MIXED CONVECTION FROM AN ACCELERATING ELLIPTIC CYLINDER

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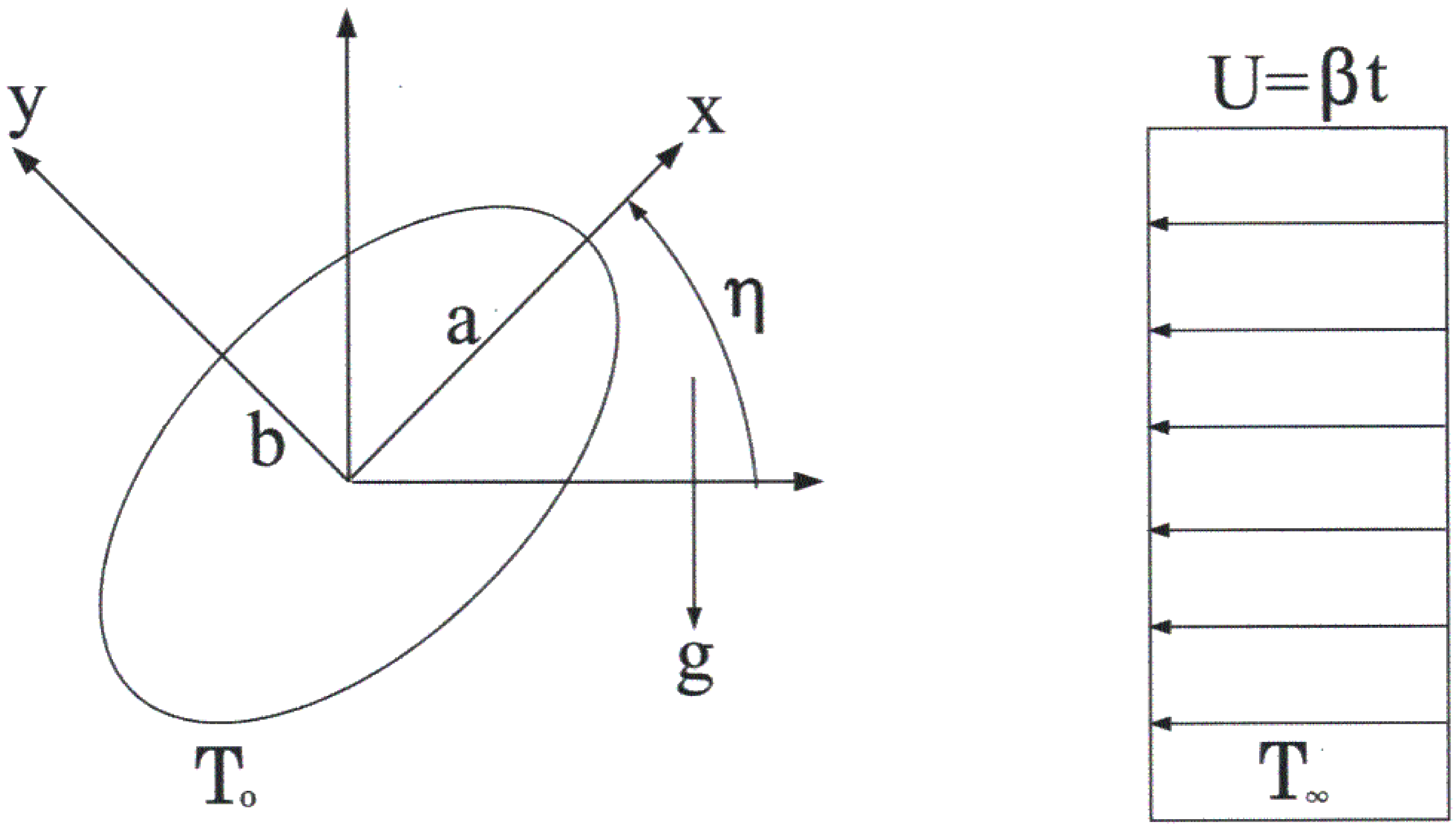
INTRODUCTION

We consider the unsteady two-dimensional thermal-fluid problem of mixed convection resulting from a viscous incompressible fluid that is uniformly accelerating past an isothermal inclined elliptic cylinder shown below:



The problem can be completely characterized by the following dimensionless parameters:

$R = 2c\sqrt{c\beta}/\nu$	Reynolds Number
$Ra = \alpha g(T_0 - T_\infty)/\beta$	Rayleigh Number
$Pr = \nu/\kappa$	Prandtl Number
$r = b/a$	Geometry Parameter
η	Inclination

GOVERNING EQUATIONS

In terms of the stream function and vorticity the dimensionless unsteady equations in Cartesian coordinates become:

NAVIER-STOKES EQUATIONS

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \zeta$$

$$\frac{\partial\zeta}{\partial t} = \frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} + \frac{2}{R}\left(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2}\right)$$
$$+Ra\left(\cos\eta\frac{\partial\phi}{\partial x} - \sin\eta\frac{\partial\phi}{\partial y}\right)$$

HEAT EQUATION

$$\frac{\partial\phi}{\partial t} = \frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y} + \frac{2}{RPr}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)$$

where

$$u = -\frac{\partial\psi}{\partial y}\,,\ v = \frac{\partial\psi}{\partial x}\,,\ \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\,,\ \phi = \frac{T - T_\infty}{T_0 - T_\infty}$$

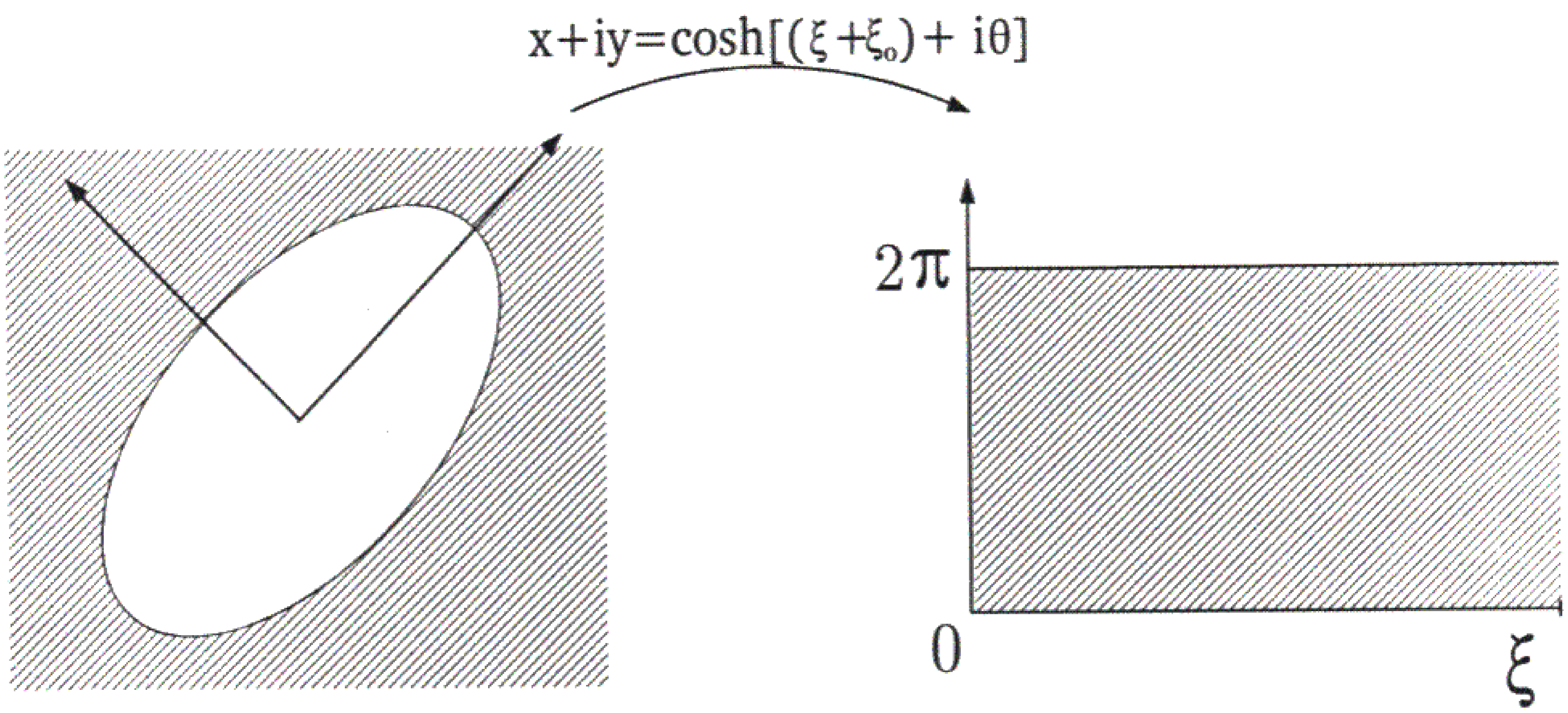
CONFORMAL MAPPING

The conformal transformation

$$x + iy = \cosh[(\xi + \xi_0) + i\theta]$$

$$\tanh \xi_0 = r$$

shown below maps the infinite region exterior to the cylinder to a semi-infinte rectangular strip bounded by $0 \leq \xi < \infty$, $0 \leq \theta \leq 2\pi$.



The equations of motion now become:

$$\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\theta^2}=M^2\zeta$$

$$\frac{\partial\zeta}{\partial t}=\frac{1}{M^2}[-\frac{\partial\psi}{\partial\xi}\frac{\partial\zeta}{\partial\theta}+\frac{\partial\psi}{\partial\theta}\frac{\partial\zeta}{\partial\xi}+\frac{2}{R}(\frac{\partial^2\zeta}{\partial\xi^2}+\frac{\partial^2\zeta}{\partial\theta^2})$$

$$+Ra(A\frac{\partial\phi}{\partial\xi}-B\frac{\partial\phi}{\partial\theta})]$$

$$\frac{\partial\phi}{\partial t}=\frac{1}{M^2}[-\frac{\partial\psi}{\partial\xi}\frac{\partial\phi}{\partial\theta}+\frac{\partial\psi}{\partial\theta}\frac{\partial\phi}{\partial\xi}+\frac{2}{RPr}(\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial\theta^2})]$$

where

$$M^2(\xi,\theta)=\frac{1}{2}[\cosh 2(\xi+\xi_0)-\cos 2\theta]$$

$$A(\xi,\theta)=\sinh(\xi+\xi_0)\cos\eta\cos\theta-\cosh(\xi+\xi_0)\sin\eta\sin\theta$$

$$B(\xi,\theta)=\cosh(\xi+\xi_0)\cos\eta\sin\theta+\sinh(\xi+\xi_0)\sin\eta\cos\theta$$

BOUNDARY CONDITIONS

Boundary conditions include the no-slip and isothermal conditions:

$$\psi=\frac{\partial\psi}{\partial\xi}=0 \text{ and } \phi=1 \text{ on } \xi=0$$

the periodicity conditions:

$$\psi(\xi,\theta,t)=\psi(\xi,\theta+2\pi,t)$$

$$\zeta(\xi,\theta,t)=\zeta(\xi,\theta+2\pi,t)$$

$$\phi(\xi,\theta,t)=\phi(\xi,\theta+2\pi,t)$$

and the far-field conditions:

$$e^{-\xi}\frac{\partial\psi}{\partial\xi}\to\frac{1}{2}te^{\xi_0}\sin(\theta+\eta)\ ,\ e^{-\xi}\frac{\partial\psi}{\partial\theta}\to\frac{1}{2}te^{\xi_0}\cos(\theta+\eta)$$

$$\zeta,\phi\to 0 \text{ as } \xi\to\infty$$

INTEGRAL CONDITIONS

The vorticity field can be shown to satisfy integral constraints. These can be derived from the boundary conditions using Green's second identity and are given by:

$$\frac{1}{\pi}\int_0^\infty \int_0^{2\pi} e^{-n\xi} M^2 \zeta \sin(n\theta) d\theta d\xi = te^{\xi_0} \cos(\eta)\delta_{1,n} \ , \ n = 1, 2, \ldots$$

$$\frac{1}{\pi}\int_0^\infty \int_0^{2\pi} e^{-n\xi} M^2 \zeta \cos(n\theta) d\theta d\xi = te^{\xi_0} \sin(\eta)\delta_{1,n} \ , \ n = 0, 1, \ldots$$

These integral conditions can be used as a means of determining the surface vorticity.

