Long-wave instability of flow with temperature dependent fluid properties down an incline

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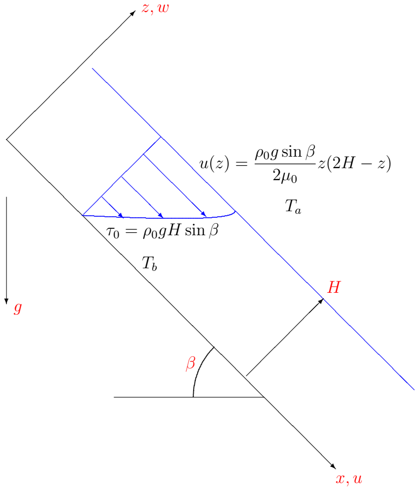
Problem Description

We consider two-dimensional gravity-driven flow of a thin fluid layer having variable fluid properties down a heated incline as shown:

Scaling based on steady flow with constant fluid properties



Length scale:

$$H = \left(\frac{3\mu_0 Q}{\rho_0 g \sin\beta} \right)^{1/3}$$

Velocity scale:

$U = Q/H$

Temperature scale:

$\Delta T = T_b - T_a$

Temperature difference:

$T = T - T_a$

Pressure scale:

$\rho_0 U^2$

Time scale:

H/U

Previous Work

- Isothermal flow has been studied extensively experimentally (initially by Kapitza & Kapitza, 1949), theoretically (initially by Benjamin, 1957) and using DNS (initially by Ramaswamy, Chippada & Joo, 1996). Commonly used mathematical models include the integral-boundary-layer model (Shkadov, 1967) and the weighted residual model (Ruyer-Quil & Manneville, 2002).
- Non-isothermal flow has received much less attention. The focus has been on the variation of surface tension with temperature which gives rise to the Marangoni effect. A key contribution was by Trevelyan *et al.* (2007). The studies by Goussis & Kelly (1985) and Hwang & Weng (1988) considered variations in viscosity only, while Kabova & Kuznetsov (2002) accounted for variable viscosity and surface tension.

In the current study we investigate the influence that variable surface tension (σ), density (ρ), viscosity (μ), thermal conductivity (K) and specific heat (c_p) have on the stability of the flow.

Mathematical Formulation

For flow with variable fluid properties the governing equations in the absence of viscous dissipation are (Spurk & Aksel, 2008):

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = 0$$

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + g\rho\sin\beta + \frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - g\rho\cos\beta + \frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]$$

$$\rho\frac{D(c_p T)}{Dt} = \frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right) - p\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)$$

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Variable Fluid Properties

The fluid properties are assumed to vary linearly with temperature as follows:

$$\rho = \rho_0 - \hat{\alpha}(T - T_a)$$

$$\mu = \mu_0 - \hat{\lambda}(T - T_a)$$

$$c_p = c_{p0} + \hat{S}(T - T_a)$$

$$K = K_0 + \hat{\Lambda}(T - T_a)$$

$$\sigma = \sigma_0 - \gamma(T - T_a)$$

where ρ_0, μ_0, c_{p0}, K_0 and σ_0 are reference values at $T = T_a$.

Dimensionless Equations

Using the Boussinesq approximation and the proposed scaling we obtain:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$Re\frac{Du}{Dt} = -Re\frac{\partial p}{\partial x} + 3(1-\alpha T) + \frac{\partial}{\partial x}\left((1-\lambda T)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial z}\left((1-\lambda T)\frac{\partial u}{\partial z}\right)$$

$$-\lambda\frac{\partial T}{\partial x}\frac{\partial u}{\partial x} - \lambda\frac{\partial T}{\partial z}\frac{\partial w}{\partial x}$$

$$Re\frac{Dw}{Dt} = -Re\frac{\partial p}{\partial z} - 3\cot\beta(1-\alpha T) + \frac{\partial}{\partial x}\left((1-\lambda T)\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial z}\left((1-\lambda T)\frac{\partial w}{\partial z}\right)$$

