Bénard Convection with Rotation and a Periodic Temperature Distribution

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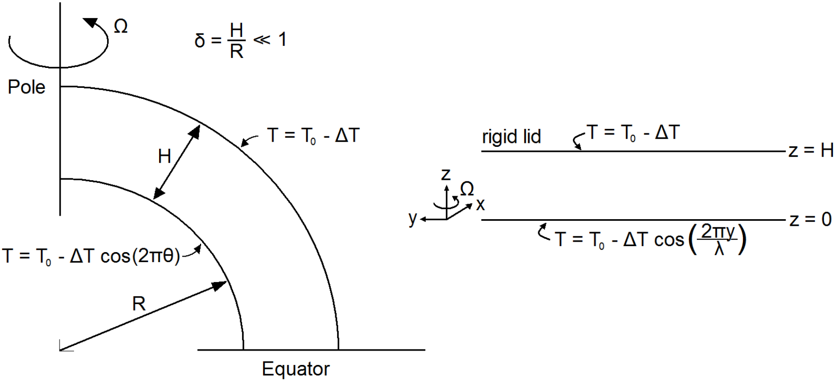
AFM 2012, June 26-28

Problem description



Previous work

- Pascal and D'Alessio [1] studied the stability of a flow with rotation and a quadratic density variation.
- Schmitz and Zimmerman [2] studied the effects of a spatially varying temperature boundary condition as well as wavy boundaries, without rotation and assuming a very large Prandtl number.
- Malashetty and Swamy [3] considered the effects of a temperature boundary condition varying in time.
- Basak et al. [4], studied the flow resulting from a spatially varying temperature boundary condition in a square cavity without rotation.

The current work takes advantage of the thinness of the fluid layer, and investigates flow in a rectangular domain with rotation and a varying bottom temperature.

Governing equations and boundary conditions

Governing equations:

$$\vec{\nabla} \cdot \vec{v} = 0$$

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla} \right) \vec{v} + f\hat{k} \times \vec{v} = \frac{-1}{\rho_0} \vec{\nabla} p - \frac{\rho}{\rho_0} g\hat{k} + \nu \nabla^2 \vec{v}$$

$$\frac{\partial T}{\partial t} + \left(\vec{v} \cdot \vec{\nabla} \right) T = \kappa \nabla^2 T$$

Boundary conditions:

$$u = v = w = 0 \ \text{ at } \ z = 0, H \ \text{ and } \ y = 0, \lambda$$

$$T = T_0 - \Delta T \ \text{ at } \ z = H \ \text{ and } \ y = 0, \lambda$$

$$T = T_0 - \Delta T \cos\left(2\pi \frac{y}{\lambda}\right) \ \text{ at } \ z = 0$$

Other conditions

Initial conditions:

$$u = v = w = 0 \ , \quad T = T_0 - \frac{\Delta T}{H} z - \Delta T \cos\left(2\pi \frac{y}{\lambda}\right)\left(1 - \frac{z}{H}\right) \quad \text{at} \quad t = 0$$

Density is assumed to vary according to:

$$\rho = (1 - \alpha\,[T - T_0])$$

The flow is assumed to be uniform in the x-direction. This allows a stream function and vorticity to be defined as:

$$v = \frac{\partial \psi}{\partial z} \ , \quad w = -\frac{\partial \psi}{\partial y} \ , \quad \zeta = -\left(\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right)$$

Other conditions

Integral constraints are imposed on the vorticity and can derived using Green's Second Identity:

$$\int_V \left(\phi\nabla^2\chi - \chi\nabla^2\phi\right) dV = \int_S \left(\phi\frac{\partial\chi}{\partial n} - \chi\frac{\partial\phi}{\partial n}\right) dS$$

Here ϕ and χ denote arbitrary differentiable functions, $\frac{\partial}{\partial n}$ is the normal derivative, and S is the surface enclosing the volume V. Choosing ϕ to satisfy $\nabla^2\phi = 0$ and letting $\chi \equiv \psi$, then $\nabla^2\chi = \nabla^2\psi = -\zeta$. Applying the boundary conditions $\frac{\partial\psi}{\partial n} = \psi = 0$ on S, the above leads to

$$\int_0^H \int_0^\lambda \phi_n \zeta \, dydz = 0$$

where

$$\phi_n(y,z) = e^{\pm 2n\pi z} \left\{ \begin{array}{c} \sin(2n\pi y) \\ \cos(2n\pi y) \end{array} \right\} \quad \text{for} \quad n = 1, 2, 3, \cdots$$

