Stability of differentially heated flow from a rotating sphere

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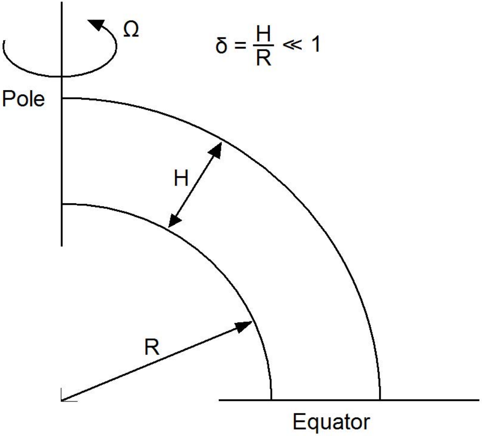
The stability of a thin fluid layer flowing over a differentially heated rotating sphere has been investigated assuming azimuthal and equatorial symmetry and using the Boussinesq approximation.



Some key previous studies include:

- Isothermal flow:
  Marcus & Tuckerman (J. Fluid Mech. - 1987)
- Non-isothermal flow:
  Hart *et al.* (J. Fluid Mech. - 1986),
  Lesueur *et al.* (Geophys. Astrophys. Fluid Dyn. - 1999)
- Stability:
  Lewis & Langford (SIAM J. Appl. Dyn. Sys. - 2008),
  Walton (Q. J. Mech. Appl. Math. - 1982)

In dimensionless form and in spherical coordinates the governing Navier-Stokes and energy equations can be compactly formulated in terms of the stream function, ψ, vorticity, ω, zonal velocity, W, and temperature, T:

$$\omega = -\delta D^2 \psi$$

$$\frac{\partial \omega}{\partial t} + \frac{\delta}{r^2 \sin\theta} \frac{\partial(\psi, \omega)}{\partial(\theta, r)} + \delta Pr Ra \sin\theta \frac{\partial T}{\partial \theta} + \frac{2\delta\omega}{r^2 \sin^2\theta} \left(\cos\theta \frac{\partial \psi}{\partial r} - \frac{\sin\theta}{r} \frac{\partial \psi}{\partial \theta} \right)$$
$$- \left(\frac{2\delta^2 W}{r^2 \sin^2\theta} + \frac{2\delta^2}{Ro} \right) \left(\cos\theta \frac{\partial W}{\partial r} - \frac{\sin\theta}{r} \frac{\partial W}{\partial \theta} \right) = \delta^2 Pr D^2 \omega$$

$$\delta^2 PrD^2 W - \frac{\partial W}{\partial t} = \frac{\delta}{r^2 \sin\theta} \frac{\partial(\psi, W)}{\partial(\theta, r)} - \frac{2\delta}{Ro} \left(\cos\theta \frac{\partial \psi}{\partial r} - \frac{\sin\theta}{r} \frac{\partial \psi}{\partial \theta} \right)$$

$$\frac{\partial T}{\partial t} + \frac{\delta}{r^2 \sin\theta} \frac{\partial(\psi, T)}{\partial(\theta, r)} = \delta^2 \nabla^2 T$$

where

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{\cot\theta}{r^2} \frac{\partial}{\partial \theta}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cot\theta}{r^2} \frac{\partial}{\partial \theta}$$

$$\frac{\partial(A, B)}{\partial(x, y)} = \frac{\partial A}{\partial x} \frac{\partial B}{\partial y} - \frac{\partial A}{\partial y} \frac{\partial B}{\partial x}$$

The dimensionless parameters include:

$$Ra = \frac{\alpha g_0 H^3 \Delta T}{\nu\kappa} \quad \text{Rayleigh number}$$

$$Ro = \frac{\kappa}{H\Omega R} \quad \text{Rossby number}$$

$$Pr = \frac{\nu}{\kappa} \quad \text{Prandtl number}$$

$$\delta = \frac{H}{R} \quad \text{Shallowness parameter}$$

Time and length are scaled as $\tilde{t} \to \frac{H^2}{\kappa} t$, $\tilde{r} \to Rr$

The adopted scaling for the flow variables is given by

$$(\tilde{\psi}, \tilde{\omega}, \tilde{W}) \to (\frac{\kappa R^2}{H}\psi, \frac{\kappa R}{H^2}\omega, \frac{\kappa R}{H} W)$$

where the tilde denotes a dimensional quantity.