$$-\lambda\frac{\partial T}{\partial x}\frac{\partial u}{\partial z} - \lambda\frac{\partial T}{\partial z}\frac{\partial w}{\partial z}$$

$$PrRe\frac{D}{Dt}\left[\left(1+\frac{S}{\Delta T_r}\right)T + ST^2\right] = \frac{\partial}{\partial x}\left[(1+\Lambda T)\frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial z}\left[(1+\Lambda T)\frac{\partial T}{\partial z}\right]$$

Boundary Conditions

Along the free surface $z = h$:

$$p = \frac{2(1-\lambda T)}{ReF}\left(\left[\frac{\partial h}{\partial x}\right]^2\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}-\frac{\partial h}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial h}{\partial x}\frac{\partial w}{\partial x}\right)-\frac{(We-MaT)}{F^{3/2}}\frac{\partial^2 h}{\partial x^2}$$

$$-MaRe\sqrt{F}\left(\frac{\partial T}{\partial x}+\frac{\partial h}{\partial x}\frac{\partial T}{\partial z}\right)=(1-\lambda T)\left[G\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)-4\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}\right]$$

$$w=\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}$$

$$-Bi\sqrt{F}T=(1+\Lambda T)\left(\frac{\partial T}{\partial z}-\frac{\partial h}{\partial x}\frac{\partial T}{\partial x}\right)$$

$$\text{where}\quad F=1+\left[\frac{\partial h}{\partial x}\right]^2,\quad G=1-\left[\frac{\partial h}{\partial x}\right]^2$$

On the bottom $z = 0$: $u = w = 0\,,\ T = 1$

Dimensionless Parameters

$$Re = \frac{\rho_0 U H}{\mu_0}$$ Reynolds number

$$We = \frac{\sigma_0}{\rho_0 U^2 H}$$ Weber number

$$Ma = \frac{\gamma \Delta T}{\rho_0 U^2 H}$$ Marangoni number

$$Pr = \frac{\mu_0 c_{p0}}{K_0}$$ Prandtl number

$$Bi = \frac{\alpha_g H}{K_0}$$ Biot number

$$\Delta T_r = \frac{T_b - T_a}{T_a}$$ Relative Temperature Difference

Also, $\alpha, \lambda, \Lambda, S$ represent dimensionless rates of change of density, viscosity, thermal conductivity and specific heat with respect to temperature.

Steady State Equations

Steady uniform flow in the streamwise direction given by $h \equiv 1$, $w \equiv 0$, $u = u_s(z)$, $p = p_s(z)$ and $T = T_s(z)$ satisfies the following boundary-value problems:

$$\frac{d}{dz}\left[(1+\Lambda T_s)\frac{dT_s}{dz}\right] = 0\ ,\ (1+\Lambda T_s)\frac{dT_s}{dz} + BiT_s = 0 \ \text{ at } \ z = 1\ ,\ T_s(0) = 1$$

$$\frac{d}{dz}\left[(1-\lambda T_s)\frac{du_s}{dz}\right] + 3(1-\alpha T_s) = 0\ ,\ \frac{du_s}{dz} = 0 \ \text{ at } \ z = 1\ ,\ u_s(0) = 0$$

$$Re\frac{dp_s}{dz} = -3\cot\beta(1-\alpha T_s)\ ,\ p_s(1) = 0$$

Steady State Solutions

The steady state solutions are given by:

$$T_s(z) = \sqrt{a - bz} - \frac{1}{\Lambda}$$

$$u_s(z) = a_0 \ln\left(\frac{A - \lambda\sqrt{a - bz}}{A - \lambda\sqrt{a}}\right) + a_1 z - \frac{\alpha}{\lambda} z^2 + a_2(\sqrt{a - bz} - \sqrt{a})$$
$$+ a_3[(a - bz)^{3/2} - a^{3/2}]$$

$$p_s(z) = \frac{3\cot\beta}{Re}\left(1 + \frac{\alpha}{\Lambda}\right)(1 - z) + \frac{2\alpha\cot\beta}{bRe}[(a - b)^{3/2} - (a - bz)^{3/2}]$$

where the constants a, b, a_0, a_1, a_2, a_3 and A are related to the parameters Λ, Bi, λ and α.

