The flow of a power-law fluid down a heated incline

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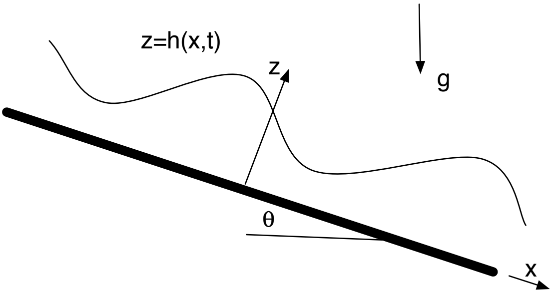
We consider the two-dimensional gravity-driven flow of a power-law fluid flowing along a heated incline as shown below:



Previous studies on the stability of non-isothermal power-law film flow include:

- Hu et al. (Phys. Fluids - 2017)
  They considered a horizontal layer with a non-deformable surface consisting of a shear-thinning fluid. Buoyancy effects were included by assuming a temperature-dependent density.
- Sadiq & Usha (J. Fluid Eng. - 2009)
  They assumed constant fluid properties and applied a thermal insulation condition along the free surface. Consequently, Marangoni stresses are not generated.
- Bernabeu et al. (Geo. Soc. London - 2016)
  They studied lava flow with temperature dependence but do not include the Marangoni effect.

The conservation of mass and momentum equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + \rho g \sin\theta$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} - \rho g \cos\theta$$

where

$$\tau_{xx} = 2\mu_n\eta\frac{\partial u}{\partial x}\ ,\ \tau_{zx} = \tau_{xz} = \mu_n\eta\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\ ,\ \tau_{zz} = 2\mu_n\eta\frac{\partial w}{\partial z}$$

$$\eta = \left[2\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right) + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2\right]^{\frac{(n-1)}{2}}$$

The consistency, μ_n, and surface tension, σ, are assumed to vary linearly with temperature, T, according to:

$$\mu_n = \mu_a - \lambda_a(T - T_a)\ ,\ \sigma = \sigma_a - \gamma(T - T_a)$$

Conservation of energy yields the following equation for the temperature:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} = \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

The equations are scaled using the Nusselt thickness corresponding to steady isothermal flow having a flow rate Q:

$$H = \left(\frac{\mu_a}{\rho g \sin\theta}\right)^{\frac{1}{2n+1}} Q^{\frac{n}{2n+1}} \left(\frac{2n+1}{n}\right)^{\frac{n}{2n+1}}$$

The scaled quantities become

$$(x,z) = H\left(\frac{x^*}{\delta}, z^*\right) \ , \ h = Hh^* \ , \ (u,w) = U(u^*, \delta w^*)$$

$$t = \frac{H}{U\delta} t^* \ , \ p - p_a = \rho U^2 p^* \ , \ T = T_a + \Delta T T^*$$

where $U = Q/H, \Delta T = T_w - T_a, \delta = H/L$ and $\lambda = \lambda_a \Delta T/\mu_a$.

The dimensionless equations become (dropping the asterisk for notational convenience)

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$Re\delta \frac{Du}{Dt} = -Re\delta \frac{\partial p}{\partial x} + 2\delta^2 \frac{\partial}{\partial x}\left[(1-\lambda T)\eta \frac{\partial u}{\partial x}\right]$$

$$+\frac{\partial}{\partial z}\left[(1-\lambda T)\eta\left(\frac{\partial u}{\partial z} + \delta^2 \frac{\partial w}{\partial x}\right)\right] + \left(\frac{2n+1}{n}\right)^n$$

$$Re\delta^2 \frac{Dw}{Dt} = -Re \frac{\partial p}{\partial z} + 2\delta \frac{\partial}{\partial z}\left[(1-\lambda T)\eta \frac{\partial w}{\partial z}\right]$$

$$+\delta \frac{\partial}{\partial x}\left[(1-\lambda T)\eta\left(\frac{\partial u}{\partial z} + \delta^2 \frac{\partial w}{\partial x}\right)\right] - \left(\frac{2n+1}{n}\right)^n \cot\theta$$

$$\delta RePr \frac{DT}{Dt} = \delta^2 \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}$$

$$p = \frac{2\delta}{ReF^2}\eta(1-\lambda T)\left[\delta^2\left(\frac{\partial h}{\partial x}\right)^2\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} - \frac{\partial h}{\partial x}\left(\frac{\partial u}{\partial z} + \delta^2\frac{\partial w}{\partial x}\right)\right]$$