The equations are to be solved in the region

$$0 \le \theta \le \frac{\pi}{2} \; , \; 1 \le r \le 1 + \delta$$

subject to the no-slip and impermeability boundary conditions

$$\psi = \frac{\partial \psi}{\partial r} = W = 0 \;\text{ on }\; r = 1 \;\text{ and }\; r = 1 + \delta$$

The assumed symmetry requires imposing the following conditions at the pole and equator

$$\psi = \omega = W = 0 \;\text{ along }\; \theta = 0 \;\text{ and }\; \psi = \omega = \frac{\partial W}{\partial \theta} = 0 \;\text{ along }\; \theta = \frac{\pi}{2}$$

Note that the stream function is overspecified while the vorticity is underspecified.

The surface temperature is allowed to vary sinusoidally

$$\tilde{T} = T_{ave} - \Delta T \cos(2\theta)$$

with T_{ave} denoting the average surface temperature. The scaled temperature is defined as $T = \dfrac{\tilde{T} - T_{edge}}{T_{ave} + \Delta T - T_{edge}}$ where T_{edge} is the prescribed temperature along the top of the fluid layer. In dimensionless form the temperature satisfies

$$T = 1 - \gamma \cos^2 \theta \;\text{ on }\; r = 1 \;\text{ and }\; T = 0 \;\text{ on }\; r = 1 + \delta$$

where $\gamma = \dfrac{2\Delta T}{T_{ave} + \Delta T - T_{edge}}$
At the pole and equator zero heat-flux conditions are applied

$$\frac{\partial T}{\partial \theta} = 0 \;\text{ along }\; \theta = 0 \;\text{ and }\; \theta = \frac{\pi}{2}$$

Introduce the change of variables (z, μ) where $r = 1 + \delta z$ and $\mu = \cos\theta$. This maps the domain to the unit square: $0 \leq z, \mu \leq 1$. The transformed equations become:

$$\delta\omega = -\hat{D}^2\psi$$

$$\frac{\partial \omega}{\partial t} + \frac{1}{(1+\delta z)^2}\frac{\partial(\psi,\omega)}{\partial(z,\mu)} + \frac{2\omega}{(1-\mu^2)(1+\delta z)^2}\left[\mu\frac{\partial \psi}{\partial z} + \frac{\delta(1-\mu^2)}{(1+\delta z)}\frac{\partial \psi}{\partial \mu}\right]$$
$$- \frac{2\delta W}{(1-\mu^2)(1+\delta z)^2}\left[\mu\frac{\partial W}{\partial z} + \frac{\delta(1-\mu^2)}{(1+\delta z)}\frac{\partial W}{\partial \mu}\right]$$
$$- \frac{2\delta}{R_0}\left[\mu\frac{\partial W}{\partial z} + \frac{\delta(1-\mu^2)}{(1+\delta z)}\frac{\partial W}{\partial \mu}\right] = \delta Pr Ra(1-\mu^2)\frac{\partial T}{\partial \mu} + Pr\hat{D}^2\omega$$

$$\frac{\partial T}{\partial t} + \frac{1}{(1+\delta z)^2}\frac{\partial(\psi, T)}{\partial(z,\mu)} = \hat{\nabla}^2 T$$

$$Pr\hat{D}^2 W - \frac{\partial W}{\partial t} = \frac{1}{(1+\delta z)^2}\frac{\partial(\psi, W)}{\partial(z,\mu)} - \frac{2}{R_0}\left[\mu\frac{\partial \psi}{\partial z} + \frac{\delta(1-\mu^2)}{(1+\delta z)}\frac{\partial \psi}{\partial \mu}\right]$$

where

$$\hat{D}^2 = \frac{\partial^2}{\partial z^2} + \frac{\delta^2(1-\mu^2)}{(1+\delta z)^2}\frac{\partial^2}{\partial \mu^2}$$

$$\hat{\nabla}^2 = \frac{\partial^2}{\partial z^2} + \frac{2\delta}{(1+\delta z)}\frac{\partial}{\partial z} - \frac{2\mu\delta^2}{(1+\delta z)^2}\frac{\partial}{\partial \mu} + \frac{\delta^2(1-\mu^2)}{(1+\delta z)^2}\frac{\partial^2}{\partial \mu^2}$$