INITIAL CONDITIONS

Since the fluid motion starts from rest, the initial conditions for ψ and ζ are simply:

$$\psi(\xi, \theta, t = 0) = \zeta(\xi, \theta, t = 0) = 0$$

The initial temperature distribution is given by:

$$\phi(\xi, \theta, t = 0) = \begin{cases} 1 \text{ on } \xi = 0 \\ 0 \text{ for } \xi \neq 0 \end{cases}$$

ANALYTICAL SOLUTION PROCEDURE

By making boundary-layer type approximations, we expect the leading order terms ψ_0, ζ_0, ϕ_0 to be dictated by the equations:

$$\frac{\partial^2 \psi_0}{\partial \xi^2} = M_0^2 \zeta_0$$

$$\frac{\partial \zeta_0}{\partial t} = \frac{1}{M_0^2}[\frac{2}{R}\frac{\partial^2 \zeta_0}{\partial \xi^2} + RaA_0 \frac{\partial \phi_0}{\partial \xi}]$$

$$\frac{\partial \phi_0}{\partial t} = \frac{2}{M_0^2 RPr}\frac{\partial^2 \phi_0}{\partial \xi^2}$$

where

$$M_0^2(\theta) = \frac{1}{2}[\cosh(2\xi_0) - \cos(2\theta)]$$

$$A_0(\theta) = \sinh \xi_0 \cos \eta \cos \theta - \cosh \xi_0 \sin \eta \sin \theta$$

Similarity solutions for ϕ_0, ζ_0 with $Pr = 1$ satisfying the boundary conditions are found to be:

$$\phi_0 = \mathrm{erfc}(M_0 z)$$

$$\zeta_0 = \frac{t}{\lambda}[\frac{1}{\sqrt{\pi} M_0} e^{-M_0^2 z^2} - z\mathrm{erfc}(M_0 z)]H(\theta)$$

$$+\frac{2RaA_0 t}{\lambda}[\frac{1}{\sqrt{\pi} M_0} e^{-M_0^2 z^2} - 2z\mathrm{erfc}(M_0 z)]$$

where

$$\mathrm{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \ , \ z = \frac{\xi}{\lambda} \ , \ \lambda = \sqrt{\frac{8t}{R}}$$

and $H(\theta)$ is an arbitrary function.

BOUNDARY-LAYER VARIABLES

The similarity solution suggests that we consider rescaling the coordinate ξ and flow variables ψ, ζ as follows:

$$\xi = \lambda z \; , \; \psi = \lambda \Psi \; , \; \zeta = \omega / \lambda \; , \; \lambda = \sqrt{\frac{8t}{R}}$$

TRANSFORMED EQUATIONS

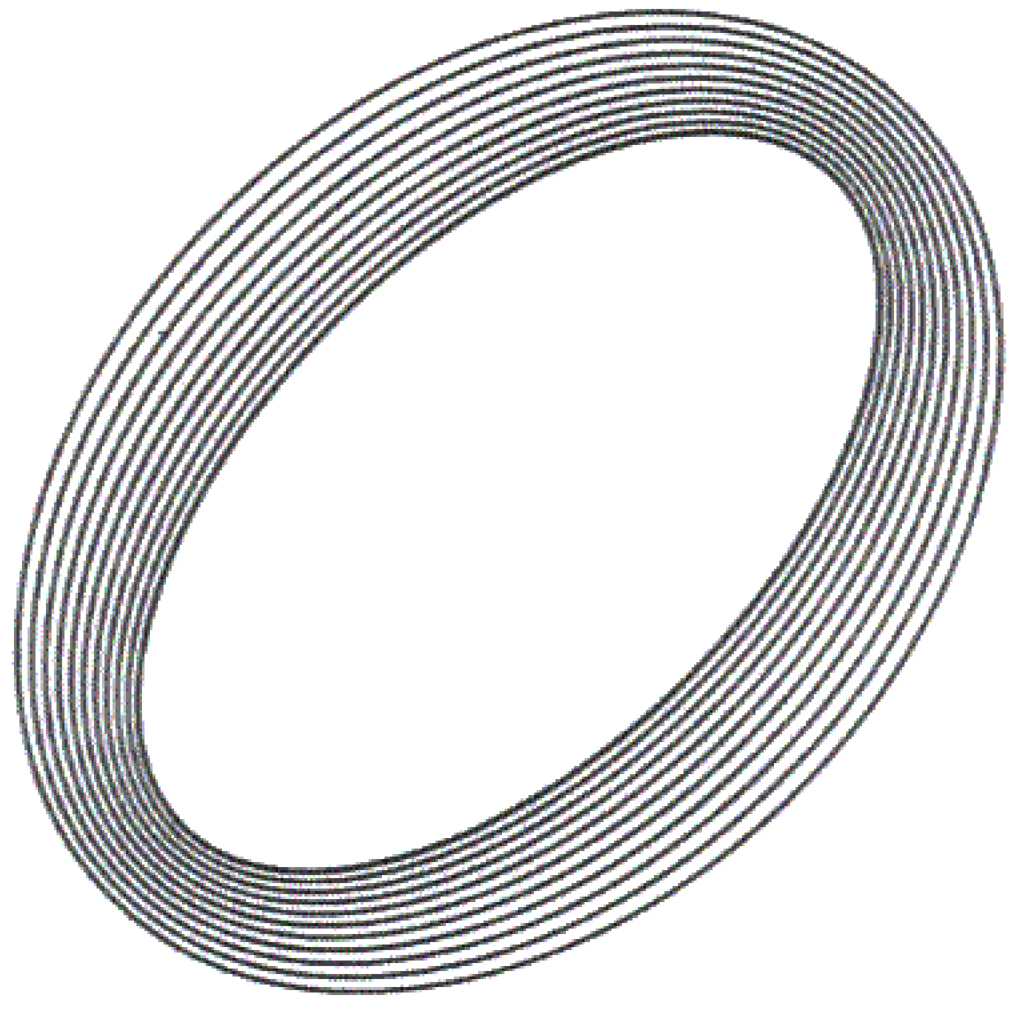
$$\frac{\partial^2 \Psi}{\partial z^2} + \lambda^2 \frac{\partial^2 \Psi}{\partial \theta^2} = M^2 \omega$$

$$\frac{1}{M^2}\frac{\partial^2 \omega}{\partial z^2} + 2z\frac{\partial \omega}{\partial z} + 2\omega = 4t\frac{\partial \omega}{\partial t} - \frac{\lambda^2}{M^2}\frac{\partial^2 \omega}{\partial \theta^2}$$
$$-\frac{4t}{M^2}\left(\frac{\partial \Psi}{\partial \theta}\frac{\partial \omega}{\partial z} - \frac{\partial \Psi}{\partial z}\frac{\partial \omega}{\partial \theta}\right)$$
$$-\frac{4t}{M^2}Ra\left(A\frac{\partial \phi}{\partial z} - B\lambda\frac{\partial \phi}{\partial \theta}\right)$$

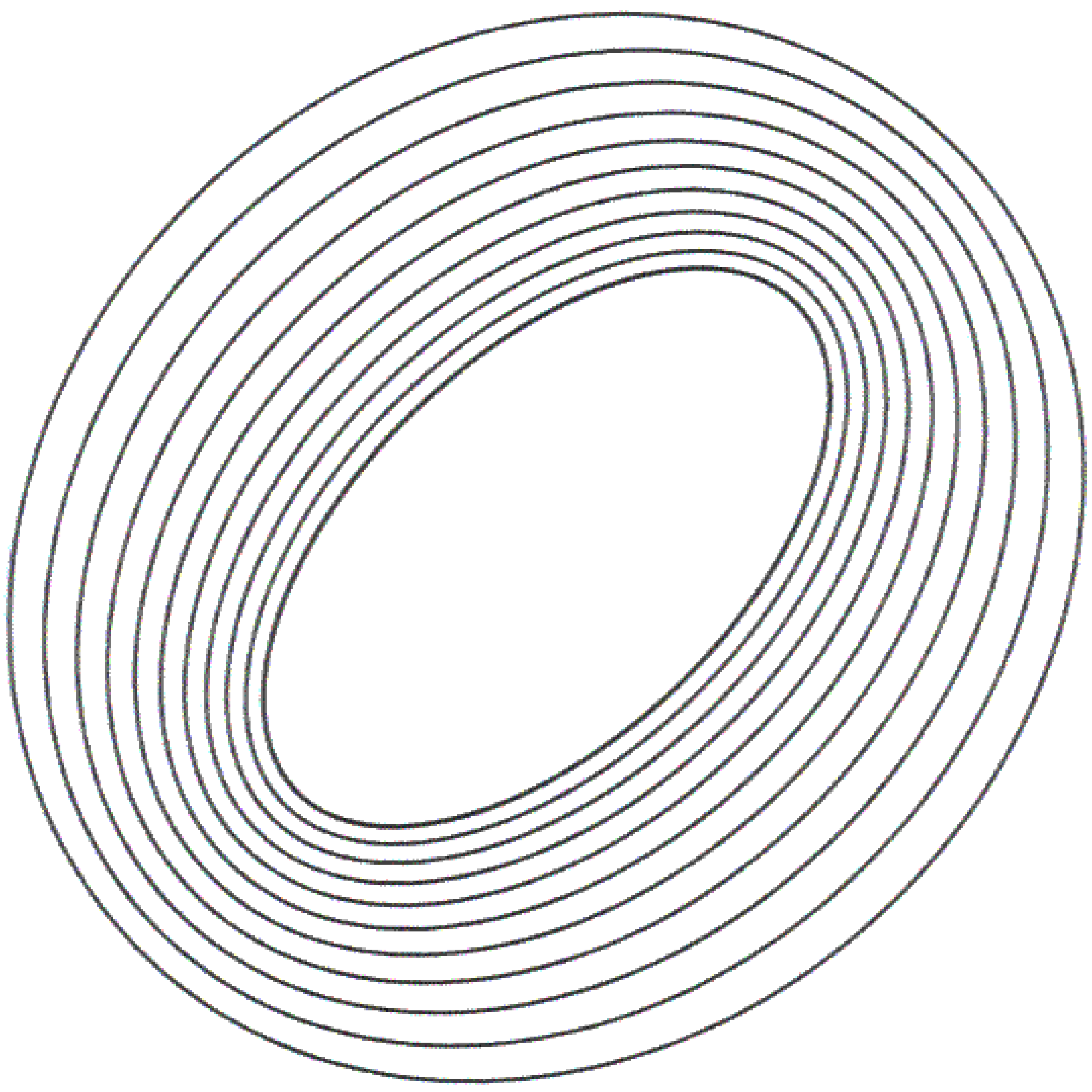
$$\frac{1}{PrM^2}\frac{\partial^2 \phi}{\partial z^2} + 2z\frac{\partial \phi}{\partial z} = 4t\frac{\partial \phi}{\partial t} - \frac{\lambda^2}{PrM^2}\frac{\partial^2 \phi}{\partial \theta^2}$$
$$-\frac{4t}{M^2}\left(\frac{\partial \Psi}{\partial \theta}\frac{\partial \phi}{\partial z} - \frac{\partial \Psi}{\partial z}\frac{\partial \phi}{\partial \theta}\right)$$

Diagram illustrating the expanding grid.

t_1



$t_2 > t_1$



MULTIPLE SERIES EXPANSION

For large R and small t, λ is also small, and it is possible to expand the variables in a double series in both λ and t as follows:

$$\Psi = (\Psi_{00} + t\Psi_{01} + \cdots) + \lambda(\Psi_{10} + t\Psi_{11} + \cdots) + O(\lambda^2)$$

$$\omega = (\omega_{00} + t\omega_{01} + \cdots) + \lambda(\omega_{10} + t\omega_{11} + \cdots) + O(\lambda^2)$$

$$\phi = (\phi_{00} + t\phi_{01} + t^2\phi_{02} \cdots) + \lambda(\phi_{10} + t\phi_{11} + \cdots) + O(\lambda^2)$$

It follows that the similarity solutions will emerge naturally from this expansion procedure. To proceed we need to expand all other quantities such as $e^{-n\lambda z}$, A, B and M^2 in a Taylor series:

$$e^{-n\lambda z} = 1 - n\lambda z + \frac{n^2\lambda^2 z^2}{2} - \cdots$$

$$A(z,\theta) = A_0(\theta) + \lambda z A_1(\theta) + \frac{\lambda^2 z^2}{2} A_0(\theta) + \cdots$$

$$B(z,\theta) = B_0(\theta) + \lambda z B_1(\theta) + \frac{\lambda^2 z^2}{2} B_0(\theta) + \cdots$$

$$M^2(z,\theta) = M_0^2(\theta) + \sinh(2\xi_0)\lambda z + \cosh(2\xi_0)\lambda^2 z^2 + \cdots$$