Scaling and dimensionless parameters

$$t \to \frac{H^2}{\kappa} t\ ,\ y \to \lambda y\ ,\ z \to Hz\ ,\ \psi \to \kappa\psi\ ,\ \zeta \to \frac{\kappa}{H^2}\zeta$$

$$T \to (T_0 - \Delta T) + \Delta T T\ ,\ u \to \frac{\kappa}{H} u$$

$$Ra = \frac{\alpha g H^3 \Delta T}{\nu\kappa}$$ Rayleigh number

$$Ro = \frac{\kappa}{Hf\lambda}$$ Rossby number

$$Pr = \frac{\nu}{\kappa}$$ Prandtl number

$$\delta = \frac{H}{\lambda}$$ Aspect ratio

Dimensionless equations

$$\frac{\partial \zeta}{\partial t} - \delta \frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial z}\frac{\partial^2 \psi}{\partial y \partial z} - \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial z^2}\right) + \delta^3 \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial z}\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial y \partial z}\right) - \frac{\delta}{Ro}\frac{\partial u}{\partial z}$$

$$= \delta PrRa \frac{\partial T}{\partial y} + Pr\left(\delta^2 \frac{\partial^2 \zeta}{\partial y^2} + \frac{\partial^2 \zeta}{\partial z^2}\right)$$

$$\zeta = -\left(\delta^2 \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right)$$

$$\frac{\partial u}{\partial t} + \delta\left(\frac{\partial \psi}{\partial z}\frac{\partial u}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial u}{\partial z}\right) - \frac{\delta}{Ro}\frac{\partial \psi}{\partial z} = Pr\left(\delta^2 \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial T}{\partial t} + \delta\left(\frac{\partial \psi}{\partial z}\frac{\partial T}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial T}{\partial z}\right) = \delta^2 \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

Approximate analytical solution

For small δ, an approximate analytical solution can be constructed:

$$\psi = \psi_0 + \delta\psi_1 + \cdots \quad , \quad \zeta = \zeta_0 + \delta\zeta_1 + \cdots$$

$$u = u_0 + \delta u_1 + \cdots \quad , \quad T = T_0 + \delta T_1 + \cdots$$

The leading-order problem becomes:

$$\frac{\partial T_0}{\partial t} = \frac{\partial^2 T_0}{\partial z^2} \quad , \quad \frac{\partial \zeta_0}{\partial t} = Pr\frac{\partial^2 \zeta_0}{\partial z^2} + Pr\delta Ra\frac{\partial T_0}{\partial y}$$

$$\frac{\partial^2 \psi_0}{\partial z^2} = -\zeta_0 \quad , \quad \frac{\partial u_0}{\partial t} = Pr\frac{\partial^2 u_0}{\partial z^2}$$

Approximate analytical solution

Applying the boundary, initial and integral conditions yields:

$$\zeta_0(y,z,t) = -\pi\delta Ra\left(z^2 - \frac{z^3}{3} - \frac{7z}{10} + \frac{1}{10}\right)\sin(2\pi y)$$

$$+\sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 Prt}\cos(n\pi z)$$

$$\psi_0(y,z,t) = \pi\delta Ra\left(\frac{z^4}{12} - \frac{z^5}{60} - \frac{7z^3}{60} + \frac{z^2}{20}\right)\sin(2\pi y)$$

$$-\sum_{n=1}^{\infty} \frac{a_n}{n^2\pi^2} e^{-n^2\pi^2 Prt}(1-\cos(n\pi z))$$

where

$$a_n = 2\pi\delta Ra\sin(2\pi y)\int_0^1 \left(z^2 - \frac{z^3}{3} - \frac{7z}{10} + \frac{1}{10}\right)\cos(n\pi z)dz\ ,\quad n = 1, 2, \cdots$$

Steady-state solutions

As $t \to \infty$, the following steady-state solutions emerge:

$$T_s = (1 - z)\,(1 - \cos\,(2\pi y))$$

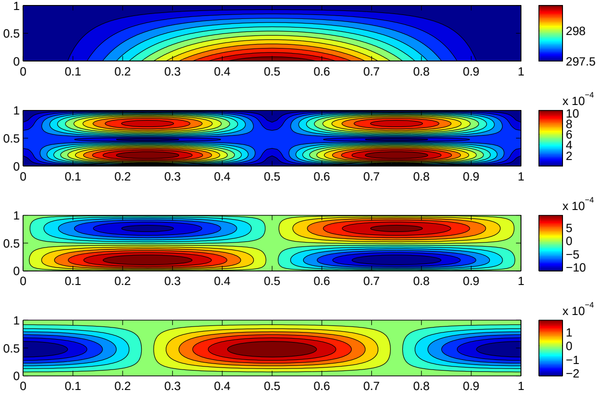
$$\zeta_s = -\pi\delta Ra\left(z^2 - \frac{z^3}{3} - \frac{7z}{10} + \frac{1}{10}\right)\sin(2\pi y)$$

$$\psi_s = \pi\delta Ra\left(\frac{z^4}{12} - \frac{z^5}{60} - \frac{7z^3}{60} + \frac{z^2}{20}\right)\sin(2\pi y)$$

$$u_s = 0$$

Plotted on the next slide are the leading-order temperature and velocities ($Ro = 0.0548$, $Pr = 0.7046$ and $Ra = 388.7$).

