ADVANCES IN FLUID MECHANICS - 2006

Rear Shock Formation in Gravity Currents

Serge D'Alessio (sdalessio@uwaterloo.ca)
Department of Applied Mathematics
University of Waterloo
Waterloo, Ontario, N2L 3G1, CANADA

Coauthors: T.B. Moodie (University of Alberta, Edmonton, CANADA)
J.P. Pascal (Ryerson University, Toronto, CANADA)

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Talk Outline

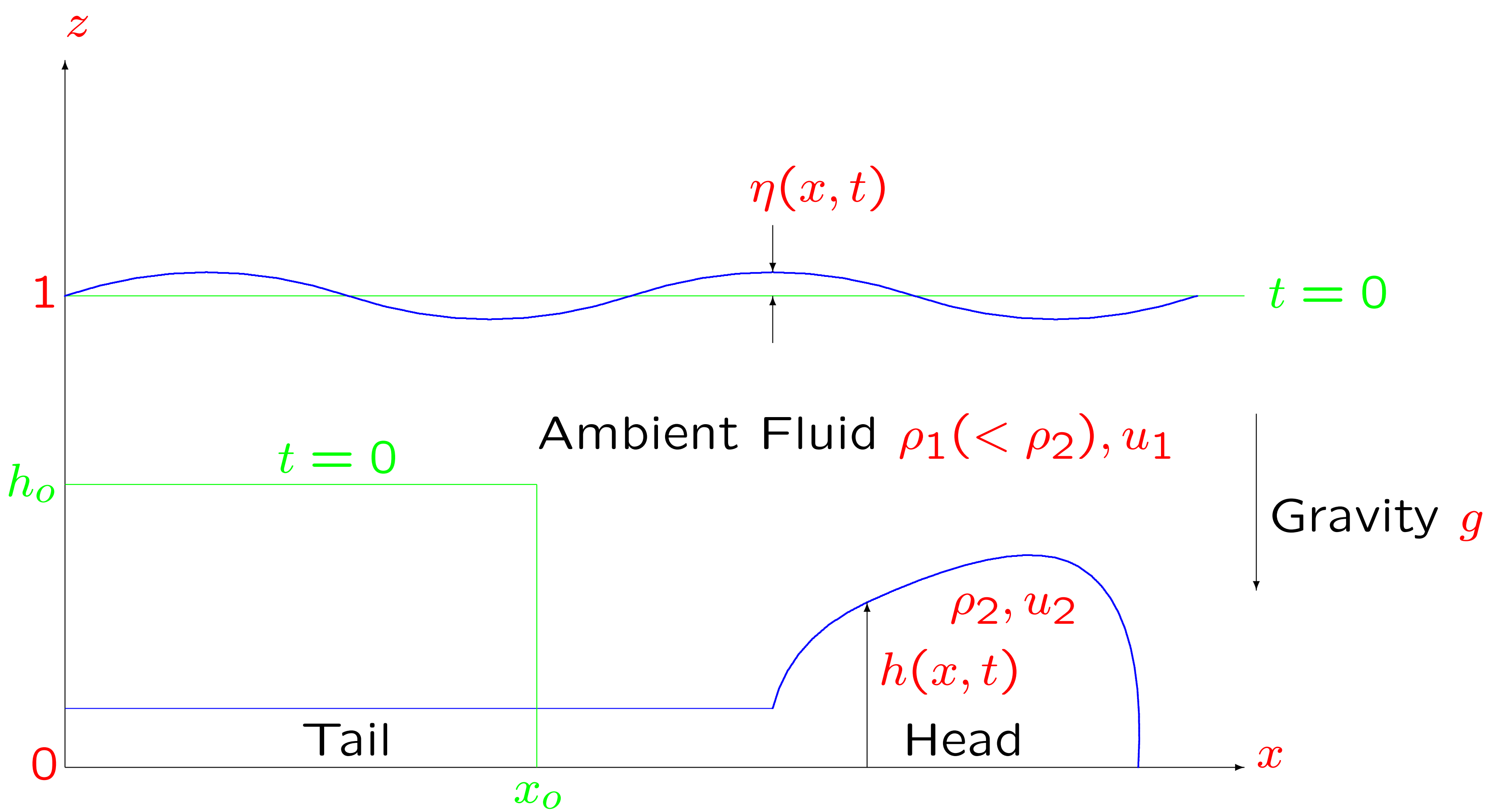
- Introduction
- Governing Equations
- Weakly Nonlinear Analysis
- Numerical Solution Procedure
- Results & Comparisons
- Conclusions