Substituting these expansions and equating like powers of λ and t leads to a hierarchy of problems at various levels of approximation. We have explicitly determined the nonzero terms:

$$\Psi_{01}\ ,\ \omega_{01}\ ,\ \phi_{00}\ ,\ \phi_{02}\ ,\ \phi_{10}$$

Also, we have deduced that:

$$\Psi_{00} = \Psi_{10} = \omega_{00} = \omega_{10} = \phi_{01} = 0$$

NUMERICAL METHOD

The computational domain $0 \le z \le z_\infty$, $0 \le \theta \le 2\pi$ is discretized into an $N \times L$ grid as follows:

$$z_i = ih \ , \ i = 0, 1, \ldots, N$$

$$\theta_j = jk \ , \ j = 0, 1, \ldots, L$$

$$h = \frac{z_\infty}{N} \ , \ k = \frac{2\pi}{L}$$

Solution of the Stream Function

Expand Ψ into a Fourier series

$$\Psi(z, \theta, t) = \frac{1}{2} F_0(z, t) + \sum_{n=1}^{\infty} [F_n(z, t) \cos n\theta + f_n(z, t) \sin n\theta]$$

$$\text{then } \frac{\partial^2 \Psi}{\partial z^2} + \lambda^2 \frac{\partial^2 \Psi}{\partial \theta^2} = M^2 \omega \text{ becomes}$$

$$\frac{\partial^2 f_n}{\partial z^2} - n^2 \lambda^2 f_n = r_n(z, t) \ ; \ n = 1, 2, \ldots$$

$$\frac{\partial^2 F_n}{\partial z^2} - n^2 \lambda^2 F_n = s_n(z, t) \ ; \ n = 0, 1, \ldots$$

$$r_n(z, t) = \frac{1}{\pi} \int_0^{2\pi} M^2 \omega \sin n\theta d\theta \ , \ s_n(z, t) = \frac{1}{\pi} \int_0^{2\pi} M^2 \omega \cos n\theta d\theta$$

Boundary Conditions

$$\text{no-slip}: \quad F_0 = F_n = f_n = 0 \text{ at } z = 0$$

$$\frac{\partial F_0}{\partial z} = \frac{\partial F_n}{\partial z} = \frac{\partial f_n}{\partial z} = 0 \text{ at } z = 0$$

$$\text{far-field}: \quad e^{-\lambda z} F_0 \to 0 \ , \ e^{-\lambda z} F_n \to \frac{t}{2\lambda} e^{\xi_0} \sin \eta \delta_{1,n}$$

$$e^{-\lambda z} f_n \to \frac{t}{2\lambda} e^{\xi_0} \cos \eta \delta_{1,n} \text{ as } z \to \infty$$

$$e^{-\lambda z} \frac{\partial F_0}{\partial z} \to 0 \ , \ e^{-\lambda z} \frac{\partial F_n}{\partial z} \to \frac{t}{2} e^{\xi_0} \sin \eta \delta_{1,n}$$

$$e^{-\lambda z} \frac{\partial f_n}{\partial z} \to \frac{t}{2} e^{\xi_0} \cos \eta \delta_{1,n} \text{ as } z \to \infty$$

Integral Conditions

$$\int_0^\infty e^{-n\lambda z} r_n(z,t) dz = t e^{\xi_0} \cos \eta \delta_{1,n}$$

$$\int_0^\infty e^{-n\lambda z} s_n(z,t) dz = t e^{\xi_0} \sin \eta \delta_{1,n}$$

$$\int_0^{z_\infty} s_0(z,t) dz = 0$$

Solution of the Vorticity and Temperature

Due to singularity in intial temperature, we rescale ϕ as $\Phi = \lambda\phi$ to avoid numerical instabilities. We use finite differences to solve for ω, Φ. Rewrite these equations in generic form as follows:

$$t\frac{\partial\chi}{\partial t} = q(z, \theta, t)$$

and use the Crank-Nicolson implicit scheme to advance the solution in time from t to $t + \tau$:

$$\chi t|_t^{t+\tau} - \int_t^{t+\tau} \chi dt = \int_t^{t+\tau} q dt$$

Next, approximate integrals using the trapezoidal rule to arrive at:

$$\chi(z, \theta, t + \tau) = \chi(z, \theta, t) + (\frac{\tau}{2t + \tau})[q(z, \theta, t + \tau) + q(z, \theta, t)]$$

this equation is solved iteratively using the Gauss-Seidel procedure.

Boundary Conditions

surface conditions: $\Phi(0, \theta, t) = \lambda$

$$\omega(0, \theta, t) = \frac{1}{M_0^2}\{\frac{1}{2}s_0(0, t) + \sum_{n=1}^{\infty}[r_n(0, t)\sin n\theta + s_n(0, t)\cos n\theta]\}$$

$$\text{periodicity}: \ \omega(z, \theta, t) = \omega(z, \theta + 2\pi, t)$$

$$\Phi(z, \theta, t) = \Phi(z, \theta + 2\pi, t)$$

$$\text{far-field}: \ \omega(z_\infty, \theta, t) = \Phi(z_\infty, \theta, t) = 0$$

RESULTS & COMPARISONS

Computational Parameters

Outer Boundary: $z_\infty = 10$

Grid Sizes Used: $N \times L = 100 \times 80$ course grid

$N \times L = 160 \times 120$ fine grid

Time Steps Used: $\tau = .0001$ till $t = .001$

then $\tau = .001$ till $t = .01$

then $\tau = .01$ for $t > .01$

Relaxation Parameter Used: $.5 \leq \gamma \leq .75$

Convergence Criterion Adopted: $|\omega^{(k+1)}(0,\theta,t) - \omega^{(k)}(0,\theta,t)| < \varepsilon$

Number of Terms in Fourier Series: $20 \leq n \leq 30$

Numerical Validation

The numerics can be verfied against the analytical solutions found. We compare the following quantities:

$$\zeta_0(\theta,t)\ ,\ C_D(t)\ ,\ C_L(t)\ ,\ Nu(\theta,t)\ ,\ \overline{Nu}(t)$$

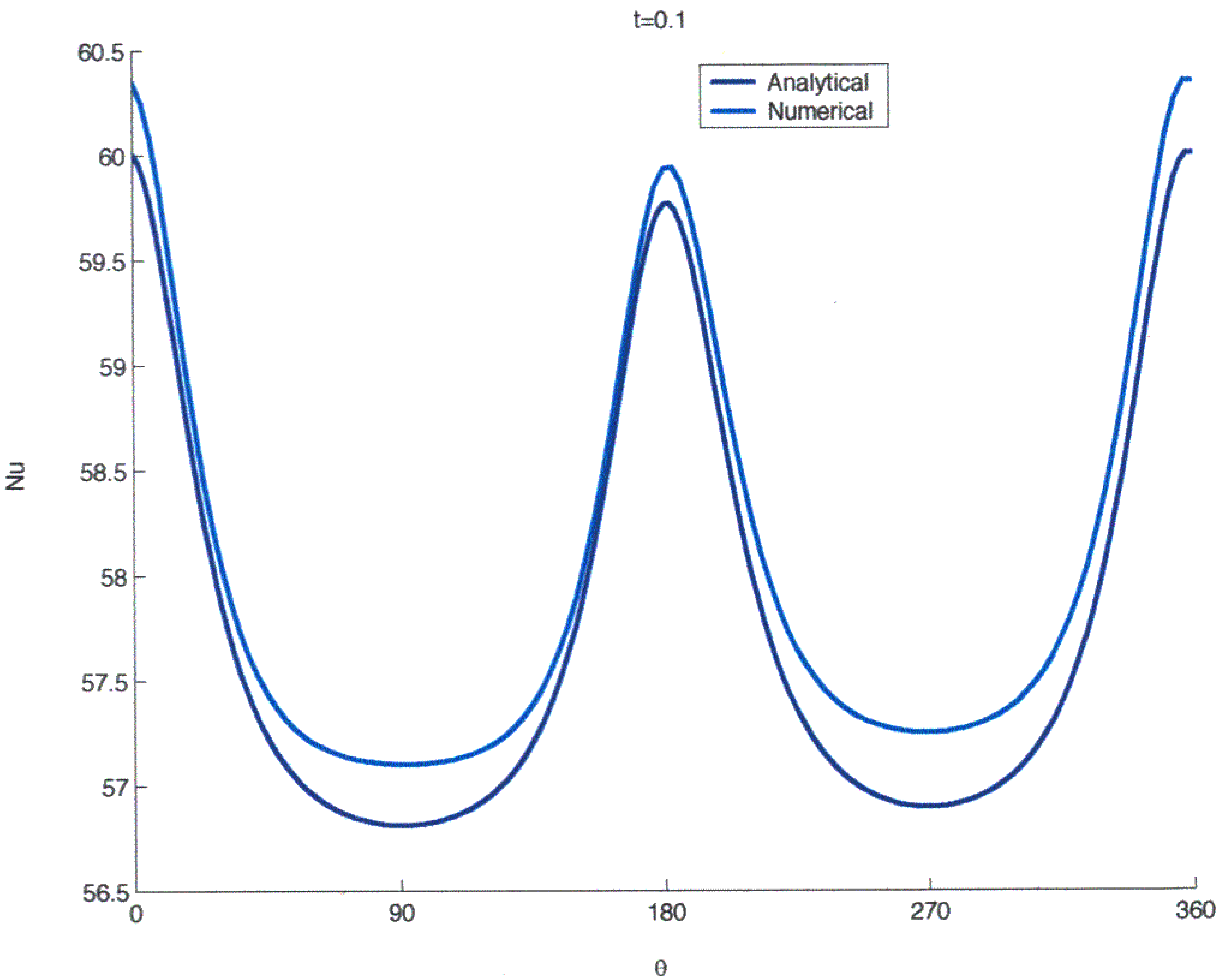
Numerical results have been obtained for times in the interval $0 \leq t \leq 3$ and for fixed values of Pr, r, η given by:

$$Pr = 1\ ,\ r = \frac{1}{2}\ ,\ \eta = \frac{\pi}{4}$$

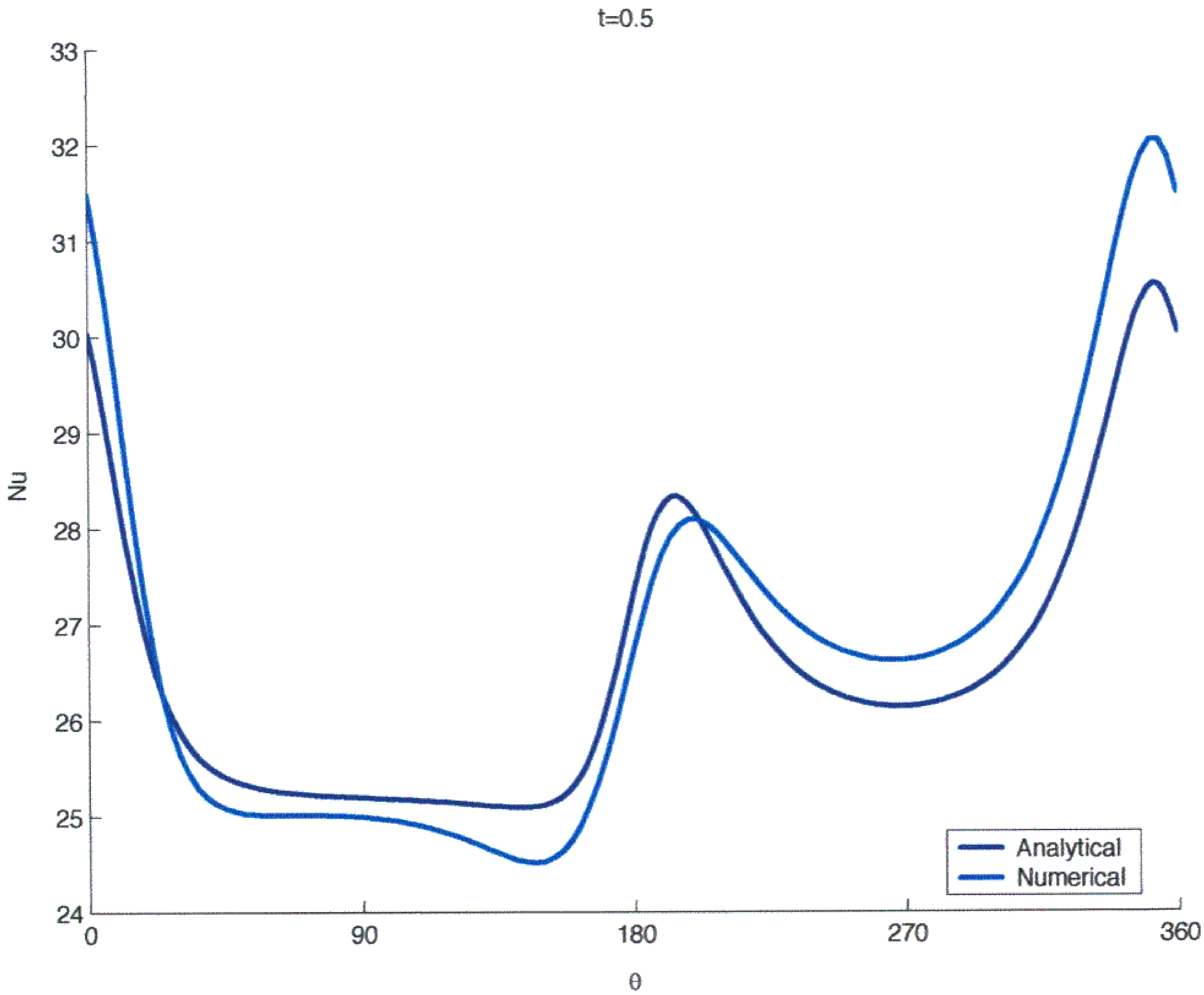
and for the following parameter ranges for R, Ra:

