Liquid Film Flow Over Heated Porous Surfaces

By: Serge D'Alessio[1],
Kelly Ogden[1] and J. P. Pascal[2]

[1]Department of Applied Mathematics
University of Waterloo, Waterloo, Canada

[2]Department of Mathematics
Ryerson University, Toronto, Canada

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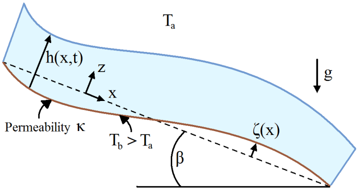
Problem description



Applications

- Heat Exchangers
- Manufacturing Coatings
- Environmental Flows

$Re = \frac{Q}{\nu}$	Reynolds number
$We = \frac{\sigma_0 H}{\rho Q^2}$	Weber number
$\delta = \frac{H}{\lambda_b}$	Shallowness parameter
$\zeta(x) = \frac{A_b}{H} \cos\left(\frac{2\pi x}{\lambda_b}\right)$	Bottom topography; $\frac{A_b}{H} = a_b$
$Bi = \frac{\alpha_g H}{\rho c_p \kappa}$	Biot number
$Ma = \frac{\gamma \Delta T}{\rho U^2 H}$	Marangoni number
$\delta_1 = \frac{\sqrt{\kappa}}{\alpha H}$	Permeability parameter
$Pr, Pe = RePr$	Prandtl and Peclet Numbers

Previous work

- Heating effects were investigated by Kalliadasis et al. [1] and Trevelyan et al. [2]., using constant temperature and a specified heat flux boundary conditions, and an even bottom.
- Heating effects with a constant temperature & wavy bottom were investigated using the weighted residual model by D'Alessio et al. [3].
- A permeable bottom was considered by Pascal & Pascal and D'Alessio [4,5] using the Beavers and Joseph [6] boundary condition.
- Heating and porosity were recently combined for the even bottom case by Sadiq et al. [7].

The current study extends the weighted residual model to include the combined effects of heating, bottom waviness, and permeability.

Governing equations

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\delta Re\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) = -\delta Re\frac{\partial P}{\partial x} + 3 + \frac{\partial^2 u}{\partial z^2} + \delta^2\frac{\partial^2 u}{\partial x^2}$$

$$\delta^2 Re\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right) = -Re\frac{\partial P}{\partial z} - 3\cot\beta + \delta\frac{\partial^2 w}{\partial z^2}$$

$$\delta Pe\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z}\right) = \frac{\partial^2 T}{\partial z^2} + \delta^2\frac{\partial^2 T}{\partial x^2}$$

The model is second order accurate in δ for $O(1)$ parameters.

Free surface boundary conditions

Dynamic conditions at the free surface, $z = h + \zeta$, to $O(\delta^2)$:

$$P_a + \hat{n} \cdot \tau \cdot \hat{n} = -\sigma(T)\vec{\nabla} \cdot \hat{n} \quad \Rightarrow \quad p = \frac{2\delta}{Re} \frac{\partial w}{\partial z}$$

$$\hat{n} \cdot \tau \cdot \hat{t} = \vec{\nabla}\sigma \cdot \hat{t} \quad \Rightarrow \quad 0 = \frac{\partial u}{\partial z} + MaRe\delta \left(\frac{\partial T}{\partial x} + \frac{\partial (h+\zeta)}{\partial x} \frac{\partial T}{\partial z} \right)$$

Heat transfer at the Surface to $O(\delta^2)$:

$$\vec{\nabla} T \cdot \hat{n} = \frac{-\alpha_g}{\rho c_p k}(T - T_a) \quad \Rightarrow \quad -BiT = \frac{\partial T}{\partial z}$$

Surface and bottom boundary conditions

Kinematic condition at the free surface:

$$w = \frac{\partial h}{\partial t} + u\left(\zeta' + \frac{\partial h}{\partial x}\right) \quad \text{at } z = h + \zeta$$

Bottom boundary conditions:

$$\left.\begin{aligned} w &= \zeta' u \\ \delta_1 \frac{\partial u}{\partial z} &= u \\ T &= 1 \end{aligned}\right\} \text{at } z = \zeta$$