Introduction

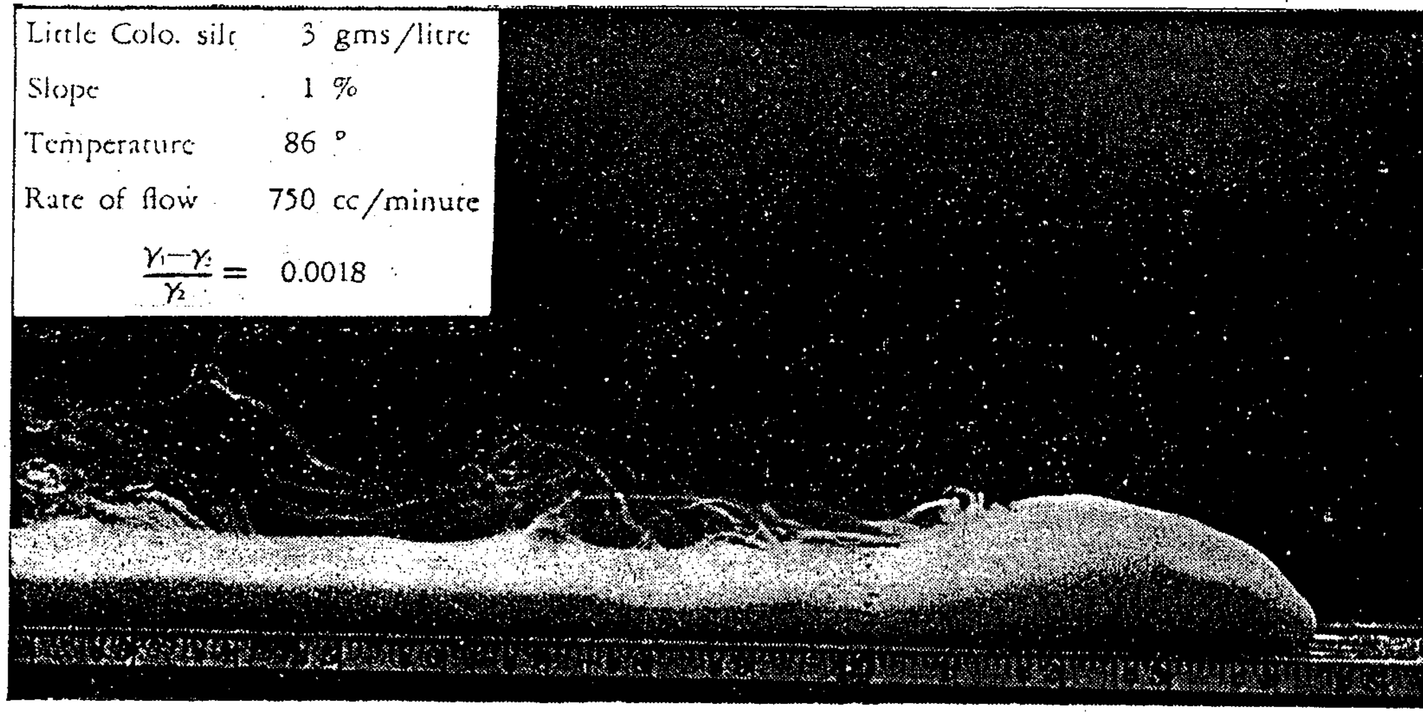
A gravity current is the flow of one fluid within another caused by the density difference between the fluids. Gravity currents occur in many natural phenomena as well as human-related activities.