$$0 \leq Ra \leq 10\ ,\ 100 \leq R \leq 1000$$

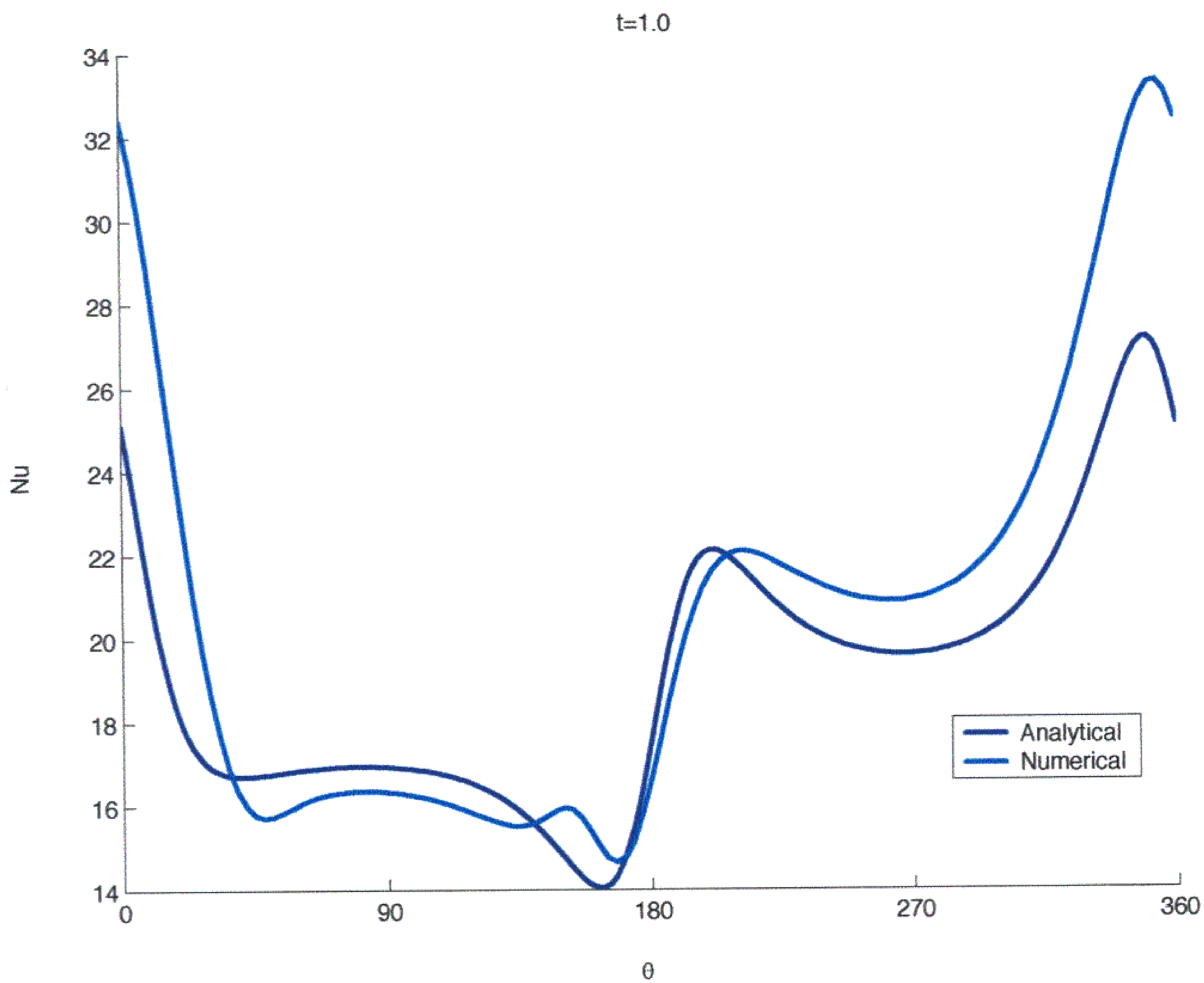
Ra=1 , R=500


t=0.1
Analytical
Numerical
Nu
θ
60.5
60
59.5
59
58.5
58
57.5
57
56.5
0
90
180
270
360

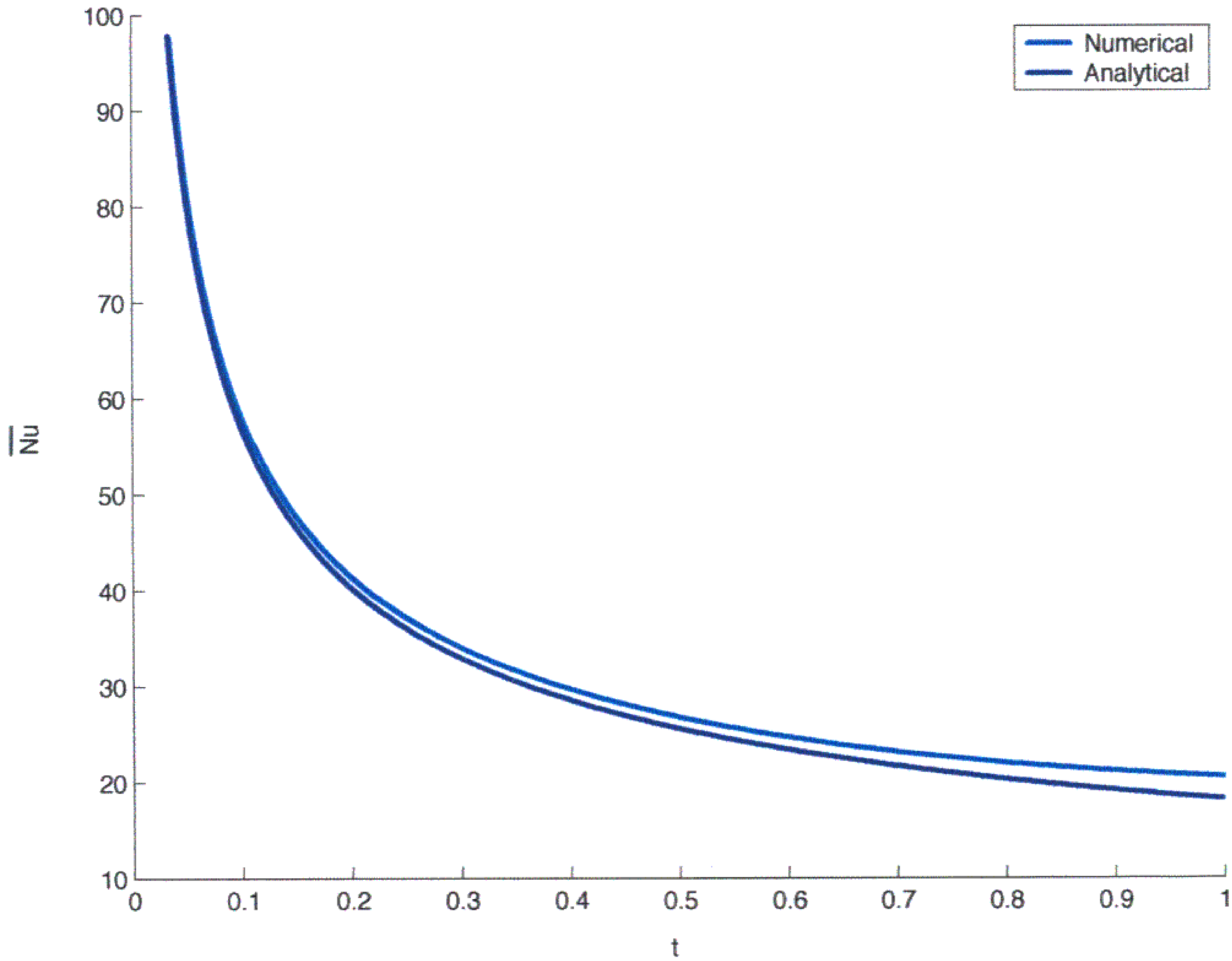
$Ra=1$, $R=500$


t=0.5
33
32
31
30
29
28
27
26
25
24
Nu
0
90
180
270
360
θ
Analytical
Numerical

Ra=1 , R=500

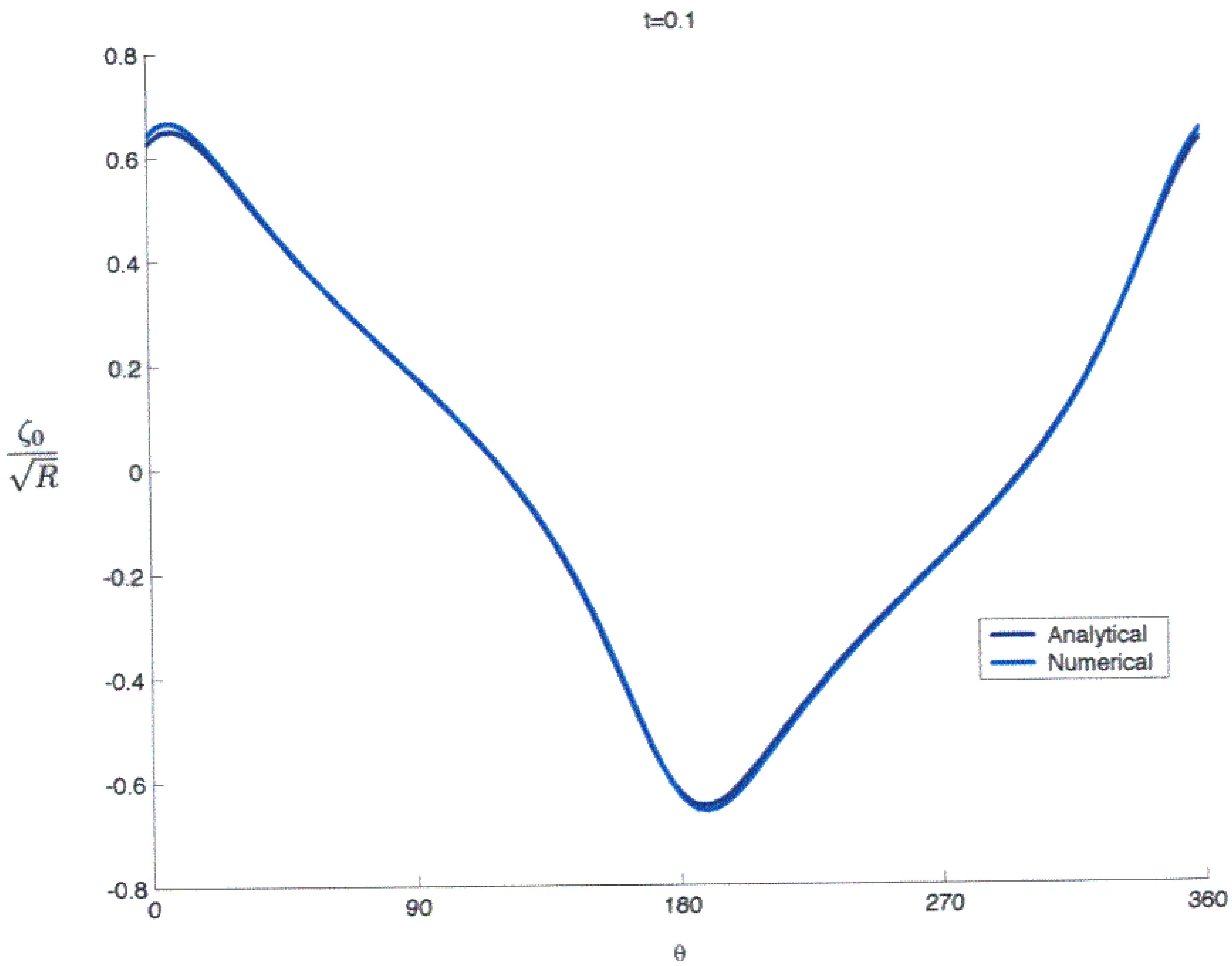

t=1.0
34
32
30
28
26
24
22
20
18
16
14
Nu
0
90
180
270
360
θ
Analytical
Numerical