For small δ approximate steady-state solutions can be constructed by expanding the flow variables in the following series:

$$\psi = \psi_0 + \delta\psi_1 + \delta^2\psi_2 + \cdots$$

$$\omega = \omega_0 + \delta\omega_1 + \delta^2\omega_2 + \cdots$$

$$W = W_0 + \delta W_1 + \delta^2 W_2 + \cdots$$

$$T = T_0 + \delta T_1 + \delta^2 T_2 + \cdots$$

The approximate solutions correct to second order in δ are:

$$\psi_s(z,\mu) \approx -2\gamma\delta^2 Ra\mu(1-\mu^2)F_1(z)$$

$$\omega_s(z,\mu) \approx 2\gamma\delta Ra\mu(1-\mu^2)\left[\frac{d^2F_1}{dz^2} + \delta F_2(z)\right]$$

$$W_s(z,\mu) \approx \frac{4\gamma\delta^2 Ra}{PrR_0}\mu^2(1-\mu^2)F_3(z)$$

where

$$F_1(z) = \frac{z^4}{24} - \frac{z^5}{120} - \frac{7z^3}{120} + \frac{z^2}{40}$$

$$F_2(z) = \frac{z^4}{12} - \frac{z^3}{6} + \frac{z}{12} - \frac{1}{60}$$

$$F_3(z) = \frac{z^5}{120} - \frac{z^6}{720} - \frac{7z^4}{480} + \frac{z^3}{120} - \frac{z}{1440}$$

and $T_s(z,\mu) \approx (1-\gamma\mu^2)(1-z)(1-\delta z) + \delta^2 T_2(z,\mu)$ with

$$T_2(z,\mu) = \gamma(1-3\mu^2)z^2\left(1-\frac{z}{3}\right) + (1-\gamma\mu^2)z^2\left(1-\frac{z^2}{3}\right)$$

$$+\gamma^2 Ra\mu^2(1-\mu^2)z^3\left(\frac{z^4}{252}-\frac{z^3}{36}+\frac{41z^2}{600}-\frac{3z}{40}+\frac{1}{30}\right)$$

$$-\gamma Ra(1-3\mu^2)(1-\gamma\mu^2)z^4\left(\frac{z^2}{360}-\frac{z^3}{2520}-\frac{7z}{1200}+\frac{1}{240}\right)$$

$$+z\left(-\frac{2}{3}(1-\gamma\mu^2)-\frac{2}{3}\gamma(1-3\mu^2)-\frac{1}{350}\gamma^2 Ra\mu^2(1-\mu^2)\right.$$

$$\left.+\frac{1}{1400}\gamma Ra(1-3\mu^2)(1-\gamma\mu^2)\right)$$

The steady-state flow is perturbed by imposing small disturbances:

$$T = T_s + T' \ , \ \ \psi = \psi_s + \psi' \ , \ \ \omega = \omega_s + \omega' \ , \ \ W = W_s + W'$$

Assuming the principle of exchange of stabilities holds, the linearized perturbation equations become:

$$\delta\omega' = -\hat{D}^2\psi'$$

$$\hat{\nabla}^2 T' = \frac{1}{(1+\delta z)^2}\left(\frac{\partial(\psi_s, T')}{\partial(z,\mu)} + \frac{\partial(\psi', T_s)}{\partial(z,\mu)}\right)$$

$$Pr\,\hat{D}^2 W' = \frac{1}{(1+\delta z)^2}\left(\frac{\partial(\psi_s, W')}{\partial(z,\mu)} + \frac{\partial(\psi', W_s)}{\partial(z,\mu)}\right) - \frac{2}{R_0}\left(\mu\frac{\partial\psi'}{\partial z} + \frac{\delta(1-\mu^2)}{(1+\delta z)}\frac{\partial\psi'}{\partial\mu}\right)$$

$$Pr\hat{D}^2\omega' + \delta PrRa(1-\mu^2)\frac{\partial T'}{\partial \mu} = \frac{1}{(1+\delta z)^2}\left(\frac{\partial(\psi_s,\omega')}{\partial(z,\mu)} + \frac{\partial(\psi',\omega_s)}{\partial(z,\mu)}\right)$$

$$+\frac{2\omega_s}{(1-\mu^2)(1+\delta z)^2}\left(\mu\frac{\partial\psi'}{\partial z} + \frac{\delta(1-\mu^2)}{(1+\delta z)}\frac{\partial\psi'}{\partial\mu}\right)$$

$$+\frac{2\omega'}{(1-\mu^2)(1+\delta z)^2}\left(\mu\frac{\partial\psi_s}{\partial z} + \frac{\delta(1-\mu^2)}{(1+\delta z)}\frac{\partial\psi_s}{\partial\mu}\right)$$