$$-\frac{\delta^2}{F^3}\frac{\partial^2 h}{\partial x^2}(We - MT) \quad \text{at} \quad z = h(x,t)$$

$$-\delta MF\left(\frac{\partial T}{\partial x} + \frac{\partial T}{\partial z}\frac{\partial h}{\partial x}\right) = \frac{\eta(1-\lambda T)}{Re}\left[-4\delta^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}\right.$$

$$\left. + \left(1-\delta^2\left(\frac{\partial h}{\partial x}\right)^2\right)\left(\delta^2\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] \quad \text{at} \quad z = h(x,t)$$

$$\frac{\partial T}{\partial z} - \delta^2\frac{\partial h}{\partial x}\frac{\partial T}{\partial x} = -BFT \quad \text{at} \quad z = h(x,t)$$

$$w = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} \quad \text{at} \quad z = h(x,t)$$

$$u = w = 0\ , \quad T = 1 \quad \text{at} \quad z = 0$$

$$Re = \frac{\rho}{\mu_a U^{2-n} H^n} \quad \text{Reynolds number}$$

$$We = \frac{\sigma_a}{\rho U^2 H} \quad \text{Weber number}$$

$$M = \frac{\gamma \Delta T}{\rho U^2 H} \quad \text{Marangoni number}$$

$$Pr = \frac{\mu_a}{\rho \kappa} \left(\frac{U}{H} \right)^{n-1} \quad \text{Prandtl number}$$

$$B = \frac{\alpha H}{k} \quad \text{Biot number}$$

Also, $F = \left[1 + \delta^2 \left(\frac{\partial h}{\partial x} \right)^2 \right]^{\frac{1}{2}}$ and

$$\eta = \left[2\delta^2 \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right) + \left(\frac{\partial u}{\partial z} + \delta^2 \frac{\partial w}{\partial x} \right)^2 \right]^{\frac{(n-1)}{2}}$$

The equations can be simplified by discarding the $O(\delta^2)$ terms:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$Re\delta \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} \right) = \frac{\partial}{\partial z} \left[(1 - \lambda T) \left(\frac{\partial u}{\partial z} \right)^n \right]$$

$$- \left(\frac{2n+1}{n} \right)^n \delta \cot\theta \frac{\partial h}{\partial x} + \left(\frac{2n+1}{n} \right)^n$$

$$PrRe\delta \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} \right) = \frac{\partial^2 T}{\partial z^2}$$

The simplified boundary conditions become:

$$-MRe\delta\left(\frac{\partial T}{\partial x}+\frac{\partial T}{\partial z}\frac{\partial h}{\partial x}\right)=(1-\lambda T)\left(\frac{\partial u}{\partial z}\right)^n \quad \text{at} \quad z=h(x,t)$$

$$\frac{\partial T}{\partial z}=-BT \quad \text{at} \quad z=h(x,t)$$

$$w=\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x} \quad \text{at} \quad z=h(x,t)$$

$$u=w=0\ , \quad T=1 \quad \text{at} \quad z=0$$

The steady-state solutions for $T = T_s$ and $w = w_s$ are:

$$T_s(z) = 1 - \left(\frac{B}{B+1}\right) z \ , \ w_s(z) = 0$$

The solution for $u = u_s$ satisfies:

$$\frac{d}{dz}\left[(1-\lambda T_s)\left(\frac{du_s}{dz}\right)^n\right] + \left(\frac{2n+1}{n}\right)^n = 0$$

Although exact solutions for selected values of n have been obtained, for other values an approximate solution based on small λ was derived. Some exact solutions are:

$$u_s = \alpha_0 \left[\frac{\alpha_1}{(h+\alpha_1)} - \frac{(z+\alpha_1)}{(h+\alpha_1)} + \ln\left(\frac{z+\alpha_1}{\alpha_1}\right)\right] \quad \text{for } n = 1$$

$$u_s = \alpha_0 \left[\frac{h(h+2\alpha_1)}{\alpha_1(h+\alpha_1)} + \frac{(z+\alpha_1)}{(h+\alpha_1)} - \frac{(h+\alpha_1)}{(z+\alpha_1)} + 2\ln\left(\frac{\alpha_1}{z+\alpha_1}\right) \right] \quad \text{for } n = \frac{1}{2}$$

$$u_s = \alpha_0 \left[\frac{\sqrt{(h-z)(z+\alpha_1)}}{h+\alpha_1} - \frac{\sqrt{\alpha_1 h}}{(h+\alpha_1)} + \arctan\left(\sqrt{\frac{h}{\alpha_1}}\right) - \arctan\left(\sqrt{\frac{h-z}{z+\alpha_1}}\right) \right] \quad \text{for } n = 2$$

where $\alpha_0 = \dfrac{(2n+1)(h+\alpha_1)}{n} \left(\dfrac{1+Bh}{\lambda B}\right)^{\frac{1}{n}}$, $\alpha_1 = \dfrac{(1-\lambda)(1+Bh)}{\lambda B}$