Ra=1 , R=500

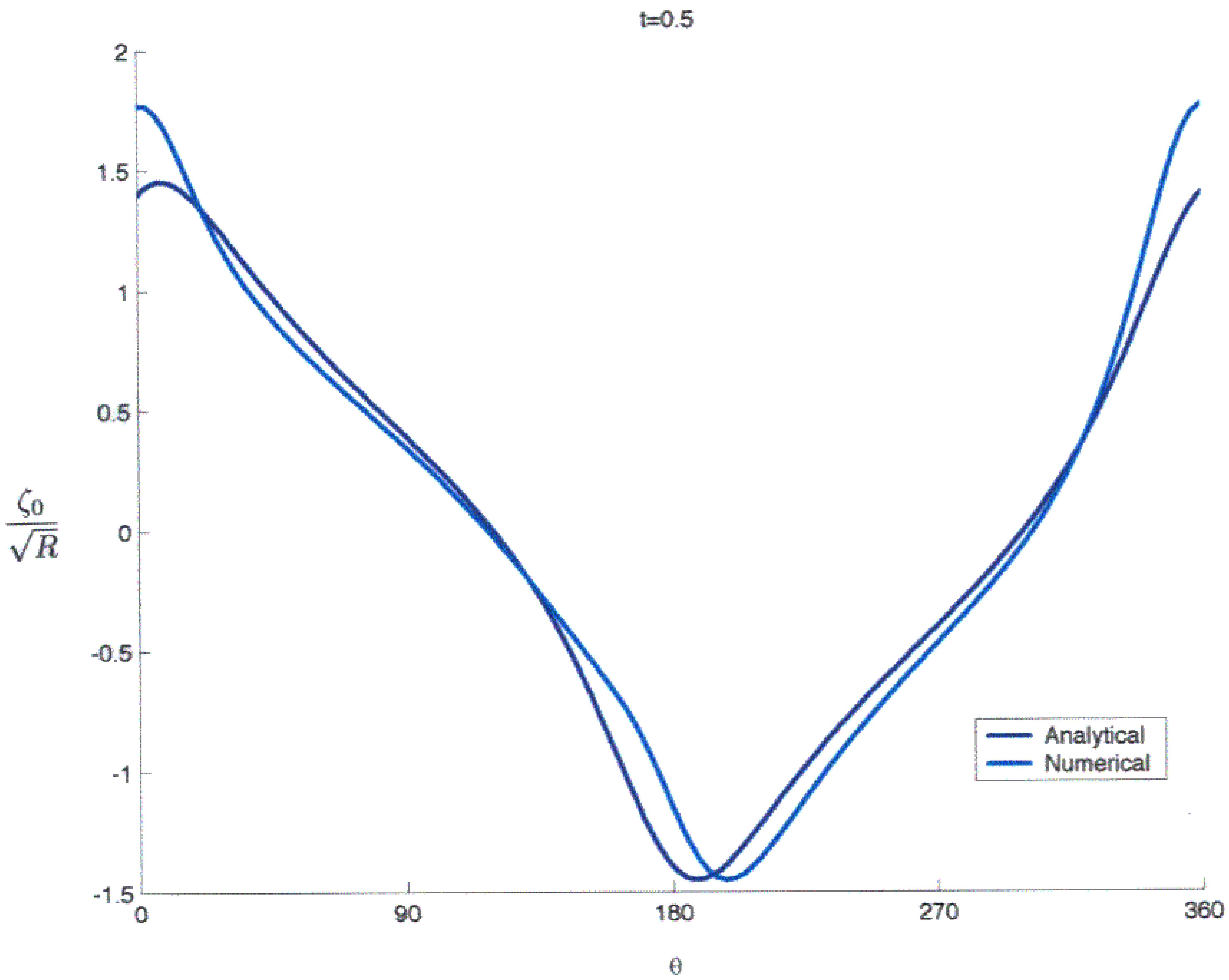


Numerical
Analytical
100
90
80
70
60
50
40
30
20
10
$\overline{Nu}$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t

Ra=1 , R=500

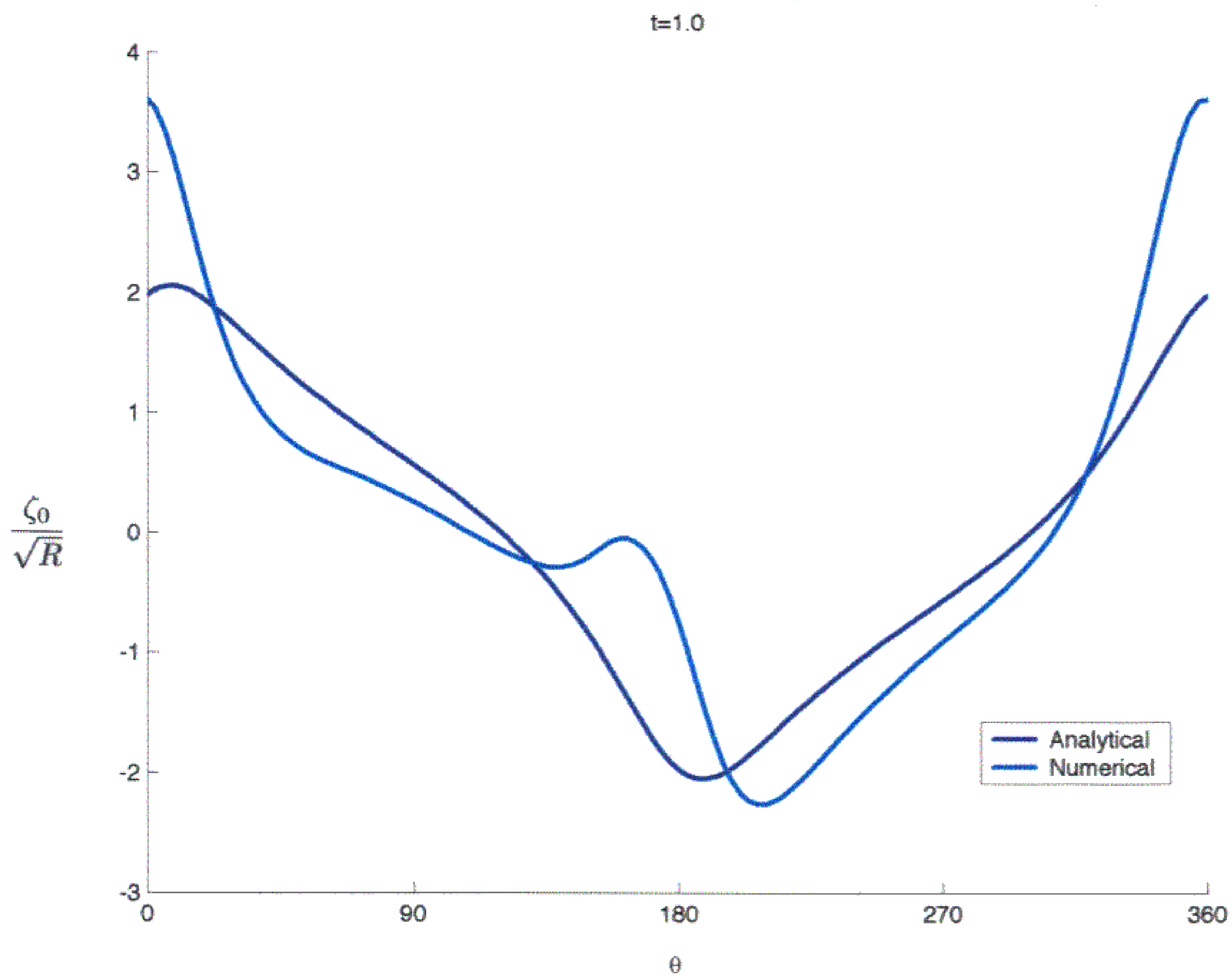

t=0.1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
$\frac{\zeta_0}{\sqrt{R}}$
0
90
180
270
360
θ
Analytical
Numerical

Ra=1 , R=500



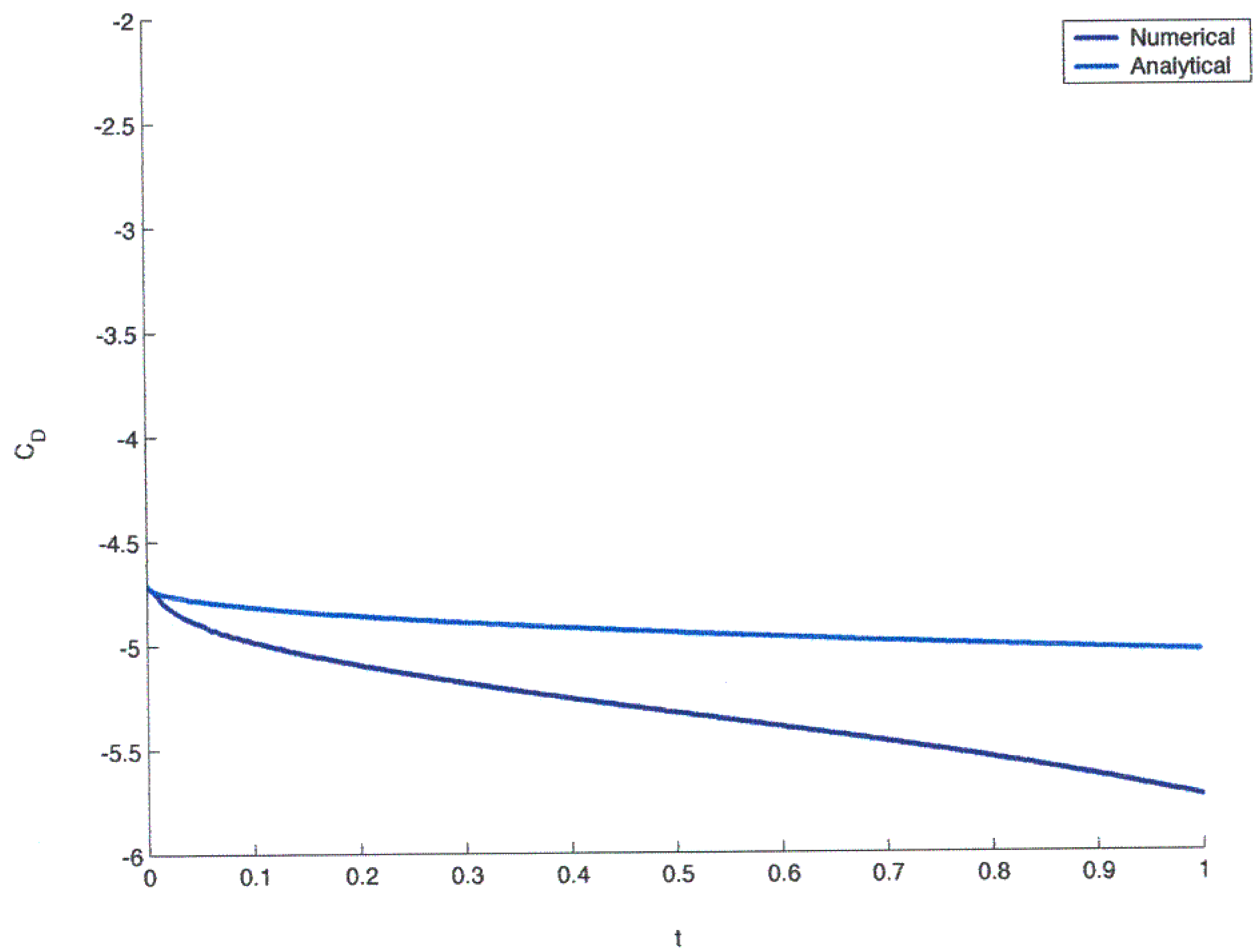
t=0.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
$\frac{\zeta_0}{\sqrt{R}}$
0
90
180
270
360
θ
Analytical
Numerical

Ra=1 , R=500



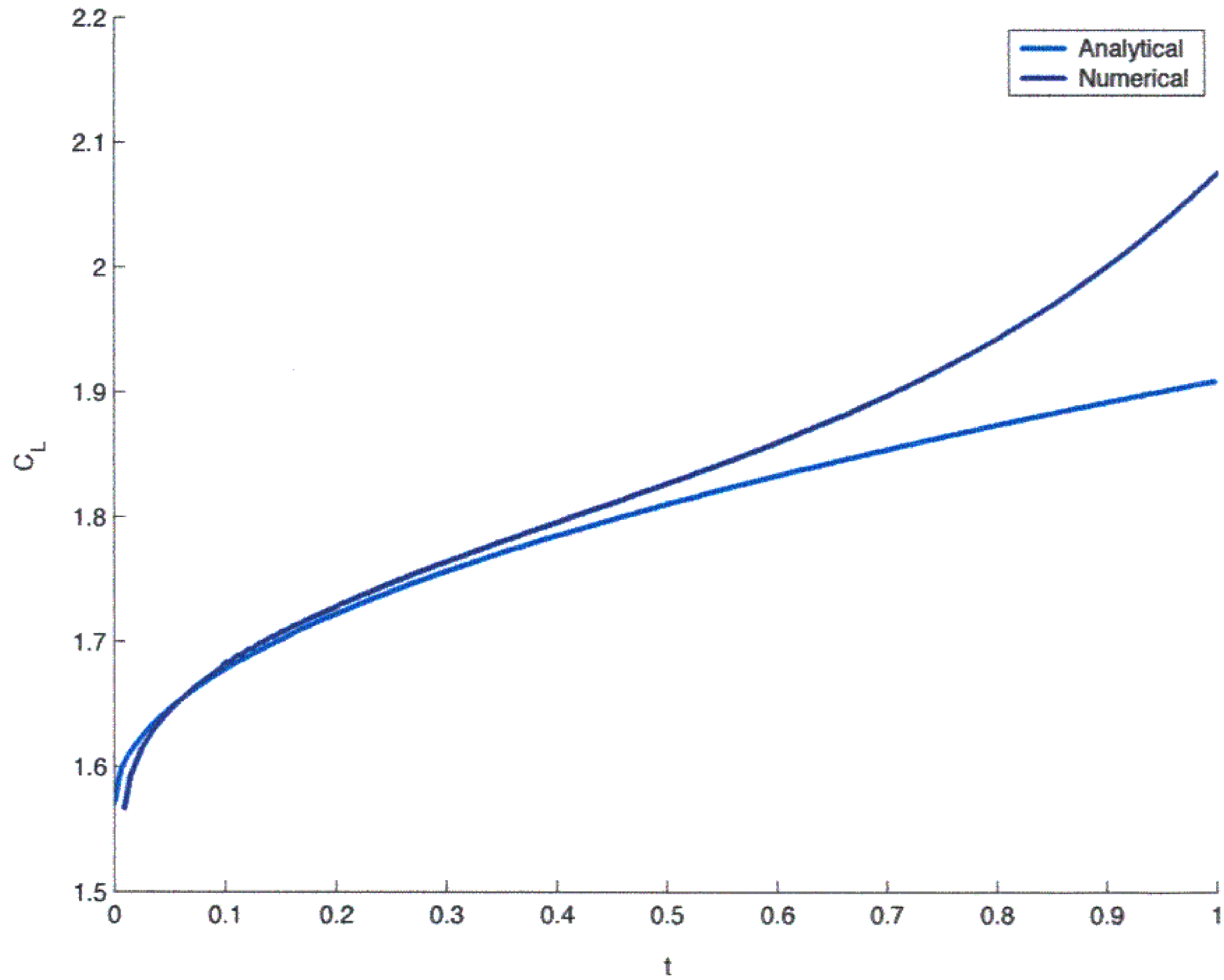
t=1.0
4
3
2
1
0
-1
-2
-3
0
90
180
270
360
θ
$\frac{\zeta_0}{\sqrt{R}}$
Analytical
Numerical