$$-\frac{2\delta W_s}{(1-\mu^2)(1+\delta z)^2}\left(\mu\frac{\partial W'}{\partial z} + \frac{\delta(1-\mu^2)}{(1+\delta z)}\frac{\partial W'}{\partial\mu}\right)$$

$$-\frac{2\delta W'}{(1-\mu^2)(1+\delta z)^2}\left(\mu\frac{\partial W_s}{\partial z} + \frac{\delta(1-\mu^2)}{(1+\delta z)}\frac{\partial W_s}{\partial\mu}\right)$$

$$-\frac{2\delta}{R_0}\left(\mu\frac{\partial W'}{\partial z} + \frac{\delta(1-\mu^2)}{(1+\delta z)}\frac{\partial W'}{\partial\mu}\right)$$

Following Walton, the disturbances are expanded in powers of δ:

$$T'(z,\mu) = (T^{(0)} + \delta T^{(1)} + \delta^2 T^{(2)} + \cdots)\exp\left(\frac{i}{\delta}\int_0^\mu k(\xi)d\xi\right)$$

$$\psi'(z,\mu) = (\psi^{(0)} + \delta\psi^{(1)} + \delta^2\psi^{(2)} + \cdots)\exp\left(\frac{i}{\delta}\int_0^\mu k(\xi)d\xi\right)$$

$$\omega'(z,\mu) = (\omega^{(0)} + \delta\omega^{(1)} + \delta^2\omega^{(2)} + \cdots)\exp\left(\frac{i}{\delta}\int_0^\mu k(\xi)d\xi\right)$$

$$W'(z,\mu) = (W^{(0)} + \delta W^{(1)} + \delta^2 W^{(2)} + \cdots)\exp\left(\frac{i}{\delta}\int_0^\mu k(\xi)d\xi\right)$$

The Rayleigh number and differential operators are also expanded in similar series:

$$Ra = Ra^{(0)} + \delta Ra^{(1)} + \delta^2 Ra^{(2)} + \cdots$$

$$\hat{D}^2 = \frac{\partial^2}{\partial z^2} + \delta^2 (1-\mu^2)\frac{\partial^2}{\partial \mu^2} + \cdots$$

$$\hat{\nabla}^2 = \frac{\partial^2}{\partial z^2} + 2\delta \frac{\partial}{\partial z} + \delta^2 \left(-2\mu \frac{\partial}{\partial \mu} + (1-\mu^2)\frac{\partial^2}{\partial \mu^2} \right) + \cdots$$

Substituting these into the perturbation equations leads to a hierarchy of problems.

$T^{(0)}$ and $\psi^{(1)}$ satisfy the coupled system

$$\left[\frac{\partial^2}{\partial z^2} - k^2(1-\mu^2)\right] T^{(0)} = ik(1-\gamma\mu^2)\psi^{(1)}$$

$$\left[\frac{\partial^2}{\partial z^2} - k^2(1-\mu^2)\right]^2 \psi^{(1)} = ikRa^{(0)}(1-\mu^2)T^{(0)}$$

and can be combined to yield

$$\left[\frac{\partial^2}{\partial z^2} - k^2(1-\mu^2)\right]^3 \psi^{(1)} = -k^2 Ra^{(0)}(1-\mu^2)(1-\gamma\mu^2)\psi^{(1)}$$

The disturbance will be concentrated near the equator, so set $\mu = 0$:

$$\left[\frac{\partial^2}{\partial z^2} - k_0^2\right]^3 \psi^{(1)} = -k_0^2 Ra^{(0)}\psi^{(1)}$$

where $k_0 = k(0)$.

Solving subject to the conditions

$$\psi^{(1)} = \frac{\partial \psi^{(1)}}{\partial z} = \left(\frac{\partial^2}{\partial z^2} - k_0^2\right)\psi^{(1)} = 0 \quad \text{at} \quad z = 0, 1$$

suggests looking for a solution of the form

$$\psi^{(1)}(z, 0) = ce^{qz}$$

where q are the roots of the equation

$$(q^2 - k_0^2)^3 = k_0^2 Ra^{(0)}$$

The problem bears a close resemblance to the classical Rayleigh-Bénard problem with rotation having no influence. The only difference lies in the allowable wavenumbers and the values of Ra_{crit} and k_{crit}.