An important parameter characterizing the problem is the reduced gravity g' defined by:

$$g' = \frac{(\rho_2 - \rho_1)}{\rho_2} g$$



The flow configuration.



Photograph of a gravity current produced in the laboratory.

The gravity current is produced by the intrusion of muddy water into a sloping channel filled with clear water. Shown is the shape of the surface separating the muddy water from the clear water.

Model Assumptions & Approximations

- Fluid is inviscid, incompressible and immiscible
- Small aspect ratio, $\delta = \frac{H}{L}$, $0 < \delta \ll 1$
- Pressure is hydrostatic to $O(\delta^2)$
- Boussinesq approximation
- Ignore effects of surface tension

Governing Equations

The planar shallow water equations in dimensionless form are:

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \frac{\partial \eta}{\partial x} = 0$$

$$\frac{\partial}{\partial t}(h - \frac{g'}{g}\eta) + \frac{\partial}{\partial x}[(1 + \frac{g'}{g}\eta - h)u_1] = 0$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + \left(1 - \frac{g'}{g}\right) \frac{\partial \eta}{\partial x} + \frac{\partial h}{\partial x} = 0$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h u_2) = 0$$

Boundary & Initial Conditions

Initial conditions:

$$u_1(x,0) = \eta(x,0) = 0 \ , \ u_2(x,0) = u_{20} \ , \ h(x,0) = \begin{cases} h_o & \text{if } 0 \leq x \leq x_o \\ 0 & \text{if } x > x_o \end{cases}$$

Impermeability, slope and far-field boundary conditions:

$$u_1(0,t) = 0 \ , \ u_2(0,t) = 0$$

$$\frac{\partial \eta}{\partial x}(0,t) = \frac{\partial h}{\partial x}(0,t) = 0$$

$$u_1(x,t), u_2(x,t), \eta(x,t), h(x,t) \to 0 \text{ as } x \to \infty$$

Weak Stratification Equations

For small density differences we can neglect terms of $O(g'/g)$. The equations then simplify to:

$$\frac{\partial u}{\partial t} + \frac{(1-3h)}{(1-h)} u \frac{\partial u}{\partial x} + \left(1 - h - \frac{u^2}{(1-h)^2}\right) \frac{\partial h}{\partial x} = 0$$