Ra=1 , R=500



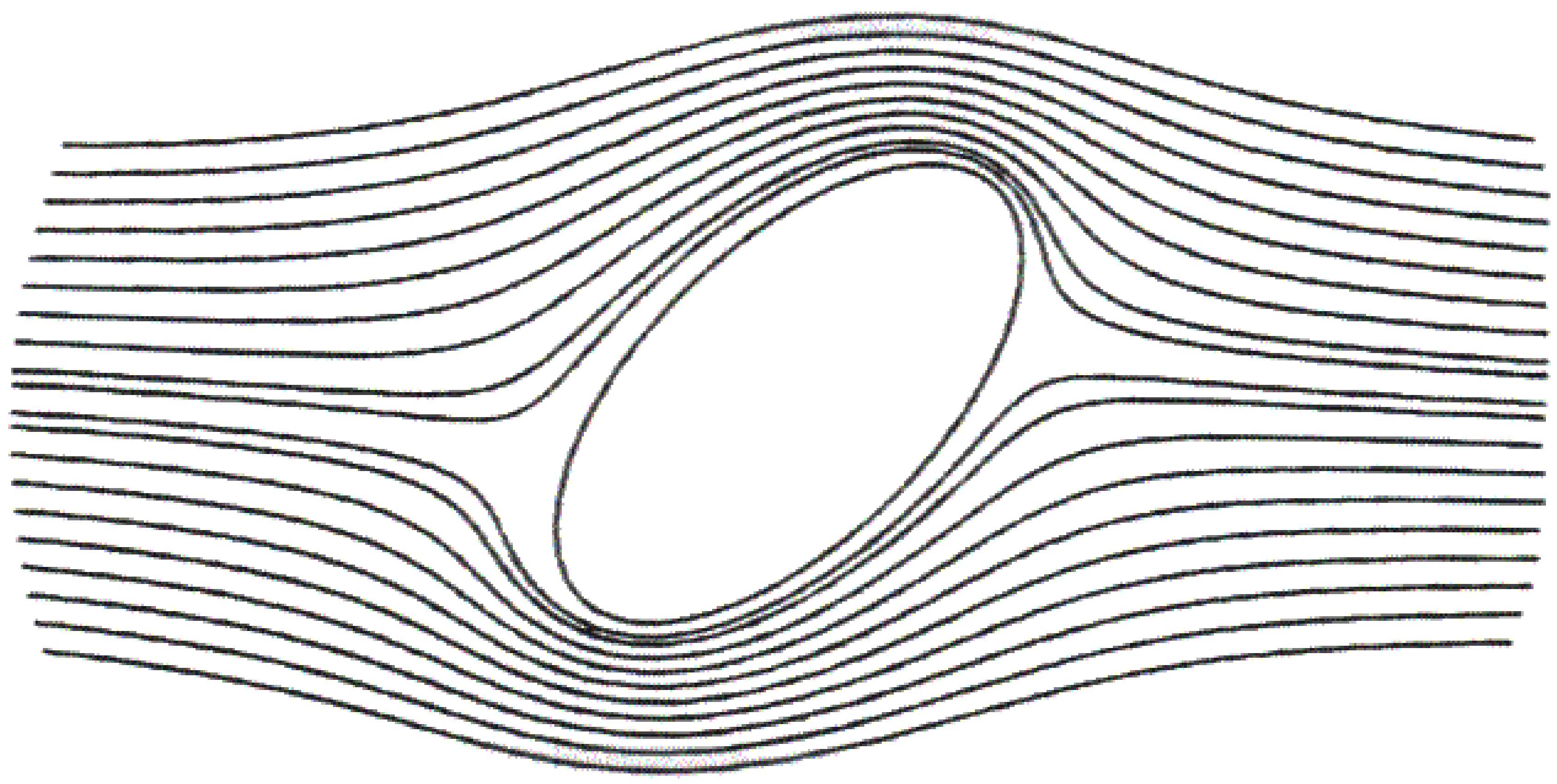
Numerical
Analytical
-2
-2.5
-3
-3.5
-4
-4.5
-5
-5.5
-6
C_D
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t

Ra=1 , R=500

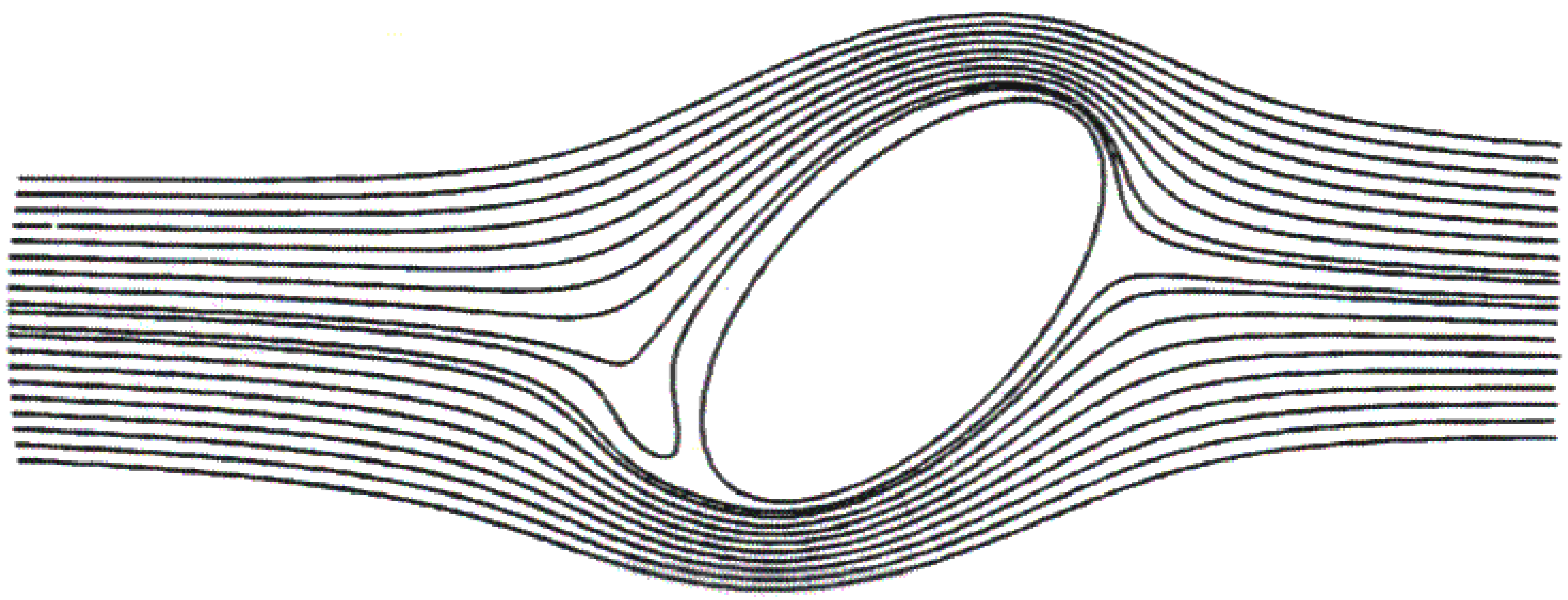


Analytical
Numerical
2.2
2.1
2
1.9
1.8
1.7
1.6
1.5
C_L
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t

