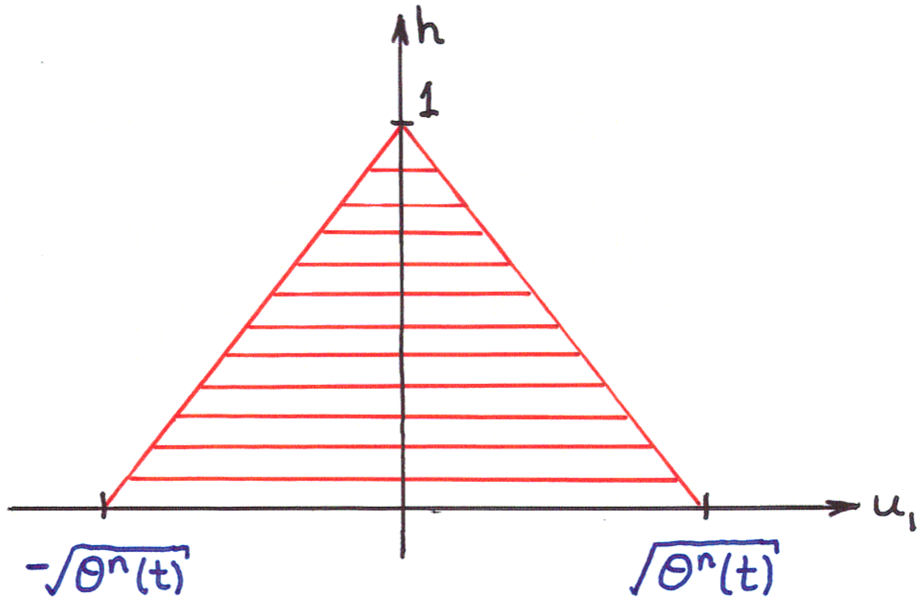
where

$$\mathbf{U} = \begin{pmatrix} u_1 \\ h \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \frac{1}{2}u_1^2 + \eta(u_1, h, \theta) \\ u_1 h \end{pmatrix}, \quad \eta = -\frac{\theta^n}{2}(1-h)^2 - \frac{u_1^2 h}{1-h}$$

The eigenvalues associated with the Jacobian matrix $\partial\mathbf{F}/\partial\mathbf{U}$ are:

$$\lambda^{\pm} = \frac{u_1(1-2h)}{1-h} \pm \frac{1}{1-h}\sqrt{h(1-h)[(1-h)^2\theta^n - u_1^2]}$$

The region of hyperbolicity satisfies: $(1-h)^2\theta^n - u_1^2 \geq 0$



The weakly stratified system can be cast in characteristic form:

$$\frac{du_1}{dt} + \left\{ \frac{u_1}{1-h} \pm \sqrt{\frac{1-h}{h}} \sqrt{\theta^n - \frac{u_1^2}{(1-h)^2}} \right\} \frac{dh}{dt} = 0$$

$$\text{along} \quad \frac{dx}{dt} = \frac{u_1(1-2h)}{1-h} \pm \sqrt{h(1-h)} \sqrt{\theta^n - \frac{u_1^2}{(1-h)^2}}$$

which can be further simplified to:

$$\frac{d}{dt}\left\{ \arcsin\left( \frac{u_1}{(1-h)\sqrt{\theta^n}} \right) \pm \arcsin\left( 2(h - \frac{1}{2}) \right) \right\} = -\frac{n u_1 Q(t)}{2\theta \sqrt{(1-h)^2\theta^n - u_1^2}}$$

$$\text{along} \quad \frac{dx}{dt} = \frac{u_1(1-2h)}{1-h} \pm \sqrt{h(1-h)} \sqrt{\theta^n - \frac{u_1^2}{(1-h)^2}}$$

The nonlinear nature of the characteristic fields is assessed using:

$$\text{grad}_{\mathbf{U}} \lambda^{\pm} \cdot \mathbf{v}^{\pm}$$

where $\mathbf{v}^{\pm}$ are the eigenvectors of the Jacobian matrix.

The characteristic fields, $\lambda^{\pm}(\mathbf{U})$, are locally linearly degenerate about the state $u_1 = 0\ ,\ h = 1/2$.
This may signal a bifurcation in the flow behaviour.
We explore this further using a weakly nonlinear analysis.