Next, impose small disturbances on the steady-state flow:

$$u = u_s + \tilde{u}\ ,\ w = \tilde{w}\ ,\ T = T_s + \tilde{T}\ ,\ h = 1 + \zeta$$

Then, substitute these into the long-wave equations, linearize and assume the disturbances of the form:

$$(\tilde{u}, \tilde{w}, \tilde{T}, \zeta) = (\hat{u}(z), \hat{w}(z), \hat{T}(z), \hat{\zeta})e^{ik(x-ct)}$$

where k (real & positive) represents the wavenumber of the perturbation and c is a complex quantity with the real part denoting the phase speed of the perturbation while the imaginary part is related to the growth rate.

The perturbation equations were solved numerically for arbitrary k using a collocation method based on polynomial interpolation with Chebyshev points. In addition, the perturbation equations were solved analytically for small wavenumbers by expanding in powers of k as follows:

$$\hat{u} = u_0 + k u_1 \ , \ \ \hat{w} = w_0 + k w_1 \ , \ \ \hat{T} = T_0 + k T_1$$

$$\hat{\zeta} = \zeta_0 + k \zeta_1 \ , \ \ c = c_0 + k c_1$$

Nonlinear effects were also investigated by implementing a first-order IBL model. The IBL equations were obtained by integrating the long-wave equations across the fluid layer, and hence, eliminating the z dependence. In terms of the flow rate, q, where

$$q = \int_0^h u dz$$

the continuity equation becomes

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0$$

Next, we introduce the interfacial temperature, $\phi(x,t) = T(x, z = h, t)$, the temperature profile

$$T = 1 + \frac{(\phi - 1)}{h} z$$

and the velocity profile given by

$$u = \frac{q}{Q_0} b_0 \quad \text{where} \quad Q_0(x,t) = \int_0^h b_0 dz$$

and

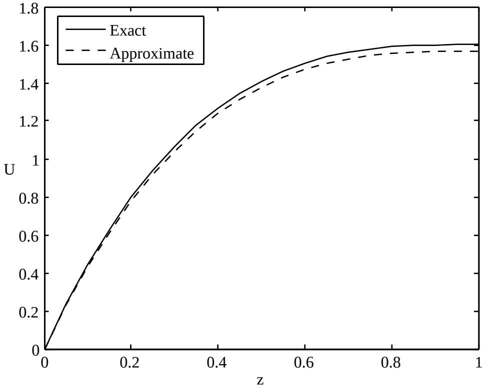
$$b_0(x,z,t) = \left(\frac{2n+1}{n+1}\right) A_0 \left[h^{\frac{n+1}{n}} - (h-z)^{\frac{n+1}{n}}\right]$$

$$+A_1 \left[h^{\frac{2n+1}{n}} - (h-z)^{\frac{2n+1}{n}}\right] + \left(\frac{2n+1}{3n+1}\right) A_2 \left[h^{\frac{3n+1}{n}} - (h-z)^{\frac{3n+1}{n}}\right]$$

$$A_0 = \frac{n^2(1+Bh)^2 + \lambda n(1+Bh) + (n+1)\lambda^2}{n^2(1+Bh)^2}$$

$$A_1 = \frac{\lambda B[n(1+Bh) + 2(n+1)\lambda]}{n^2(1+Bh)^2}, \quad A_2 = \frac{(n+1)\lambda^2 B^2}{n^2(1+Bh)^2}$$

Comparison between the assumed and exact velocity profiles for $n = 1/2$, $\lambda = 0.1$, $h = 1$ and $B = 1$.



