The dynamics of the globe fountain

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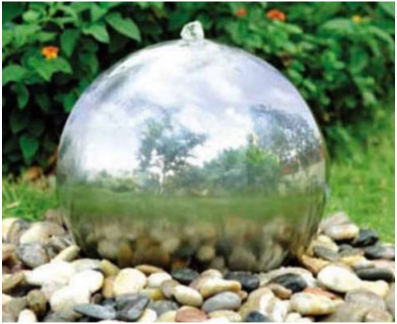
The steady flow of a thin fluid layer over a sphere occurs naturally in a globe fountain as shown. A globe fountain consists of a sphere with a hole at the top whereby water is pumped out at a constant rate. An analytical study of the flow has been carried out in this study.



The kugel fountain shown here involves a massive granite sphere which floats and spins on a thin film of flowing water which is pumped out of a hole at the base of the fountain.

- Takagi & Huppert (JFM, 2010) studied a constant volume of fluid released at the top of a sphere using lubrication theory. Their analytical results agreed well with their experiments. They also investigated the onset of instability of the advancing front as it split into a series of rivulets.
- The dynamics of the kugel fountain was recently analysed by Snoeijer & van der Weele (Am. J. Phys., 2014), again using lubrication theory.

Formulated in spherical coordinates (r, θ, ϕ) with the hole oriented about the polar axis $\theta = 0$ and assuming azimuthal symmetry, the governing steady-state Navier-Stokes equations become:

$$\frac{\partial}{\partial r}\left(r^2 v \sin\theta\right) + \frac{\partial}{\partial \theta}\left(r u \sin\theta\right) = 0 \; ,$$

$$v\frac{\partial v}{\partial r} + \frac{u}{r}\frac{\partial v}{\partial \theta} - \frac{u^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} - g\cos\theta$$

$$+\frac{\nu}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v}{\partial \theta}\right) - \frac{2}{\sin\theta}\frac{\partial}{\partial \theta}\left(u\sin\theta\right) - 2v\right] \; ,$$

$$v\frac{\partial u}{\partial r} + \frac{u}{r}\frac{\partial u}{\partial \theta} + \frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial P}{\partial \theta} + g\sin\theta$$

$$+\frac{\nu}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + 2\frac{\partial v}{\partial \theta} - \frac{u}{\sin^2\theta}\right] \; .$$

Here, u, v denote the velocity components in the θ and radial directions, respectively, P refers to the pressure, $\nu = \mu/\rho$ represents the kinematic viscosity while μ is the dynamic viscosity and ρ is the fluid density whereas g is the acceleration due to gravity. The thickness of the fluid layer is scaled by the Nusselt thickness, H, which for a vertical incline is given by

$$H^3 = \frac{3\nu Q}{g} \ ,$$

where Q is the constant flow rate exiting the small hole. For a fluid layer having a width of unity $Q = UH$ where U is the velocity scale.

The coordinate y is introduced which is related to r through the relation $r = R + y$ with R referring to the radius of the sphere. This can be scaled as

$$\frac{r}{R} = 1 + \delta \left(\frac{y}{H} \right) \,,$$

where the dimensionless parameter $\delta = H/R \ll 1$ denotes the shallowness parameter. With this scaling the dimensionless flow variables and coordinate y are given by

$$(u, v, P, y) \rightarrow (U\tilde{u}, \delta U\tilde{v}, \rho U^2 \tilde{P}, H\tilde{y}) \,,$$

where the tilde denotes a dimensionless quantity.

In dimensionless form and suppressing the tildes the governing equations become:

$$\frac{\partial}{\partial y}\left[(1+\delta y)^2 v\sin\theta\right]+\frac{\partial}{\partial\theta}\left[(1+\delta y)u\sin\theta\right]=0\ ,$$

$$\delta^2 v\frac{\partial v}{\partial y}+\frac{\delta^2 u}{(1+\delta y)}\frac{\partial v}{\partial\theta}-\frac{\delta u^2}{(1+\delta y)}=-\frac{\partial P}{\partial y}-\frac{3\cos\theta}{Re}+\frac{\delta}{Re(1+\delta y)^2}\times$$