# Weakly Nonlinear Analysis

We begin by expanding the variables about the basic state $(u_1, h) = (0, h_*)$ and work with the quadratically nonlinear system:

$$\frac{\partial \hat{u}}{\partial t} + \frac{(1-3h_*)}{(1-h_*)}\hat{u}\frac{\partial \hat{u}}{\partial x} + \theta^n(1 - h_* - \hat{h})\frac{\partial \hat{h}}{\partial x} = 0$$

$$\frac{\partial \hat{h}}{\partial t} + (h_* + \hat{h})\frac{\partial \hat{u}}{\partial x} + \hat{u}\frac{\partial \hat{h}}{\partial x} = 0$$

These equations can be combined to yield the single equation:

$$\hat{h}_{tt} - h_*(1-h_*)\theta^n \hat{h}_{xx} = -(\hat{u}\hat{h})_{xt} - h_*\theta^n(\hat{h}\hat{h}_x)_x + \frac{(1-3h_*)h_*}{(1-h_*)}(\hat{u}\hat{u}_x)_x$$

We illustrate the procedure for the specific case $Q(t) = Q_0$ and $n = 2$ with $0 < Q_0 = \varepsilon \ll 1$. Thus, $\theta(t) = 1 + \varepsilon t$.
Next, we introduce the coordinates and series expansions given by:

$$\xi = x - c(t)t \ , \ \eta = x + c(t)t \ , \ T = \varepsilon t$$

where $c(t) = c_0(1 + \frac{\varepsilon}{2}t)$ and $c_0 = \sqrt{h_*(1-h_*)}$

$$\hat{h} = \varepsilon h^{(0)}(\xi, \eta, T) + \varepsilon^2 h^{(1)}(\xi, \eta, T) + \cdots$$

$$\hat{u} = \varepsilon u^{(0)}(\xi, \eta, T) + \varepsilon^2 u^{(1)}(\xi, \eta, T) + \cdots$$

## The $O(\varepsilon)$ Problem:

$$h^{(0)}_{\eta\xi} = 0$$

$$c_0(\partial_\eta - \partial_\xi)u^{(0)} = -(1-h_*)(1+T)(\partial_\eta + \partial_\xi)h^{(0)}$$

These can be easily solved to give:

$$h^{(0)} = \phi(\xi, T) + \psi(\eta, T)$$

$$u^{(0)} = \frac{(1-h_*)(1+T)}{c_0}(\phi(\xi,T) - \psi(\eta,T))$$

## The $O(\varepsilon^2)$ Problem:

$$-4c_0^2 h^{(1)}_{\eta\xi} = A(\xi,T) + B(\xi,\eta,T) + C(\eta,T)$$

where

$$A(\xi,T) = \frac{2c_0}{(1+T)}\phi_{T\xi} + \frac{3}{2}(1-2h_*)(\phi^2)_{\xi\xi}$$

$$B(\xi,\eta,T) = -(1-2h_*)[\phi\psi_{\eta\eta} + \psi\phi_{\xi\xi} + 2\psi_\eta\phi_\xi]$$

$$C(\eta,T) = -\frac{2c_0}{(1+T)}\psi_{T\eta} + \frac{3}{2}(1-2h_*)(\psi^2)_{\eta\eta}$$

Imposing the solvability condition $A = 0$ and integrating yields:

$$\phi_T + \frac{3(1-2h_*)(1+T)}{2c_0}\phi\phi_\xi = 0$$

If $\phi(\xi, 0) = f(\xi)$ then the solution can be expressed implicitly in terms of the parameter $s$ as:

$$\phi(\xi, T) = f(s) \ \text{ along } \ \xi = \frac{3(1-2h_*)}{4c_0}[(1+T)^2 - 1]f(s) + s$$

Shock formation occurs when $|\phi_\xi| \to \infty$ where

$$\phi_\xi = \frac{4c_0 f'(s)}{4c_0 + 3(1-2h_*)f'(s)[(1+T)^2 - 1]}$$

which becomes infinite when

$$T(s) = -1 \pm \sqrt{1 - K(s)}\ , \ K(s) = \frac{4c_0}{3(1-2h_*)f'(s)}$$

Thus, a shock will form for $T > 0$ when $K(s) < 0$.
For $f'(s) > 0$ (corresponding to the back side of the rightward propagating smooth initial wave) this will happen provided $h_* > 1/2$.
A similar conclusion follows from the solvability condition $C = 0$.