The Beavers and Joseph slip condition

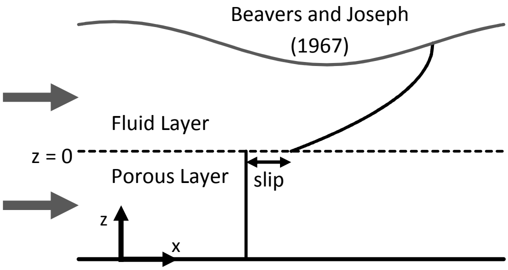


Figure modified from LeBars and Worster, 2006 [8]

$$u_D = u_S(0) - \delta_1 \frac{du_S}{dz} = u_S(-\delta_1) \;\; where \;\; \delta_1 = \frac{\sqrt{\kappa}}{\alpha H}$$

Model derivation

- Combine momentum equations to eliminate pressure.
- Assume the following profile for velocity:

$$u(x,z,t) = \frac{3q}{2\left(h^3 + 3\delta_1 h^2\right)} b + \frac{\delta MaRe}{4h} \frac{\partial \theta}{\partial x} b_1$$

where

$$b = (z-\zeta)\,(2h - z + \zeta) + 2\delta_1 h \quad , \quad b_1 = (z-\zeta)\,(2h - 3\,(z-\zeta))$$

$$q = \int_{\zeta}^{\zeta+h} u \, dz$$

- Assume the following profile for temperature:

$$T = 1 + \frac{\theta - 1}{h}\,(z-\zeta) \quad , \quad \theta(x,t) = T(z = h+\zeta, x, t)$$

- Multiply by weight functions and integrate in z. Flow variables become h, q, θ.

Final Model equations: continuity and momentum

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0$$

$$\delta\left(h + 2\delta_1\right)\frac{\partial q}{\partial t} = \delta^3 hWe\left(\frac{5}{2}\delta_1 + \frac{5}{6}h\right)\left(\zeta''' + \frac{\partial^3 h}{\partial x^3}\right)$$

$$+\frac{\delta^2}{Re}\left(\frac{9}{2}h\frac{\partial^2 q}{\partial x^2} - \frac{9}{2}\frac{\partial q}{\partial x}\frac{\partial h}{\partial x} - 6q\frac{\partial^2 h}{\partial x^2} - \frac{15}{4}q\zeta'' + \frac{q}{h}\left(4\left(\frac{\partial h}{\partial x}\right)^2 - 5\left(\zeta'\right)^2 - \frac{5}{2}\frac{\partial h}{\partial x}\zeta'\right)\right)$$

$$+\delta^2 hReMa\left(\frac{1}{48}h^2\frac{\partial^2\theta}{\partial x\partial t} + \frac{15}{224}hq\frac{\partial^2\theta}{\partial x^2} + \frac{19}{336}h\frac{\partial q}{\partial x}\frac{\partial\theta}{\partial x} + \frac{5}{112}q\frac{\partial\theta}{\partial x}\frac{\partial h}{\partial x}\right)$$

$$+\delta\left(\frac{9}{7}\frac{q^2}{h}\frac{\partial h}{\partial x} - \frac{45}{16}\delta_1\frac{q^2}{h^2}\zeta' - \frac{5}{2}Ma\frac{\partial\theta}{\partial x}\left(\frac{h}{2} + \delta_1\right) - \frac{17}{7}q\frac{\partial q}{\partial x}\left(1 + \frac{\delta_1}{h}\right)\right)$$

$$+\delta\left(-\frac{5}{2}h\frac{\cot(\beta)}{Re}\left(\frac{\partial h}{\partial x} + \zeta'\right)\left(h + 3\delta_1\right)\right) + \frac{5}{2}\frac{h}{Re}\left(h + 3\delta_1\right) - \frac{5}{2}\frac{q}{hRe}$$