Using the assumed profiles the momentum and energy equations become:

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left[\int_0^h u^2 dz + M\phi + \left(\frac{2n+1}{n}\right)^n \frac{\cot\theta}{2Re} h^2\right] = \frac{h}{Re\delta}\left[\left(\frac{2n+1}{n}\right)^n - \left(1 + \frac{(n+1)\lambda^2}{2n}\right)\frac{q^n}{Q_0^n}\right]$$

$$h\frac{\partial \phi}{\partial t} - \frac{1}{(4n+1)(3n+1)^2(1+Bh)}\left[n(\phi-1)F_1\frac{\partial q}{\partial x} + qF_2\frac{\partial \phi}{\partial x} - \frac{n(2n+1)\lambda B(\phi-1)q}{(1+Bh)}\frac{\partial h}{\partial x}\right] = -\frac{2}{PrRe\delta h}\left[(1+Bh)\phi - 1\right]$$

where $F_1 = (2n+1)\lambda Bh - (3n+1)(4n+1)(1+Bh)$

and $F_2 = n(2n+1)\lambda Bh - (3n+1)(4n+1)^2(1+Bh)$

The IBL equations were solved using the fractional-step splitting technique.

Exact expressions for the critical Reynolds number were obtained for special cases, such as the Newtonian case ($n = 1$):

$$Re_{crit} = \frac{D_1}{D_2}$$

$$\text{where} \quad D_1 = 1680\,(B+1)^2 \cot\theta\,\left(3\,B\lambda + 4\,B + 4\,\lambda + 4\right)$$

$$\text{and} \quad D_2 = \left(8064 + \left(231 Pr + 18501\right)\lambda\right) B^3$$

$$+ \left(\left(2240 M + 339 Pr + 58797\right)\lambda + 3360 M + 24192\right) B^2$$

$$+ \left(\left(3360 M - 816 Pr + 64488\right)\lambda + 3360 M + 24192\right) B + 24192\lambda + 8064$$

Setting $\lambda = 0$ yields:

$$Re_{crit} = \frac{10 \cot\theta (1+B)^2}{12(1+B)^2 + 5MB}$$

which agrees with D'Alessio et al. (J. Fluid Mech., 2010) for the case with constant viscosity.

For the general power-law case under isothermal conditions we obtain:

$$Re_{crit} = \frac{1}{2}\left(\frac{n}{2n+1}\right)^{2-n}(3n+2)\cot\theta$$

which is in full agreement with the result reported by Fernandez-Nieto et al. (J. Non-Newtonian Fluid Mech., 2010).

The IBL model with $\lambda = 0$ and $n = 1$ predicts:

$$Re_{crit}^{IBL} = \frac{3\cot\theta(1+B)^2}{3(1+B)^2 + MB} \text{ compared to } Re_{crit}^{full} = \frac{10\cot\theta(1+B)^2}{12(1+B)^2 + 5MB}$$

Further, if $M = B = 0$ the IBL model yields:

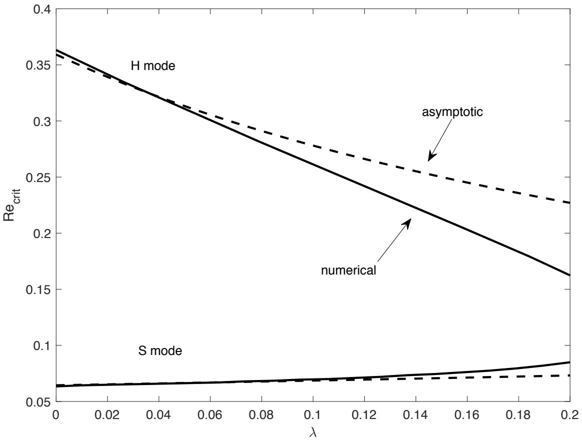
$$Re_{crit}^{IBL} = \cot\theta$$

which is in full agreement with the Shkadov IBL model (Izv. Akad. Nauk SSSP, Mekh. Zhidk Gaza, 1967). As a final check if we set $\lambda = M = B = 0$, then

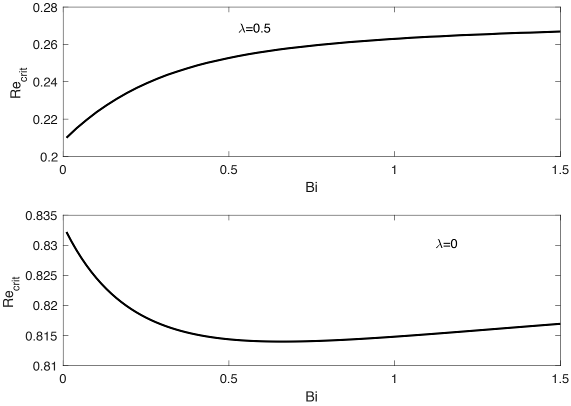
$$Re_{crit}^{IBL} = \frac{n^{2-n}\cot\theta}{(2n+1)^{1-n}}$$

which recovers the expression obtained by Ng & Mei (J. Fluid. Mech. 1994).