$$\left[\frac{\partial}{\partial y}\left((1+\delta y)^2\frac{\partial v}{\partial y}\right)+\frac{\delta^2}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)-\frac{2\delta}{\sin\theta}\frac{\partial}{\partial\theta}(u\sin\theta)-2\delta^2 v\right]\ ,$$

$$\delta v\frac{\partial u}{\partial y}+\frac{\delta u}{(1+\delta y)}\frac{\partial u}{\partial\theta}+\frac{\delta^2 uv}{(1+\delta y)}=-\frac{\delta}{(1+\delta y)}\frac{\partial P}{\partial\theta}+\frac{3\sin\theta}{Re}+\frac{1}{Re(1+\delta y)^2}\times$$

$$\left[\frac{\partial}{\partial y}\left((1+\delta y)^2\frac{\partial u}{\partial y}\right)+\frac{\delta^2}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+2\delta^3\frac{\partial v}{\partial\theta}-\frac{\delta^2 u}{\sin^2\theta}\right]\ ,$$

where $Re=Q/\nu$ denotes the Reynolds number.

Along the air-fluid interface dynamic conditions along the steady free surface $y = \eta(\theta)$ are applied:

$$P - P_a = \delta We \left[\frac{3\delta^2(\eta')^2 + 2(1+\delta\eta)^2 - \delta(1+\delta\eta)\eta''}{[(1+\delta\eta)^2 + \delta^2(\eta')^2]^{3/2}} - \frac{\delta\eta' \cot\theta}{(1+\delta\eta)\sqrt{(1+\delta\eta)^2 + \delta^2(\eta')^2}} \right] + \frac{2\delta}{Re[(1+\delta\eta)^2 + \delta^2(\eta')^2]} \left[(1+\delta\eta)^2 \frac{\partial v}{\partial y} + \frac{\delta^2(\eta')^2}{(1+\delta\eta)} \left(\delta v + \frac{\partial u}{\partial \theta} \right) - (1+\delta\eta)\eta' \left(\frac{\partial u}{\partial y} + \frac{\delta}{(1+\delta\eta)} \left(\delta \frac{\partial v}{\partial \theta} - u \right) \right) \right] ,$$

$$[(1+\delta\eta)^2 - \delta^2(\eta')^2] \left(\frac{\partial u}{\partial y} + \frac{\delta}{(1+\delta\eta)} \left(\delta \frac{\partial v}{\partial \theta} - u \right) \right) + 2\delta(1+\delta\eta)\eta' \left(\frac{\partial v}{\partial y} - \frac{\delta}{(1+\delta\eta)} \left(\frac{\partial u}{\partial \theta} + \delta v \right) \right) = 0 \, .$$

Here, $We = \sigma H/(\rho Q^2)$ is the Weber number with σ denoting surface tension, P_a the constant ambient air pressure, $\eta = h/H$ with h being the thickness of the fluid layer, and the prime denotes differentiation with respect to θ. The kinematic condition along the free surface $y = \eta(\theta)$ is given by

$$v = \frac{u}{(1+\delta\eta)}\eta' \ ,$$

and the no-slip and impermeability conditions on the surface of the sphere are

$$u = v = 0 \text{ at } y = 0 \ .$$

For small δ an approximate analytical solution can be constructed by expanding the flow variables in the following series

$$u(y,\theta) = u_0(y,\theta) + \delta u_1(y,\theta) + \cdots \ ,$$

$$v(y,\theta) = v_0(y,\theta) + \delta v_1(y,\theta) + \cdots \ ,$$

$$P(y,\theta) = P_0(y,\theta) + \delta P_1(y,\theta) + \cdots \ ,$$

$$\eta(\theta) = \eta_0(\theta) + \delta \eta_1(\theta) + \cdots \ .$$

Substituting these expansions into the equations of motion and expanding the dynamic conditions in powers of δ leads to a hierarchy of problems.