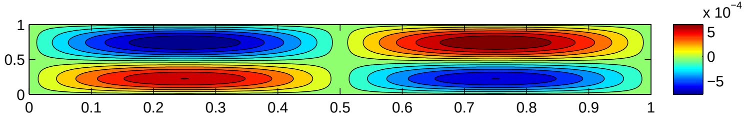
Steady-state solutions



Steady-state solutions

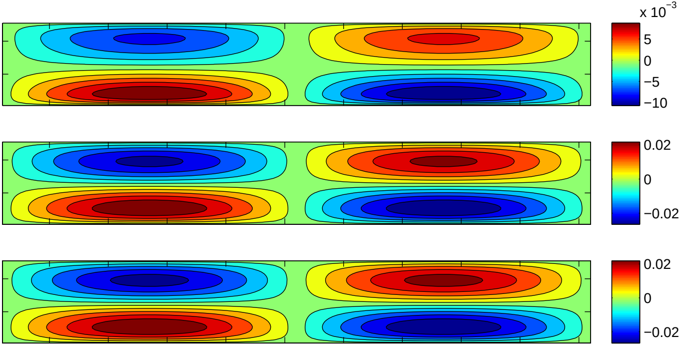
At leading order $u_s = 0$, the $O(\delta)$ solution for u_s is:

$$u_s = \frac{1}{720}\frac{\pi\delta Ra}{Ro Pr}\sin(2\pi y)z(z-1)\left(2z^4 - 10z^3 + 11z^2 - z - 1\right)$$



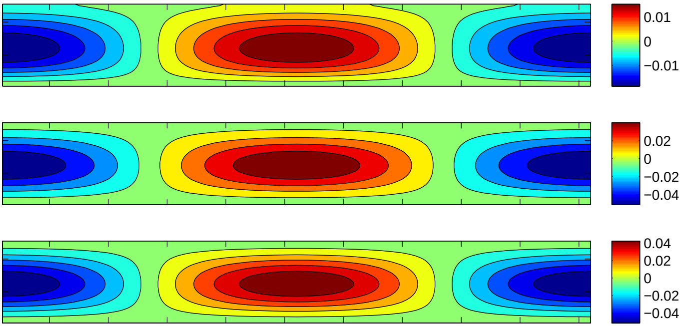
Transient development

v- velocity development with time for $Ra = 1$, $Pr = 1$ at times $t = 0.01, 0.1, 1$ from top to bottom.



Transient development

w- velocity development with time for $Ra = 1$, $Pr = 1$ at times $t = 0.01, 0.1, 1$ from top to bottom.

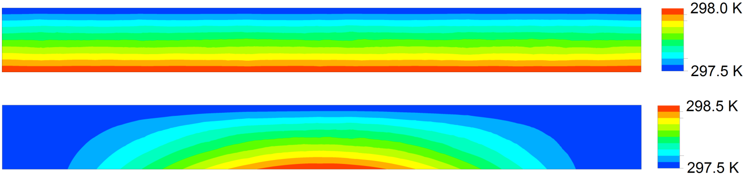


Steady-state numerical solutions

The steady-state solution was also be determined numerically using the commercial software package CFX. All results shown use air at $298K$ with a domain having a length of 20 cm and height of 2 cm with periodicity imposed at the ends. For cases with rotation, the angular velocity is $0.05s^{-1}$. The non-dimensional values are $Ro = 0.0548$ and $Pr = 0.7046$. The Rayleigh number will depend on ΔT.

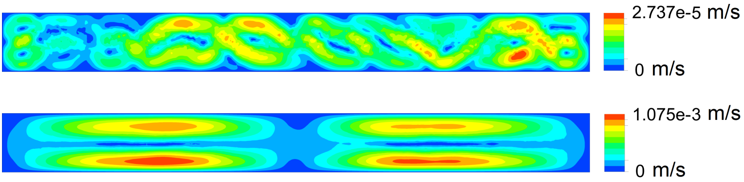
Steady-state numerical solutions

The temperature distribution for a case without rotation or modulated bottom heating (top) is compared to a case with these effects ($\Delta T = 0.5K$, $Ra = 388.7$).