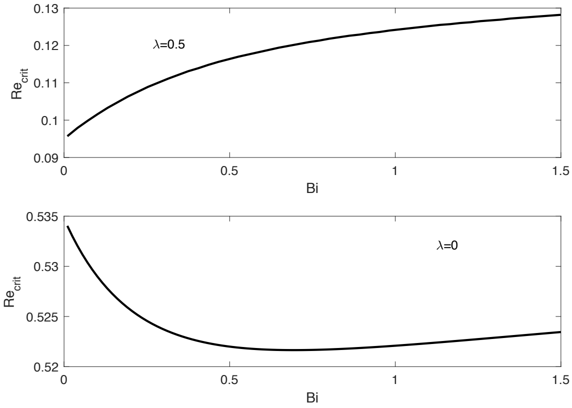
Re_{crit} as a function of λ for $\cot\theta = 1$, $Pr = 7$, $Bi = 1$, $n = 0.8$ and $Ma = 1.1$; comparison between numerical and analytical results.



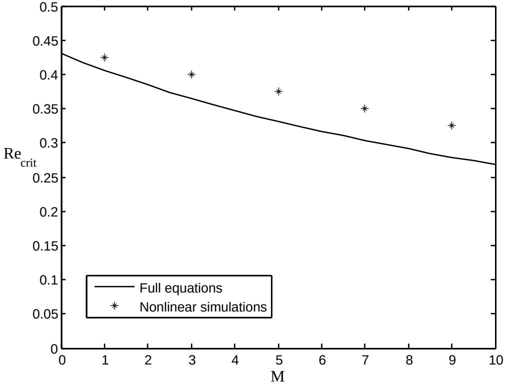
Re_{crit} for the H mode as a function of Bi for $\cot\theta = 1$, $Pr = 7$, $Ma = 0.1$, $n = 1$.



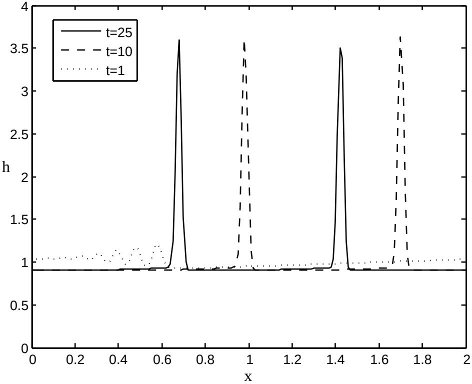
Re_{crit} for the H mode as a function of Bi for $\cot\theta = 1$, $Pr = 7$, $Ma = 0.1$, $n = 0.8$.



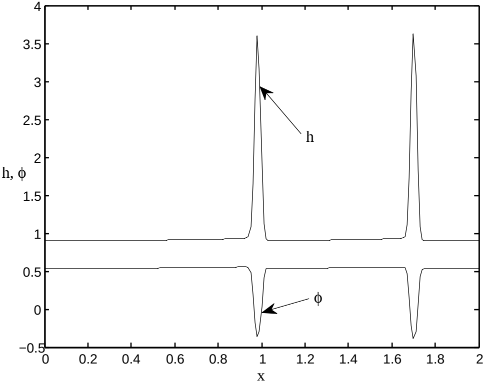
Comparison in Re_{crit} for the case $n = 0.8$, $\lambda = 0.1$, $B = 1$, $\cot\theta = 1$, $\delta = 0.1$ and $Pr = 7$.



Time evolution of the fluid thickness for the case $n = 0.8$, $\cot\theta = 1$, $M = 5$, $B = 1$, $Pr = 7$, $\delta = 0.1$, $\lambda = 0.1$ and $Re = 0.4$.



The fluid thickness and surface temperature at $t = 10$ for the case $n = 0.8$, $\cot\theta = 1$, $M = 5$, $B = 1$, $Pr = 7$, $\delta = 0.1$, $\lambda = 0.1$ and $Re = 0.4$.



- The stability of the flow of a power-law fluid down a heated incline was studied.
- The consistency and surface tension were allowed to vary linearly with temperature.
- A linear stability analysis was conducted both numerically and analytically.
- Nonlinear simulations were also carried out using a first-order IBL model.
- Reasonable agreement was found between numerical and analytical results, and also with previous investigations.
- This research has recently appeared in *AIP Advances*, **8**, 105215, 2018.