The leading-order problem is governed by the system

$$\frac{\partial P_0}{\partial y} = -\frac{3\cos\theta}{Re} ,$$

$$\frac{\partial^2 u_0}{\partial y^2} = -3\sin\theta ,$$

$$\sin\theta\left(\frac{\partial v_0}{\partial y} + \frac{\partial u_0}{\partial \theta}\right) + u_0\cos\theta = 0 ,$$

subject to

$$P_0 = P_a , \ \frac{\partial u_0}{\partial y} = 0 \ \text{ at } \ y = \eta_0 ,$$

$$u_0 = v_0 = 0 \ \text{ at } \ y = 0 .$$

The solutions are easily found to be

$$P_0(y,\theta) = P_a + \frac{3\cos\theta}{Re}(\eta_0 - y) \ ,$$

$$u_0(y,\theta) = \frac{3}{2} y \sin\theta (2\eta_0 - y) \ ,$$

$$v_0(y,\theta) = -\frac{y^2}{2}(6\eta_0 \cos\theta + 3\eta_0' \sin\theta - 2y\cos\theta) \ .$$

Thus, to leading order the pressure is hydrostatic and the velocity in the θ direction, u_0, has a parabolic profile in y which is consistent with flow down an inclined surface at an angle of θ with the horizontal. Further, the velocity component u_0 is symmetric about the plane $\theta = \pi/2$.

The leading-order term for the unknown free surface can be determined by applying the kinematic condition which when transferred from $y = \eta$ to $y = \eta_0$ takes the form

$$v_0 = u_0 \eta_0' \quad \text{at} \quad y = \eta_0 \; .$$

This leads to the differential equation and solution given by

$$\eta_0' = -\frac{2\eta_0 \cos\theta}{3\sin\theta} \;\Rightarrow\; \eta_0(\theta) = \frac{C}{\sin^{2/3}\theta} \; .$$

The constant C is found by applying $\eta_0 = h_0$ at $\theta = \theta_0$ where h_0 and θ_0 are free dimensionless parameters. Thus, $C = h_0 \sin^{2/3}\theta_0$ and the singularity at $\theta = 0$ is removed. Since we expect the flow to separate from the surface before reaching the bottom of the sphere, the singularity at $\theta = \pi$ is also resolved.

The $O(\delta)$ problem satisfies the system

$$\frac{\partial^2 u_1}{\partial y^2} = Re\frac{\partial}{\partial \theta}\left(P_0 + \frac{u_0^2}{2}\right) + (Rev_0 - 2)\frac{\partial u_0}{\partial y} \ ,$$

$$\sin\theta \left(\frac{\partial v_1}{\partial y} + \frac{\partial u_1}{\partial \theta}\right) + u_1 \cos\theta =$$

$$-\sin\theta \left(2y\frac{\partial v_0}{\partial y} + y\frac{\partial u_0}{\partial \theta} + 2v_0\right) - yu_0 \cos\theta \ ,$$

subject to

$$\frac{\partial u_1}{\partial y} + \eta_1 \frac{\partial^2 u_0}{\partial y^2} = u_0 - 2\eta_0 \frac{\partial u_0}{\partial y} - 2\eta_0' \frac{\partial v_0}{\partial y} \quad \text{at} \quad y = \eta_0 \ ,$$

$$u_1 = v_1 = 0 \quad \text{at} \quad y = 0 \ .$$

The kinematic condition applied at $y = \eta_0$ yields the differential equation for the free surface correction, η_1, given by

$$u_0\eta_1' + \left(\eta_0' \frac{\partial u_0}{\partial y} - \frac{\partial v_0}{\partial y}\right)\eta_1 = v_1 + \eta_0\eta_0' u_0 - \eta_0' u_1 \ ,$$

subject to

$$\eta_1 = 0 \ \text{ at } \ \theta = \theta_0 \ .$$

The solution to the $O(\delta)$ problem is significantly more complicated and was obtained using the Maple computer algebra system. For example, the solution for u_1 is:

$$u_1(y,\theta) = \frac{y}{40}\Big(-Rey^5\sin\theta\cos\theta + 6CRey^4\sin^{1/3}\theta\cos\theta$$

$$+15CRe\eta_0' y^3\sin^{4/3}\theta + 60y^2\sin\theta + 60y[\eta_0'\cos\theta - 3C\sin^{1/3}\theta] + F(\theta)\Big)$$

where

$$F(\theta) = 120\eta_1\sin\theta + 240C^2\sin^{-4/3}\theta[\eta_0'\cos\theta + \sin\theta] - 60C^4Re\eta_0'\sin^{-2/3}\theta$$

$$+240C(\eta_0')^2\sin^{1/3}\theta - 120C\eta_0'\sin^{-2/3}\theta\cos\theta - 24C^5Re\sin^{-7/3}\theta\cos\theta\ .$$