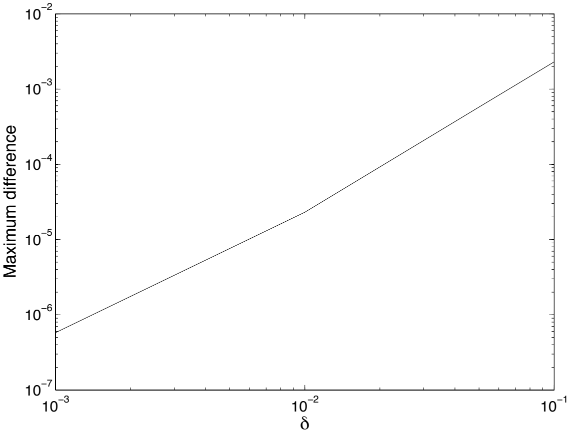
Here, the allowable perturbation wavenumbers are $k_n = 2n$ where $n = 1, 2, 3, \cdots$. $Ra^{(0)}_{crit}$ is defined as the minimum value of $Ra^{(0)}$ having a real wavenumber $k_{0,crit}$. From the table below it follows that the minimum value of $Ra^{(0)}$ occurs when $k_{0,crit} = 4$ and the numerical solution to the algebraic equation yields $Ra^{(0)}_{crit} \approx 1879$. Hence, to leading order $Ra_{crit} \approx 1879$.

k_0	$Ra^{(0)}$
2	2178
4	1879
6	3418
8	7085

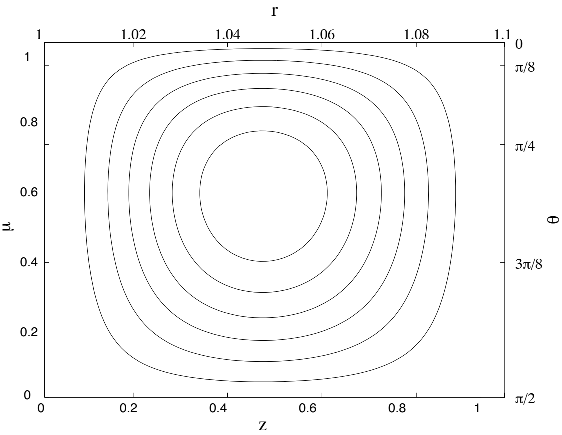
- Computations were carried out using $\gamma = 0.5$, $Ro = 1$ and $Pr = 0.7$; δ and Ra were allowed to vary.
- Computational parameters used included:
  80×80 grid with uniform spacing of $1/80$,
  predefined tolerance of $\epsilon = 10^{-6}$, and
  the uniform time step of $\Delta t = 0.01$ was used in the unsteady computations.
- The initial conditions used in the unsteady calculations were:

$$W = \psi = \omega = 0 \text{ and } T = T_s(z, \mu)$$

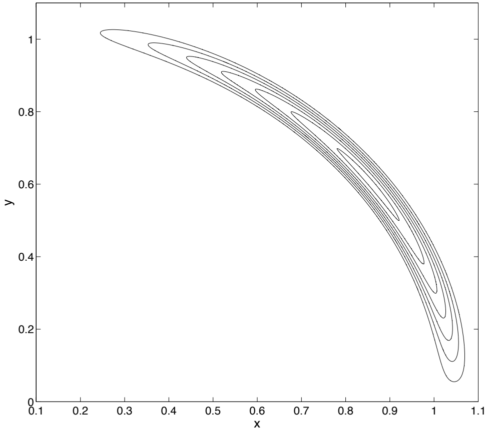
Loglog plot of the maximum difference between the analytical and numerical steady-state solutions with $Ra = 100$.



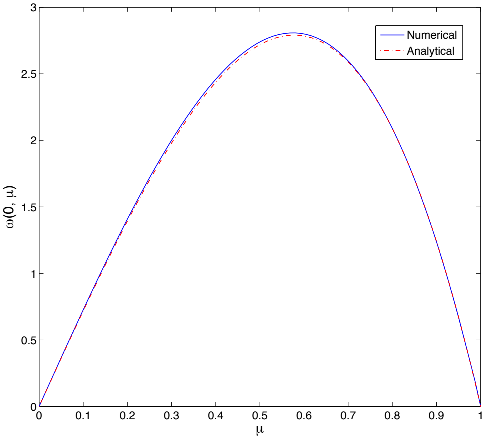
Contour plot of the steady-state stream function in (z, μ) and (r, θ) coordinates for $Ra = 1500$ and $\delta = 0.1$.