Steady-state numerical solutions

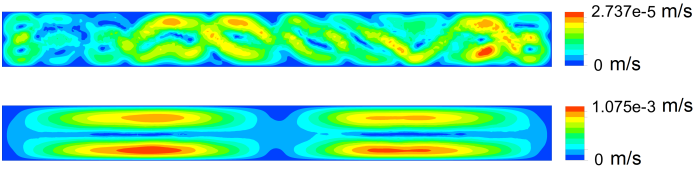
The velocity for a case without rotation or modulated bottom heating (top) is compared to a case with these effects ($\Delta T = 0.5K$, $Ra = 388.7$).



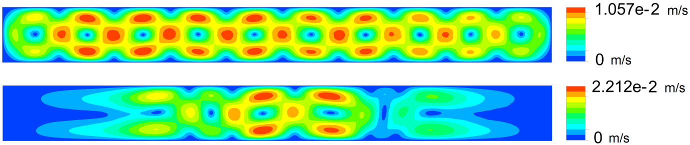
Note that the pattern and magnitude of the temperature and velocity agree well with the approximate analytical solutions.

Unstable numerical solutions

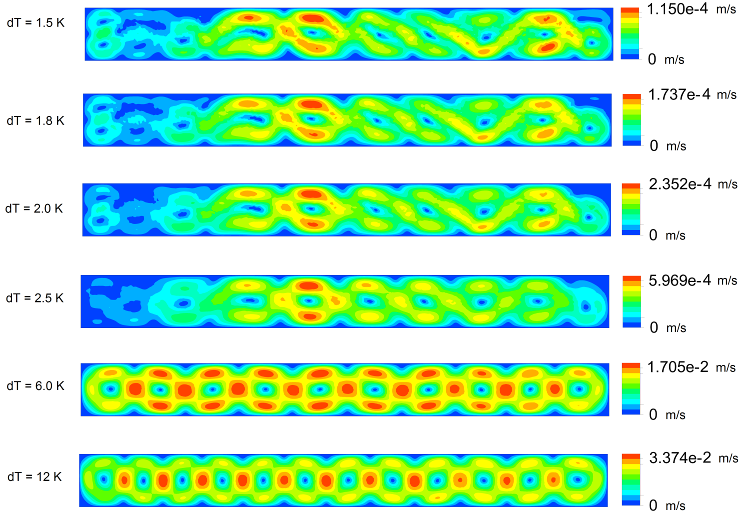
$\Delta T = 0.5K$, $Ra = 388.7$



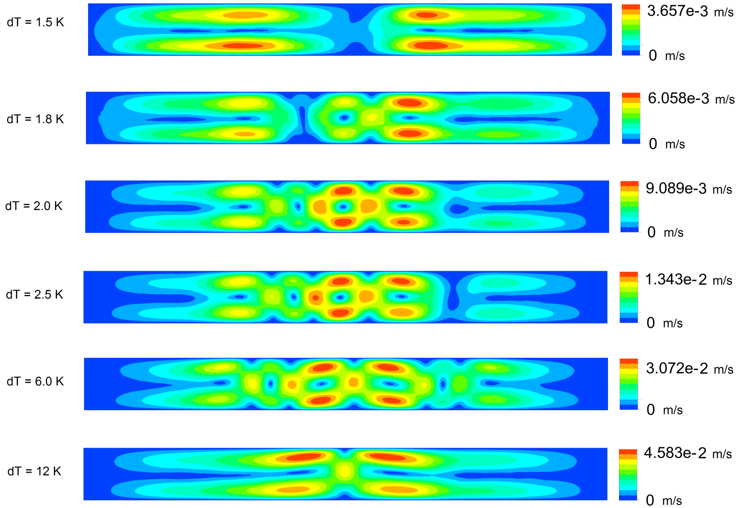
$\Delta T = 4K$, $Ra = 3109$



Unstable numerical solutions



Unstable numerical solutions



Summary

Conclusions

- Contrary to the Bénard problem, a non-zero background flow was found.
- The approximate analytical solution agrees well with the fully numerical solution.
- While rotation is known to stabilize the flow, a variable bottom temperature also influences the stability of the flow.
- Interesting features emerging from the unstable numerical simulations were observed.

References

[1] Pascal, J.P. & D'Alessio, S.J.D., The effects of density extremum and rotation on the onset of thermal instability, *International Journal of Numerical Methods for Heat & Fluid Flow* **13**, pp. 266 - 285, 2003.

[2] Schmitz, R & Zimmerman, W., Spatially periodic modulated Rayleigh-Bénard convection, *Physical Review E* **53**, pp. 5993 - 6011, 1996.

[3] Malashetty, M.S., & Swamy, M., Effect of thermal modulation on the onset of convection in a rotating fluid layer, *International Journal of Heat and Mass Transfer* **51**, pp. 2814 - 2823, 2008.

[4] Basak, T., Roy, S. & Balakrishnan, A.R., Effects of thermal boundary conditions on natural convection flows within a square cavity, *International Journal of Heat and Mass Transfer* **49**, pp. 4525 - 4535, 2006.