The solution for η_1 can be expressed in the form

$$\eta_1(\theta) = -\frac{512C^2}{945\sin^{2/3}\theta}\int_{\theta_0}^{\theta} f(\alpha)d\alpha\ ,$$

and can be integrated numerically.

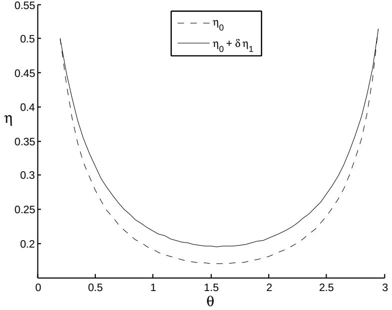
The function $f(\alpha)$ is defined as:

$$f(\alpha) = \sin^{23/3}\alpha \left(-180C^3 Re \sin^{2/3}\alpha \cos 2\alpha - 396C^3 Re \sin^{2/3}\alpha\right.$$

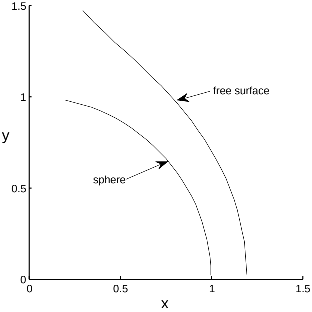
$$\left.+280C\cos 3\alpha + 2520C\cos\alpha + 385\sin^{2/3}\alpha\cos 3\alpha - 2625\sin^{2/3}\alpha\cos\alpha\right) /$$

$$(462+\cos 12\alpha - 12\cos 10\alpha + 66\cos 8\alpha - 220\cos 6\alpha + 495\cos 4\alpha - 792\cos 2\alpha)\ .$$

The leading-order, η_0, and first-order correction, η_1, to the free surface with $Re = 1$, $\delta = 0.1$, $\theta_0 = 0.2$ and $h_0 = 0.5$.



The free surface $\eta = \eta_0 + \delta\eta_1$ shown in Cartesian coordinates with $Re = 1$, $\delta = 0.1$, $\theta_0 = 0.2$ and $h_0 = 0.5$.

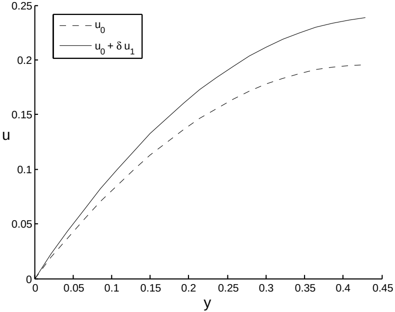


The dimensionless flow rate per unit width, $\hat{Q}$, and dimensionless average streamwise velocity, $\hat{U}$, can be computed using the leading-order solution as follows

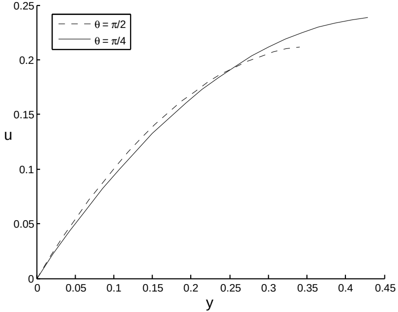
$$\hat{Q} = \int_0^{\eta_0} u_0(y,\theta)dy = \eta_0^3 \sin\theta \ , \ \hat{U} = \frac{\hat{Q}}{\eta_0} = \eta_0^2 \sin\theta \ .$$

Substituting $\eta_0 = C/\sin^{2/3}\theta$ yields $\hat{Q} = C^3/\sin\theta$ and $\hat{U} = C^2/\sin^{1/3}\theta$. Both the flow rate and average speed decrease as θ increases from θ_0 to $\pi/2$ which leads to a decrease in fluid thickness. Note that $\hat{Q}\sin\theta$ is constant!