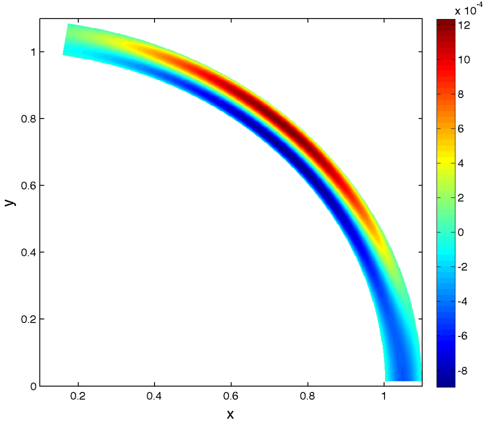
Contour plot of the steady-state stream function in Cartesian coordinates for $Ra = 1500$ and $\delta = 0.1$.



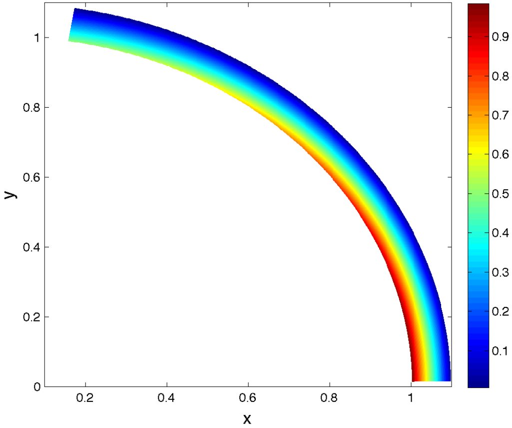
The steady-state surface vorticity distribution for $Ra = 1500$ and $\delta = 0.1$.



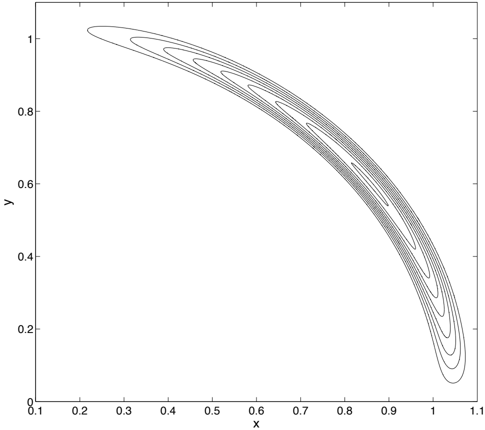
The steady-state zonal velocity distribution for $Ra = 1500$ and $\delta = 0.1$.



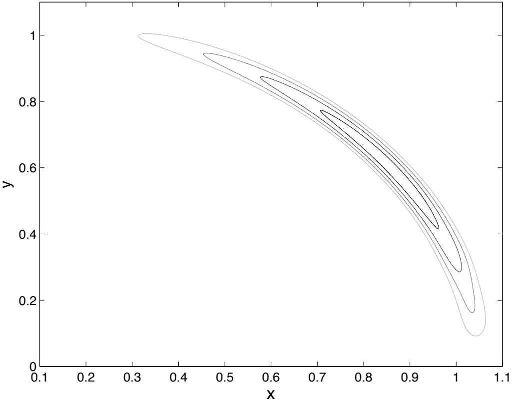
The steady-state temperature distribution for $Ra = 1500$ and $\delta = 0.1$.



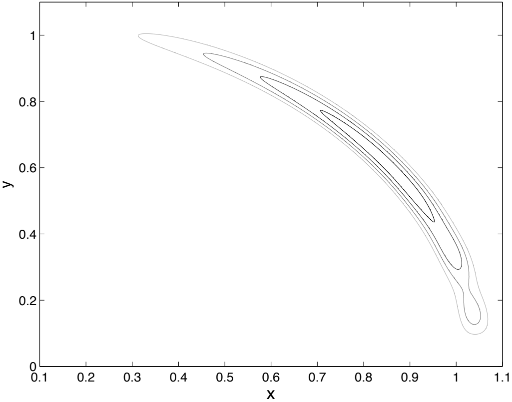
Streamline circulation pattern in Cartesian coordinates at $t = 10.6$ for $Ra = 1870$ and $\delta = 0.1$.



Streamline circulation pattern in Cartesian coordinates at $t = 8.5$ for $Ra = 1890$ and $\delta = 0.1$.



Streamline circulation pattern in Cartesian coordinates at $t = 10.7$ for $Ra = 1890$ and $\delta = 0.1$.



Streamline circulation pattern in Cartesian coordinates at $t = 12.9$ for $Ra = 1890$ and $\delta = 0.1$.