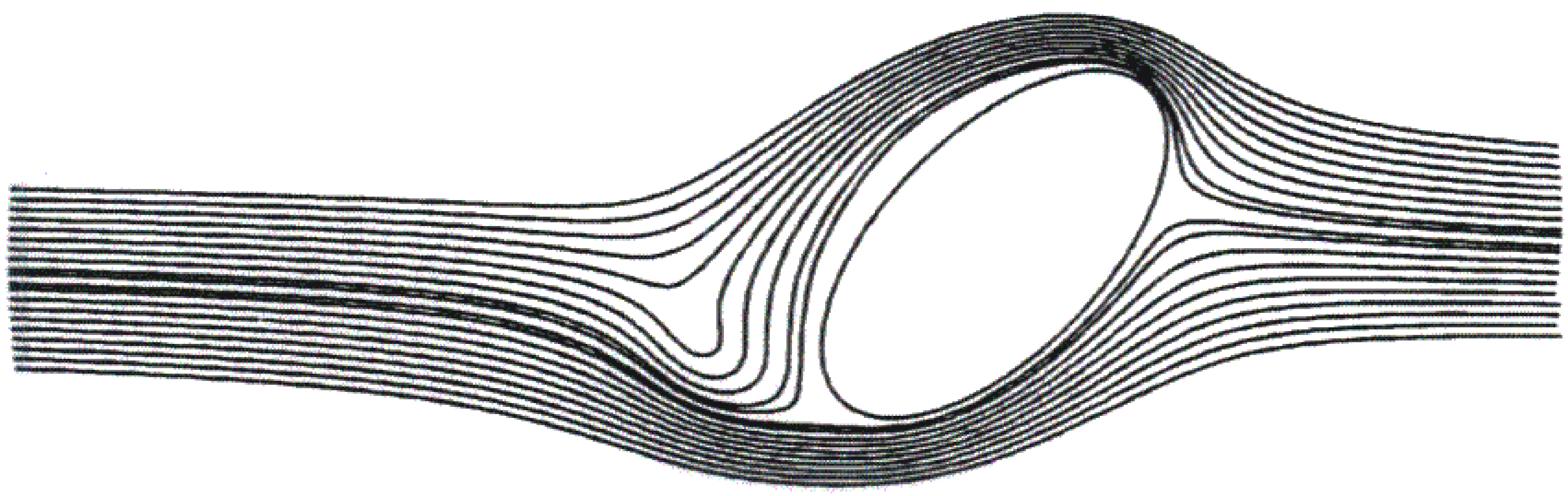
Streamline Plots for $Ra = 0, R = 500$



$t = 1$

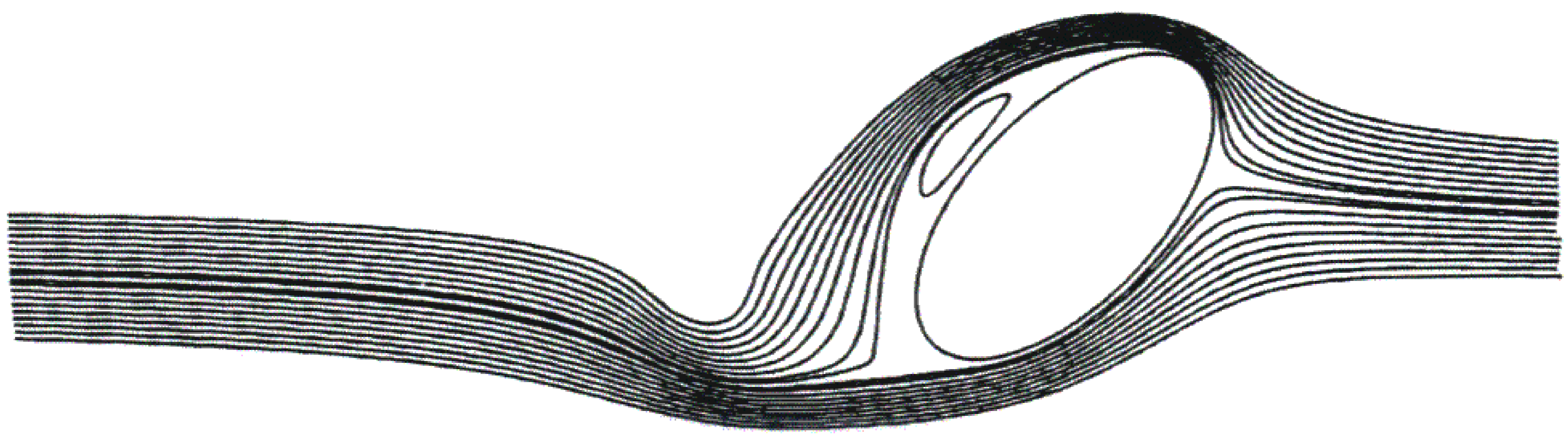


$t = 1.5$

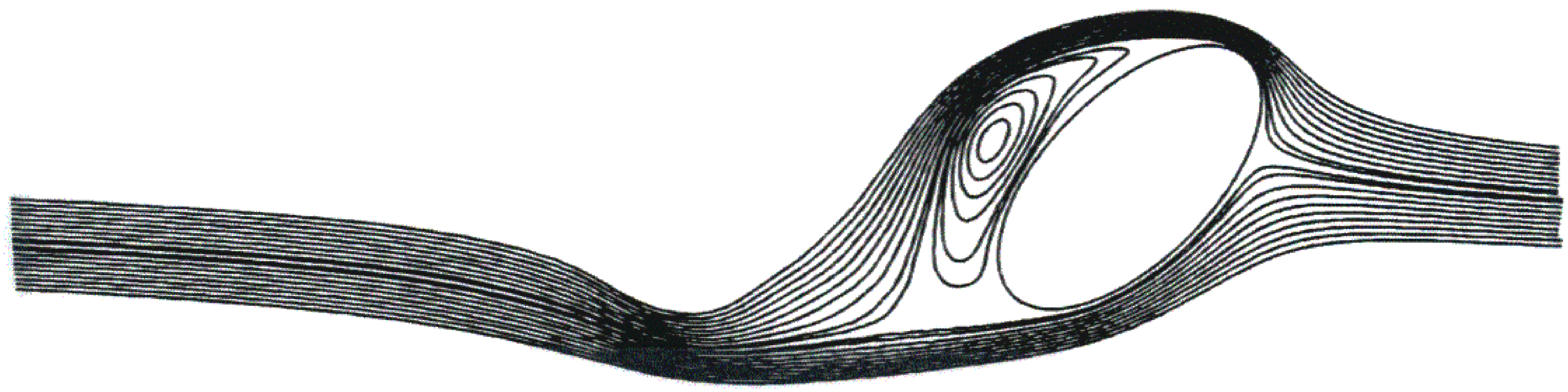


$t = 2$

Streamline Plots for $Ra = 0, R = 500$



$t = 2.5$

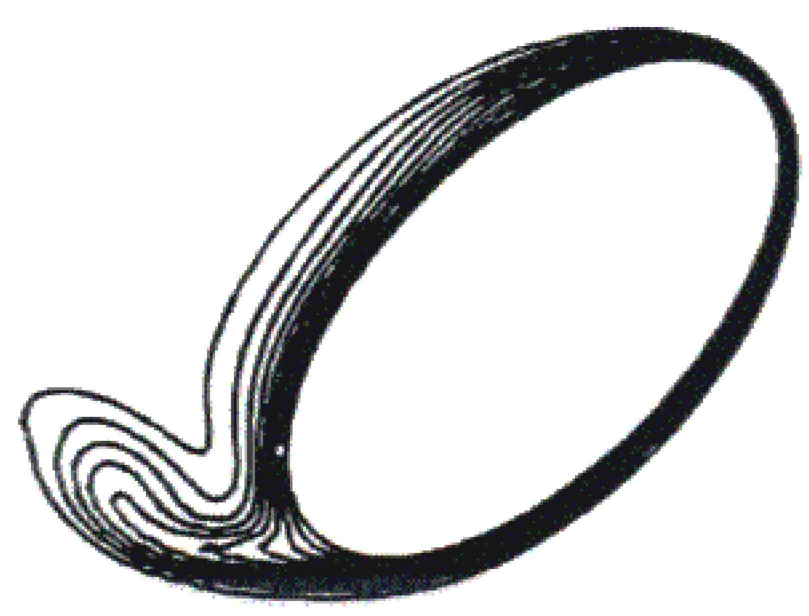


$t = 3$

Isotherm Plots for $Ra = 0, R = 500$

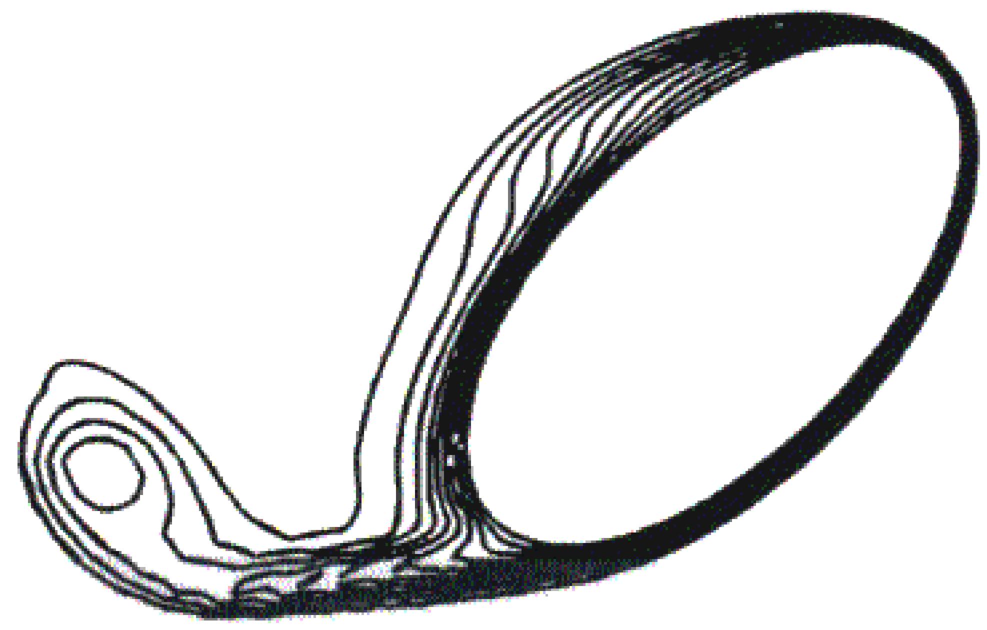
$t = 1$

$t = 1.5$



$t = 2$

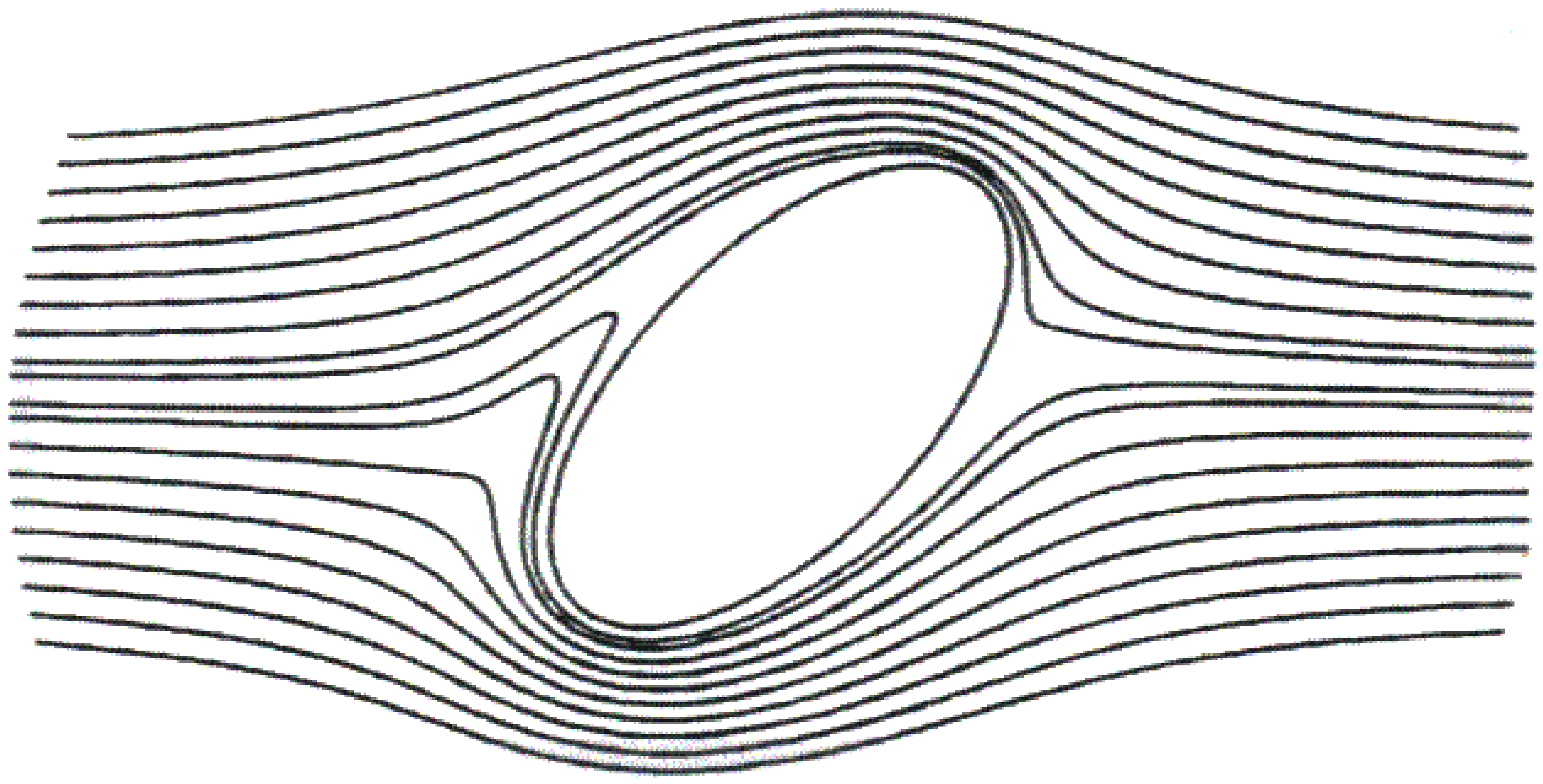
Isotherm Plots for $Ra = 0, R = 500$



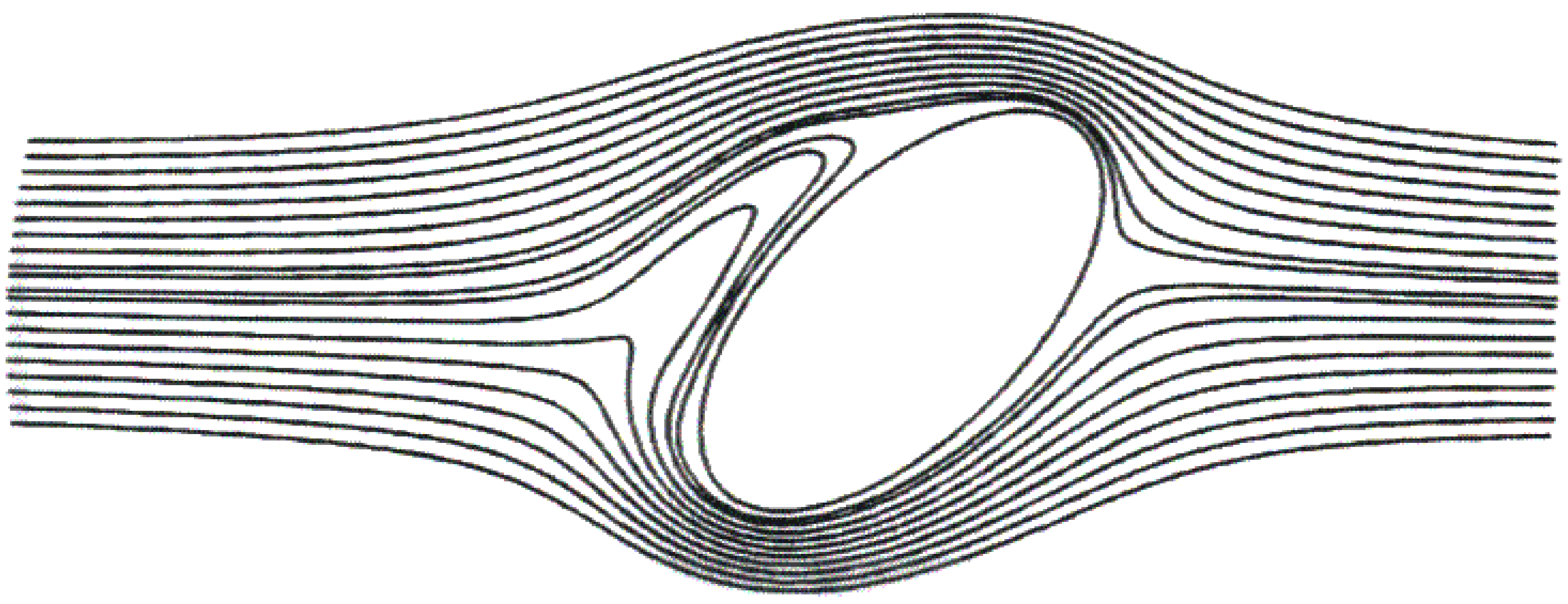