The leading-order, u_0, and first-order correction, u_1, to the velocity profile with $Re = 1$, $\delta = 0.1$, $\theta_0 = 0.2$ and $h_0 = 1$ at $\theta = \pi/4$.



The velocity profile $u = u_0 + \delta u_1$ using $Re = 1$, $\delta = 0.1$, $\theta_0 = 0.2$ and $h_0 = 1$ at $\theta = \pi/4$ and $\theta = \pi/2$.



The point of separation can be estimated by using the zero-stress condition:

$$\frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = 0 \, .$$

Using $u = u_0$ leads to $3\eta_0(\theta)\sin\theta = 0$ and yields the separation angle $\theta_s = \pi$. Including the first-order correction $u = u_0 + \delta u_1$ requires solving the equation

$$3\eta_0(\theta)\sin\theta + \frac{\delta}{40}F(\theta) = 0 \, ,$$

to determine θ_s. Here, $F(\theta)$ was previously defined. As expected, θ_s occurs near the bottom of the sphere with little dependence on Re and δ. When $h_0 = 1$ and $\theta_0 = 0.2$ separation occurs at $\theta_s \approx 3.045$ while when $h_0 = 0.5$ and $\theta_0 = 0.3$ separation occurs at $\theta_s \approx 3.085$.

The calculation was repeated for the problem of thin flow over a cylinder and very similar streamwise velocity profiles were obtained. However, the free surface was found to vary more rapidly with θ for the sphere than it does for the cylinder. Expressions for $\eta_0(\theta)$ and $\eta_1(\theta)$ for the cylinder are given by

$$\eta_0(\theta) = \frac{D}{\sin^{1/3}\theta} \quad \text{where} \quad D = h_0 \sin^{1/3}\theta_0 \ ,$$

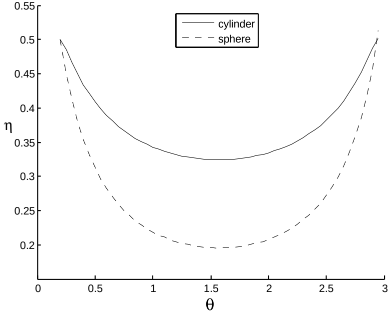
$$\eta_1(\theta) = -\frac{D^2}{7560 \sin^{1/3}\theta} \int_{\theta_0}^{\theta} \frac{1}{\sin^{11/3}\alpha} \Big(245 \sin^{1/3}\alpha \cos 3\alpha$$
$$-2205 \sin^{1/3}\alpha \cos\alpha + 70 D \cos 3\alpha + 1050 D \cos\alpha$$
$$+72 D^3 Re \sin^{4/3}\alpha \cos 2\alpha + 504 D^3 Re \sin^{4/3}\alpha \Big) \, d\alpha \ .$$

For the cylinder the dimensionless flow rate per unit width, $\hat{Q}$, and dimensionless average streamwise velocity, $\hat{U}$, are given by

$$\hat{Q} = D^3 \quad \text{and} \quad \hat{U} = \hat{Q}/\eta_0 = D^2 \sin^{1/3}\theta \ .$$

Here, the average streamwise velocity increases as θ increases from θ_0 to $\pi/2$ and since the flow rate remains constant the fluid thickness must decrease accordingly.

The free surface $\eta = \eta_0 + \delta\eta_1$ for the cylinder and sphere with $Re = 1$, $\delta = 0.1$, $\theta_0 = 0.2$ and $h_0 = 0.5$.



- Discussed in this work was an analytical investigation of the steady flow of a thin fluid layer over a sphere resulting from a constant discharge from a small hole at the top of the sphere. As an extension the cylindrical case was also solved.
- The variation in fluid layer thickness is more pronounced for the sphere than it is for the cylinder. This is because the average streamwise velocity increases as the fluid flows over the cylinder and since the flow rate is constant the thickness must therefore decrease. For the sphere both the flow rate and the average streamwise velocity decrease as the fluid flows over the surface which leads to a more rapid decrease in fluid layer thickness.
- The technique and approach adopted here can be used to model other thin flows that occur in similar settings.