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Final Model equations: energy

$$\delta h \frac{\partial \theta}{\partial t} = \delta^2 \left(-\frac{3}{2}\frac{Bi}{Pe}\theta \left(\zeta' + \frac{\partial h}{\partial x}\right)^2 + \frac{3}{40} MaReh^2 \left(\frac{\partial \theta}{\partial x}\right)^2 + \frac{h}{Pe}\frac{\partial^2 \theta}{\partial x^2} + \frac{1}{Pe}\frac{\partial \theta}{\partial x}\frac{\partial h}{\partial x} \right)$$

$$+\delta^2 (\theta - 1) \left(\frac{3}{80} MaReh \left(h\frac{\partial^2 \theta}{\partial x^2} + 2\frac{\partial \theta}{\partial x}\frac{\partial h}{\partial x} \right) + \frac{3}{Peh}\zeta' \frac{\partial h}{\partial x} + \frac{2}{Peh}\left(\frac{\partial h}{\partial x}\right)^2 - \frac{3}{2Pe}\zeta'' - \frac{1}{Pe}\frac{\partial^2 h}{\partial x^2} \right)$$

$$+\delta (\theta - 1) \left(\frac{\delta_1}{h} \left(\frac{21}{40}\frac{\partial q}{\partial x} - \frac{21}{40}\frac{q}{h}\zeta' \right) - \frac{7}{40}\frac{\partial q}{\partial x} \right)$$

$$+\delta \frac{q}{20}\frac{\partial \theta}{\partial x}\left(21\frac{\delta_1}{h} - 27\right) - \frac{3}{Peh}(\theta - 1) - 3\frac{Bi}{Pe}\theta$$

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Benney equation

Conduct linear stability analysis using first-order Benney Equation [9]. Variables u, w, P, and T are expanded in a perturbation series:

$$u = u_0 + \delta u_1 + O(\delta^2)$$

Substituting these into the governing equations, the $O(1)$ and $O(\delta)$ problems are considered, and the boundary conditions are used to form a single equation for the evolution of the free surface, $h(x,t)$. Assume perturbations of the form $\hat{h} = h_0 e^{ik(x-ct)}$ where $h = h_s + \hat{h}$. This gives the following critical Reynolds number (for $h_s = 1$):

$$Re_{crit}^{Ben} = \frac{5}{6}\cot\beta \frac{1+3\delta_1}{1+6\delta_1+\frac{25}{2}\delta_1^2+\frac{5}{12}\frac{MaBi}{(1+Bi)^2}(1+2\delta_1)}$$

Linear stability using Model equations

A linear stability analysis is also conducted using the model equations.

The steady-state solutions for the even bottom case are:

$$h_s = 1 \quad , \quad \theta_s = \frac{1}{1+Bi} \quad , \quad q_s = 1 + 3\delta_1$$

Express variables as the steady-state value plus a perturbation:

$$h = h_s + \hat{h}, \theta = \theta_s + \hat{\theta}, q = q_s + \hat{q}$$

The equations are then linearized in the perturbation.

Linear stability using Model equations (cont'd)

The form of the perturbation is assumed:

$$\hat{h} = h_0 e^{ikx+\omega t},\ \hat{\theta} = \theta_0 e^{ikx+\omega t},\ \hat{q} = q_0 e^{ikx+\omega t}$$

Requiring that $real(\omega) = 0$ gives the neutral stability curve having critical Reynolds number:

$$Re_{crit}^{WRM} = \frac{5}{6}\cot\beta \frac{1+3\delta_1}{1+6\delta_1+\frac{92}{7}\delta_1^2+\frac{5}{12}\frac{MaBi(1+2\delta_1)}{(1+Bi)^2}}$$

This critical Reynolds number matches the Benney result to $O(\delta_1)$.

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Comparison of Re_{crit} for limiting cases

Limiting Case	Re_{crit}^{WRM}	Re_{crit}^{Ben} [9]	Re_{crit}^{Theor} [3,5,10,11]
Isothermal and Impermeable $Bi = Ma = \delta_1 = 0$	$\frac{5}{6}\cot\beta$	$\frac{5}{6}\cot\beta$	$\frac{5}{6}\cot\beta$
Impermeable $\delta_1 = 0$	$\frac{\frac{5}{6}\cot\beta}{1+\frac{5MaBi}{12(1+Bi)^2}}$	$\frac{\frac{5}{6}\cot\beta}{1+\frac{5MaBi}{12(1+Bi)^2}}$	$\frac{\frac{5}{6}\cot\beta}{1+\frac{5MaBi}{12(1+Bi)^2}}$
Isothermal $Bi = Ma = 0$	$\frac{5}{6}\cot\beta\left[\frac{1+3\delta_1}{1+6\delta_1+\frac{92}{7}\delta_1^2}\right]$	$\frac{5}{6}\cot\beta\left[\frac{1+3\delta_1}{1+6\delta_1+\frac{25}{2}\delta_1^2}\right]$	$\frac{5}{6}\cot\beta\left[\frac{1+3\delta_1}{1+6\delta_1+\frac{25}{2}\delta_1^2}\right]$