$t = 2.5$

$t = 3$

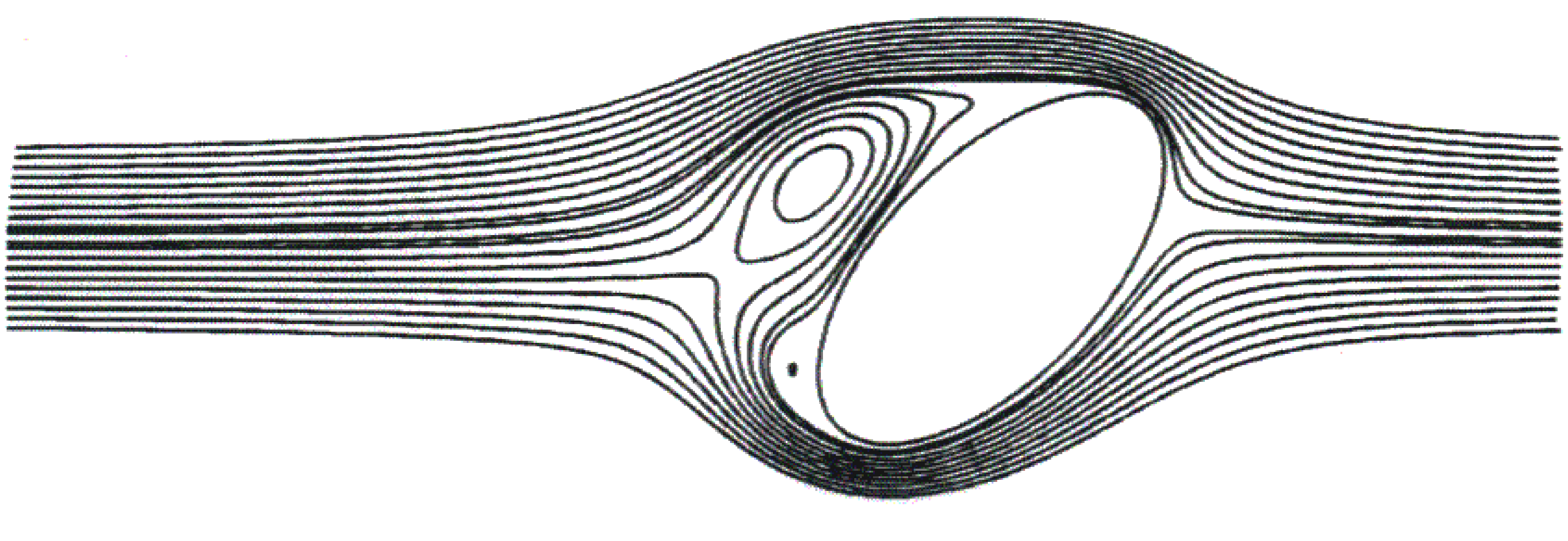
Streamline Plots for $Ra = 5, R = 500$



$t = 1$

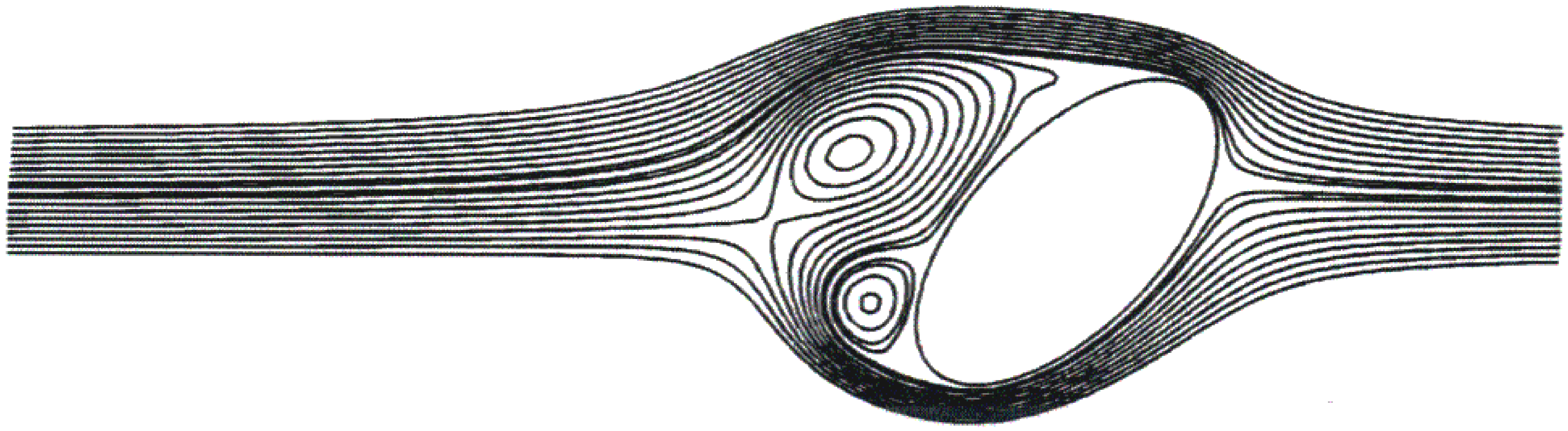


$t = 1.5$

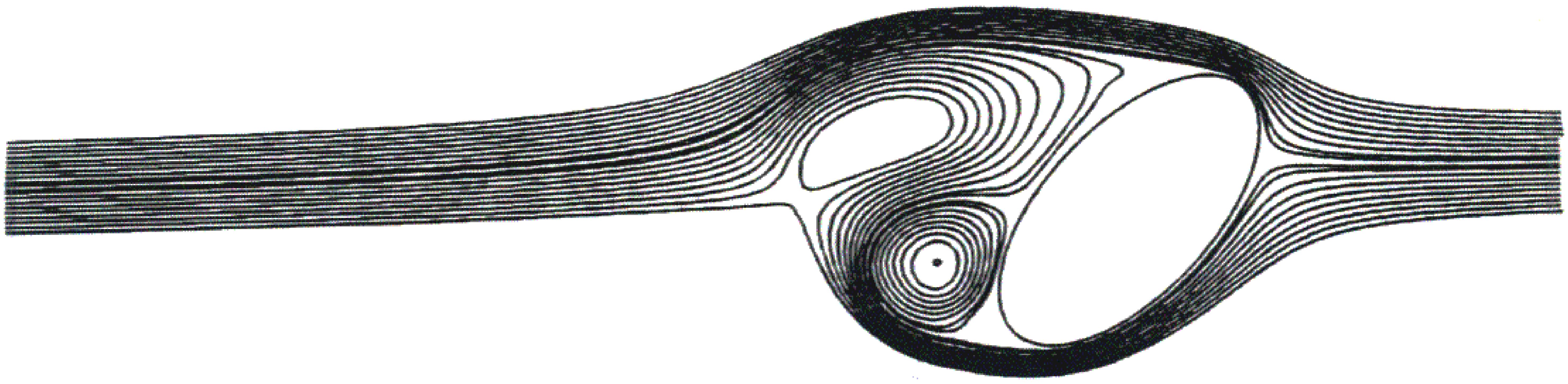


$t = 2$

Streamline Plots for $Ra = 5, R = 500$



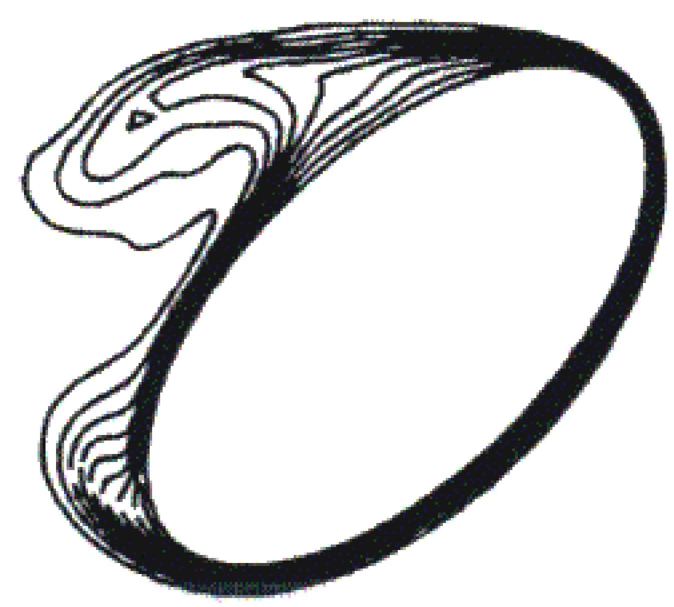
$t = 2.5$



$t = 3$

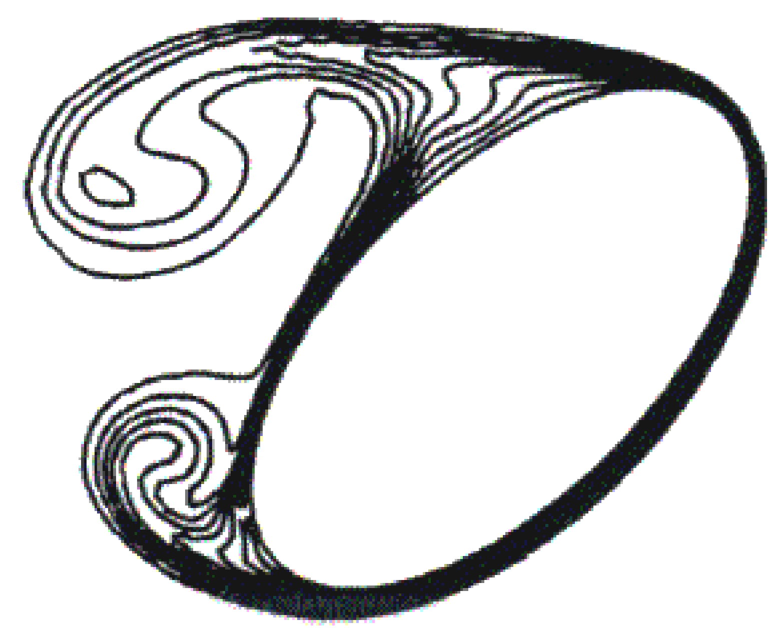
Isotherm Plots for $Ra = 5, R = 500$

$t = 1.5$

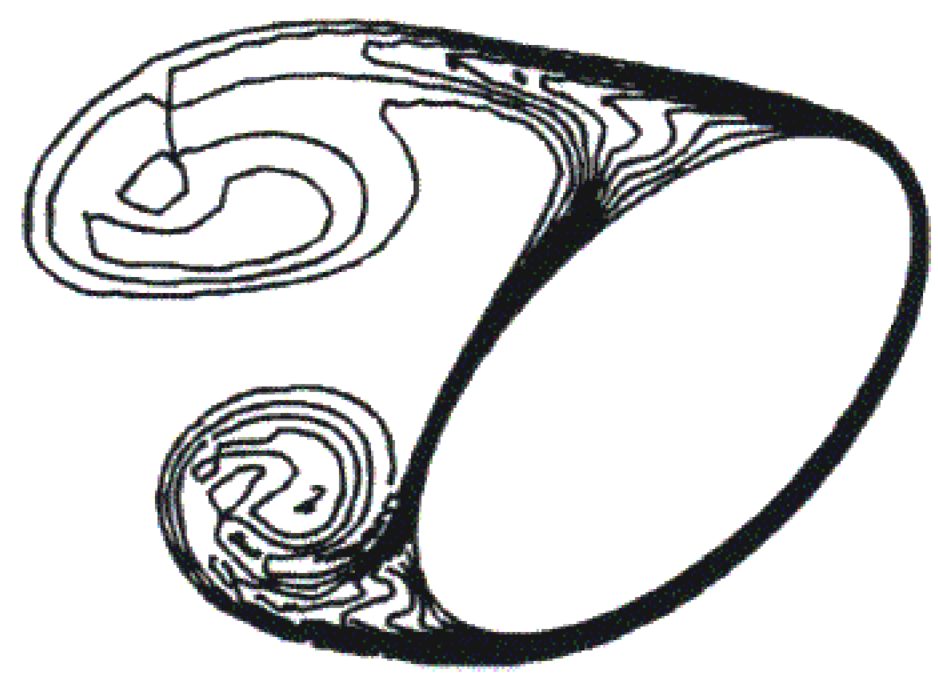


$t = 2$

Isotherm Plots for $Ra = 5, R = 500$



$t = 2.5$



$t = 3$