Stability Analysis

Now impose small disturbances on the steady state flow:

$$u = u_s(z) + \tilde{u}(x,z,t)\ ,\ w = \tilde{w}(x,z,t)\ ,\ p = p_s(z) + \tilde{p}(x,z,t)$$

$$T = T_s(z) + \tilde{T}(x,z,t)\ ,\ h = 1 + \eta(x,t)$$

Next, substitute these into the governing equations, linearize and assume the disturbances have the form:

$$(\tilde{u}, \tilde{w}, \tilde{p}, \tilde{T}, \eta) = (\hat{u}(z), \hat{w}(z), \hat{p}(z), \hat{T}(z), \hat{\eta})e^{ik(x-ct)}$$

where k (real & positive) represents the wavenumber of the perturbation and c is a complex quantity with the real part denoting the phase speed of the perturbation while the imaginary part is related to the growth rate.

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Perturbation Equations

The linearized perturbed equations become:

$$D\hat{w} + ik\hat{u} = 0$$

$$Re[ik(u_s - c)\hat{u} + \hat{w}Du_s] = -ikRe\hat{p} + k^2(\lambda T_s - 1)\hat{u}$$

$$+D[(1 - \lambda T_s)D\hat{u}] - \lambda \hat{T} D^2 u_s - \lambda Du_s D\hat{T} - ik\lambda \hat{w} DT_s - 3\alpha \hat{T}$$

$$ikRe(u_s - c)\hat{w} = -ReD\hat{p} + 3\alpha \cot\beta\, \hat{T} - k^2(1 - \lambda T_s)\hat{w}$$

$$+D[(1 - \lambda T_s)D\hat{w}] - ik\lambda \hat{T} Du_s - \lambda DT_s D\hat{w}$$

$$PrRe(1 + S/\Delta T_r + 2ST_s)[ik(u_s - c)\hat{T} + \hat{w}DT_s]$$

$$= -k^2(1 + \Lambda T_s)\hat{T} + D^2[(1 + \Lambda T_s)\hat{T}]$$

where the differential operator D is defined as:

$$D \equiv \frac{d}{dz}$$

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Boundary Conditions

Along the free surface ($z = 1$) the perturbations will satisfy:

$$\hat{p} = -\hat{\eta} D p_s + \frac{2}{Re}(1 - \lambda T_s) D\hat{w} + k^2(We - MaT_s)\hat{\eta}$$

$$(1 - \lambda T_s)(\hat{\eta} D^2 u_s + D\hat{u} + ik\hat{w}) = -ikMaRe(\hat{T} + \hat{\eta} DT_s)$$

$$D[(1 + \Lambda T_s)\hat{T}] + \hat{\eta} D[(1 + \Lambda T_s) DT_s + BiT_s] + Bi\hat{T} = 0$$

$$\hat{w} = ik(u_s - c)\hat{\eta}$$

while the bottom conditions are:

$$\hat{u}(0) = \hat{w}(0) = \hat{T}(0) = 0$$

Small Wavenumber Expansion

Since small wavenumber perturbations are expected to be the most unstable, we expand the perturbations in the following series:

$$\hat{u} = u_0(z) + ku_1(z) + O(k^2)$$

$$\hat{w} = w_0(z) + kw_1(z) + O(k^2)$$

$$\hat{p} = p_0(z) + kp_1(z) + O(k^2)$$

$$\hat{T} = T_0(z) + kT_1(z) + O(k^2)$$

$$\hat{\eta} = \eta_0 + k\eta_1 + O(k^2)$$

$$c = c_0 + kc_1 + O(k^2)$$

and proceed to solve the eigenvalue problem asymptotically as $k \to 0$.