Comparison of Re_{crit} with heating and porosity

The critical Reynolds number from each method is compared to results from Sadiq et al. [7]

$$Re_{crit}^{Ben} = \frac{5}{6}\cot\beta \frac{1+3\delta_1}{1+6\delta_1+\frac{25}{2}\delta_1^2+\frac{5}{12}\frac{MaBi}{(1+Bi)^2}(1+2\delta_1)}$$

$$Re_{crit}^{WRM} = \frac{5}{6}\cot\beta \frac{1+3\delta_1}{1+6\delta_1+\frac{92}{7}\delta_1^2+\frac{5}{12}\frac{MaBi}{(1+Bi)^2}(1+2\delta_1)}$$

$$Re_{crit}^{Sad} = \frac{5}{6}\cot\beta \frac{1+3\delta_1}{1+6\delta_1+\frac{25}{2}\delta_1^2+\frac{15}{2}\delta_1^3+\frac{5}{12}\frac{MaBi}{(1+Bi)^2}(1+2\delta_1)}$$

Neutral stability for a wavy bottom

Neutral stability curves are calculated numerically using Floquet theory. Each variable equals the steady state value plus a perturbation:

$$h = h_s(x) + \hat{h}\,,\ q = 1 + \hat{q}\,,\ \theta = \theta_s(x) + \hat{\theta}$$

The model equations are linearized in the perturbations:

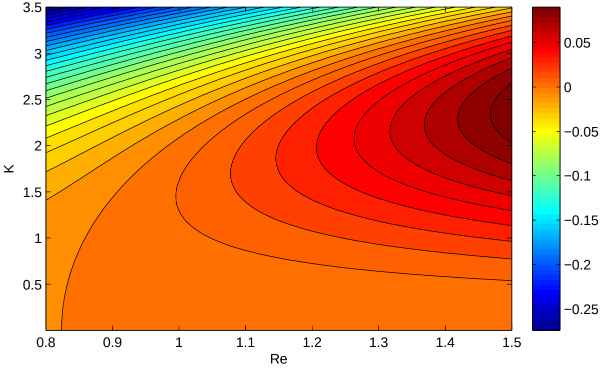
$$\frac{\partial \hat{h}}{\partial t} + \frac{\partial \hat{q}}{\partial x} = 0\ ,\quad \frac{\partial \hat{q}}{\partial t} + f_1 \frac{\partial^2 \hat{q}}{\partial x^2} + f_2 \frac{\partial \hat{q}}{\partial x} + ... = 0\ ,\quad \frac{\partial \hat{\theta}}{\partial t} + g_1 \frac{\partial^2 \hat{\theta}}{\partial x^2} + g_2 \frac{\partial \hat{\theta}}{\partial x} + ... = 0$$

The perturbations and coefficients are expanded in truncated Fourier series:

$$\begin{pmatrix} \hat{h} \\ \hat{q} \\ \hat{\theta} \end{pmatrix} = e^{\omega t} e^{iKx} \sum_{n=-N}^{N} \begin{pmatrix} \hat{h}_n \\ \hat{q}_n \\ \hat{\theta}_n \end{pmatrix} e^{i2\pi nx}\ ,\quad \begin{pmatrix} f_j \\ g_j \end{pmatrix} = \sum_{n=-N}^{N} \begin{pmatrix} f_{j,n} \\ g_{j,n} \end{pmatrix} e^{i2\pi nx}$$

This leads to a generalized eigenvalue problem given by: $\mathbf{A}\vec{V} = \omega \mathbf{B}\vec{V}$

Growth rate of flow over a wavy bottom



$\delta = 0.05$, $\cot\beta = 1$, $Bi = Ma = 0$, $We = 50$, $\delta_1 = 0.2$, $a_b = 0.3$