Special Cases:

For $Bi = 0$ the critical Reynolds number for the onset of instability can be found exactly and is given by:

$$Re_{crit} = \frac{5}{6}\cot\beta \frac{(1-\lambda)^2}{(1-\alpha)}$$

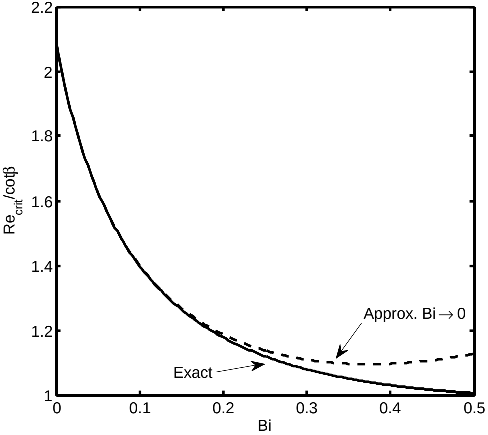
Note that if the values of α and λ are such that $1 - \alpha = (1 - \lambda)^2$, then it follows that the effects of variable fluid properties cancel and the threshold of instability is the same as that for isothermal flow.

Other special cases include $\Lambda = \lambda = 0$ and $Bi \to \infty$, both of which lead to exact, but lengthy, expressions for Re_{crit}.

For the general case, analytical expressions for the neutral stability state have been obtained in the form of asymptotic expansions as $Bi \to 0$ or as Λ, $\lambda \to 0$.

Results:

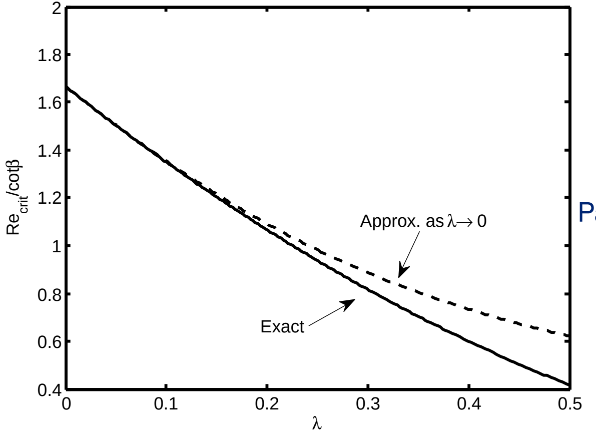
Accuracy of the expansion as $Bi \to 0$



Parameters: $\lambda = \Lambda = 0$
$\alpha = 0.6$, $S = 0$, $\Delta T_r = 1$
$Ma = 1$ and $Pr = 7$

Results:

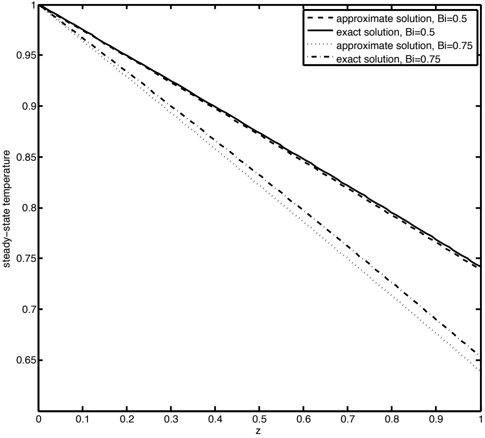
Accuracy of the expansion as $\lambda \to 0$



Parameters: $Bi = 0$, $\alpha = 0.5$

Results:

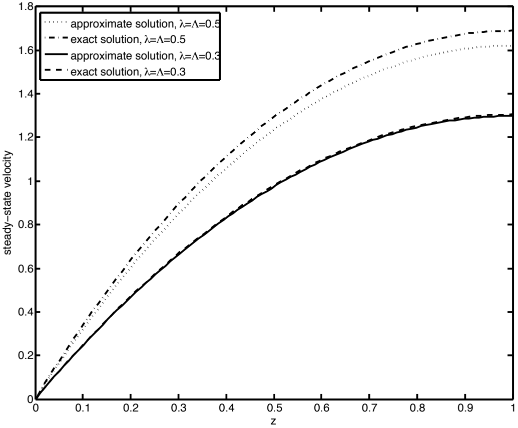
Steady-state temperature as a function of z



Parameters: $\Lambda = 0.5$

Results:

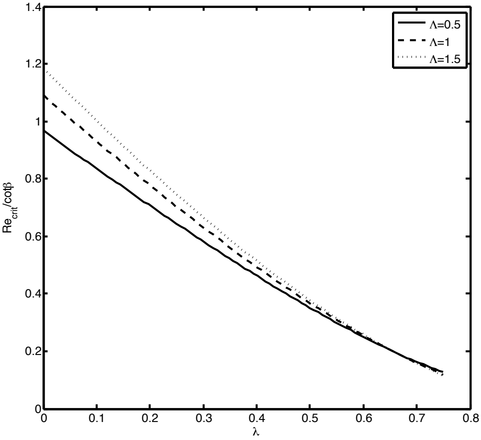
Steady-state velocity as a function of z



Parameters: $\alpha = 0.5$, $Bi = 1$

Results:

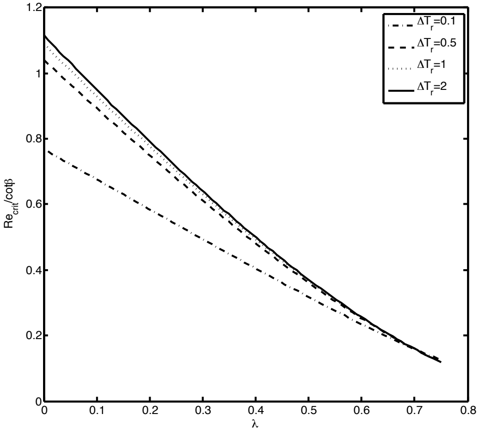
Scaled critical Reynolds number as a function of λ



Parameters: $Bi = 0.25$
$\alpha = 0.5$, $S = 1$, $Ma = 1$
$\Delta T_r = 1$ and $Pr = 7$

Results:

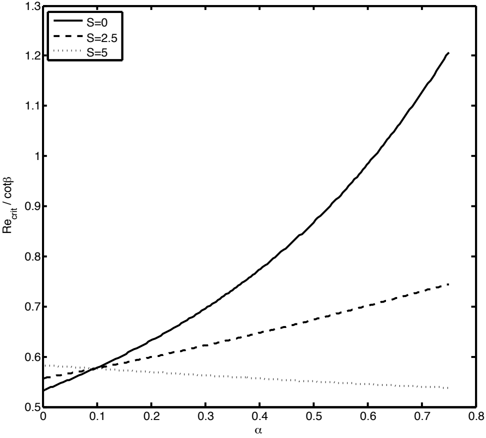
Scaled critical Reynolds number as a function of λ



Parameters: $Bi = 0.25$
$\alpha = 0.5$, $S = 1$, $Ma = 1$
$\Lambda = 1$ and $Pr = 7$

Results:

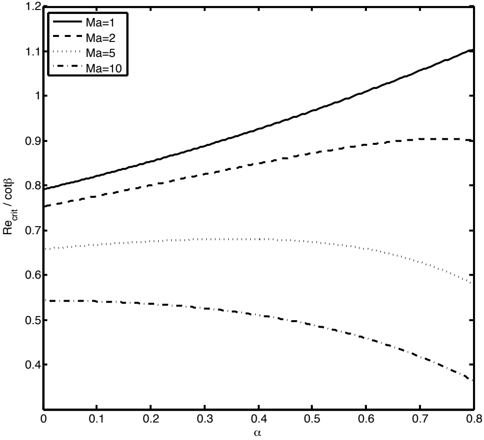
Scaled critical Reynolds number as a function of α