Neutral stability curves for a wavy bottom

Effect of surface tension:

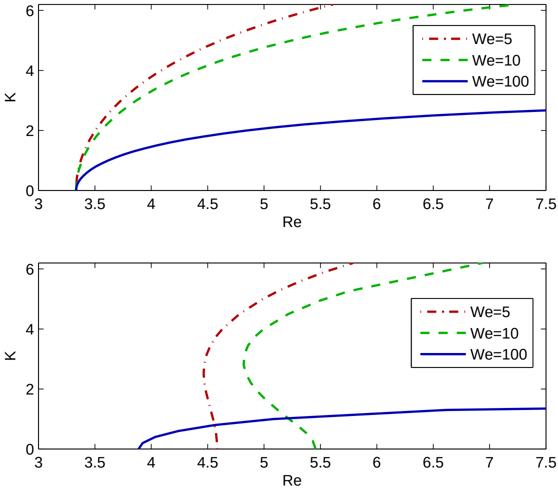
$\delta = 0.05$, $\delta_1 = 0$

$\cot\beta = 4$

$Bi = Ma = 0$

$a_b = 0$ (top)

$a_b = 0.4$ (bottom)



Neutral stability curves for a wavy bottom

Effect of bottom permeability:

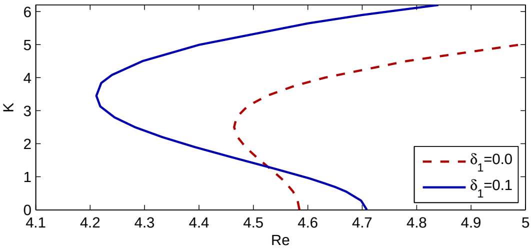
$\delta = 0.05$

$We = 5$

$a_b = 0.4$

$\cot\beta = 4$

$Bi = Ma = 0$



Neutral stability distribution for a wavy bottom

Effect of bottom permeability reverses for sufficiently large bottom amplitude and surface tension.

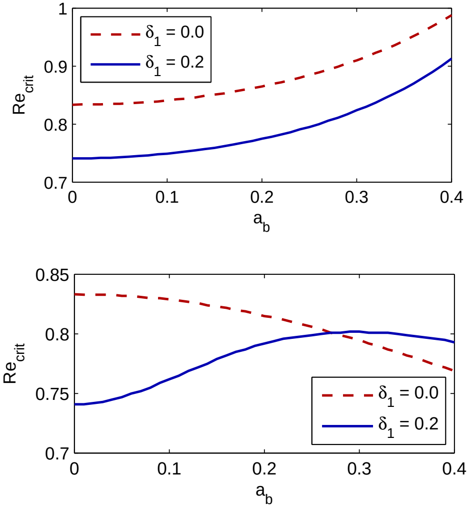
$\delta = 0.05$

$\cot\beta = 1$

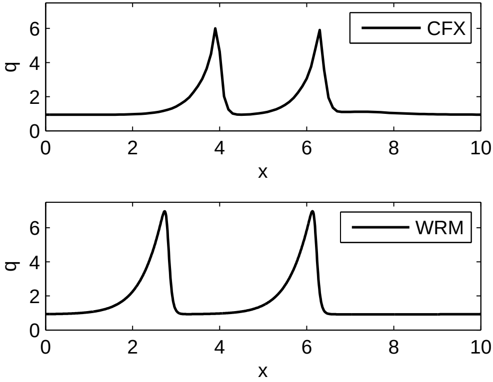
$Bi = Ma = 0$

$We = 50$ (top)

$We = 400$ (bottom)



Comparison between CFX and Model



Conclusions

- The weighted residual model has been extended to include the Marangoni effect, bottom permeability and bottom waviness.
- Both permeability and Marangoni effects destabilize flow over an even bottom, and the combined effect is to further destabilize the flow; the model equations accurately predict these effects.
- Weak surface tension and bottom topography stabilizes the flow, while strong surface tension and bottom topography have the opposite effect.
- With strong surface tension and large bottom amplitude, permeability can have a stabilizing effect.
- The model equations have been be solved numerically to predict the development of the free-surface; an unstable case with bottom permeability was shown to closely match the results of the full Navier-Stokes equations obtained using CFX.

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