Parameters: $Bi = 0.25$
$\lambda = 0.2$, $\Lambda = 1$, $Ma = 1$
$\Delta T_r = 1$ and $Pr = 7$

Results:

Scaled critical Reynolds number as a function of α



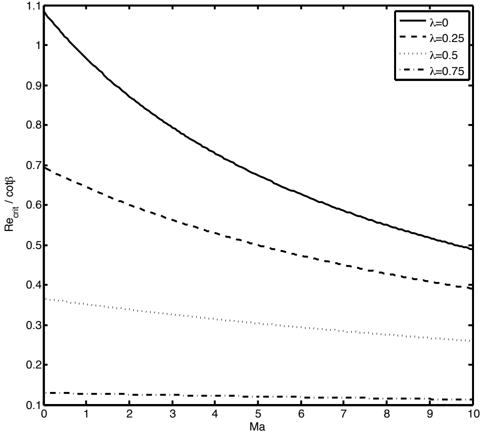
Parameters: $Bi = 0.25$

$\lambda = 0$, $S = 1$, $\Lambda = 0.5$

$\Delta T_r = 1$ and $Pr = 7$

Results:

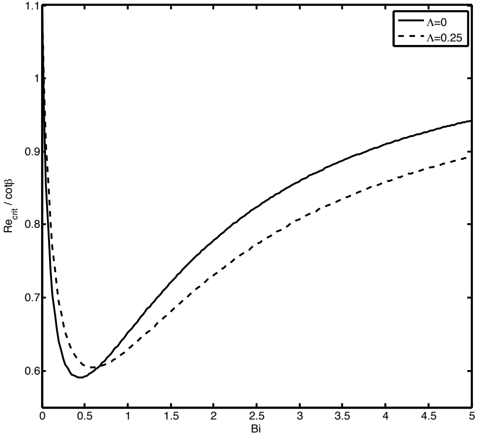
Scaled critical Reynolds number as a function of *Ma*



Parameters: $Bi = 0.25$
$\alpha = 0.5$, $S = 1$, $\Lambda = 0.5$
$\Delta T_r = 1$ and $Pr = 7$

Results:

Scaled critical Reynolds number as a function of *Bi*



Parameters: $\alpha = 0.5$, $\lambda = 0.2$
$S = 1$, $\Delta T_r = 1$, $Ma = 1$
and $Pr = 7$

Conclusions

Through a combination of special cases and asymptotic expansions focussing on long-wave perturbations, our linear stability analysis uncovered the following key results:

- If $Bi = 0$, then the critical Reynolds number for the onset of instability is given by:

$$Re_{crit} = \frac{5}{6}\cot\beta\frac{(1-\lambda)^2}{(1-\alpha)}$$

- If $\Lambda = \lambda = 0$ an exact lengthy expression for Re_{crit} was obtained which reproduces known results when appropriate limits are taken.
- If $Bi \to \infty$ another lengthy expression for Re_{crit} was obtained which can confirm the result in the limit of large Bi.

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- Increasing λ decreases viscosity which destabilizes the flow.
- Increasing α decreases density which, in general, stabilizes the flow. Exceptions to this occur in extreme cases of large variations in the specific heat and for large Marangoni numbers.
- In general, increasing Ma destabilizes the flow.
- The behaviour of Re_{crit} with Bi is more complicated. As Bi increases from zero Re_{crit} initially decreases and reaches a minimum value; as Bi is increases further Re_{crit} increases. The limiting values of Re_{crit} as $Bi \to 0$ and $Bi \to \infty$ can be found.
- This research has recently appeared in the: *International Journal of Engineering Science*, **70**, 73-90, 2013