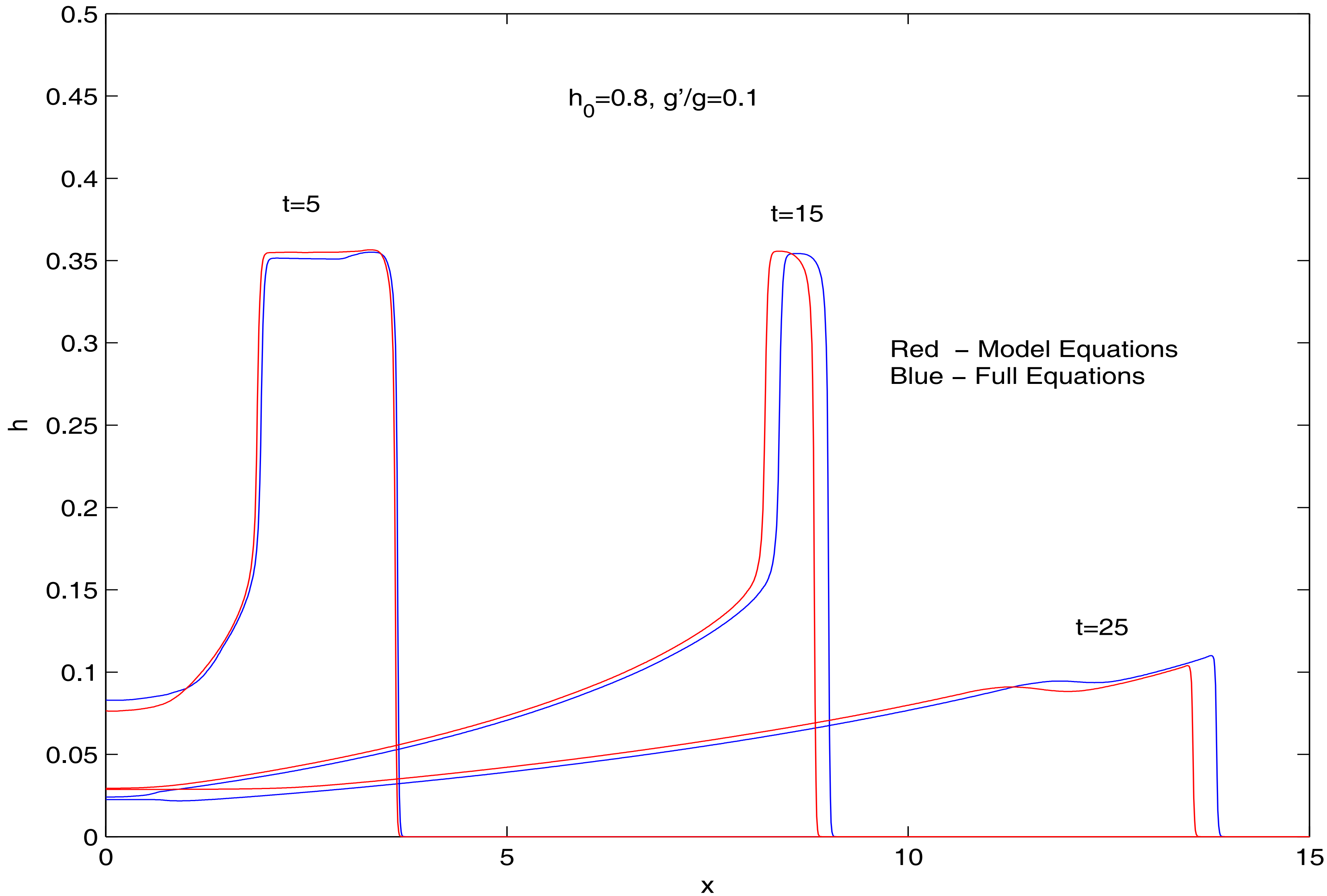
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0$$

$$\eta = -\frac{u^2 h}{1-h} - \frac{1}{2} h^2$$

$$u_1 = -\frac{hu}{1-h}$$

where $u \equiv u_2$.

Comparison between model & full equations



Weakly Nonlinear Analysis

Expand about $(u, h) = (u_o, h_o)$ by letting $u = u_o + \hat{u}, h = h_o + \hat{h}$ and retain quadratically nonlinear terms, then $\hat{u}, \hat{h}$ satisfy:

$$\frac{\partial \hat{u}}{\partial t} + \left(\frac{u_o(1-3h_o)}{(1-h_o)} + \frac{(1-3h_o)}{(1-h_o)}\hat{u} - \frac{2u_o}{(1-h_o)^2}\hat{h} \right) \frac{\partial \hat{u}}{\partial x}$$

$$+ \left(\frac{(1-h_o)^3 - u_o^2}{(1-h_o)^2} - \frac{2u_o}{(1-h_o)^2}\hat{u} - \frac{[(1-h_o)^3 + 2u_o^2]}{(1-h_o)^3}\hat{h} \right) \frac{\partial \hat{h}}{\partial x} = 0$$

$$\frac{\partial \hat{h}}{\partial t} + (h_o + \hat{h})\frac{\partial \hat{u}}{\partial x} + (u_o + \hat{u})\frac{\partial \hat{h}}{\partial x} = 0$$

which can be combined into the single equation:

$$\hat{h}_{tt} + a_1 \hat{h}_{xt} + a_2 \hat{h}_{xx} = -(\hat{u}\hat{h})_{xt} + a_3(\hat{u}\hat{u}_x)_x - a_4(\hat{h}\hat{h}_x)_x - a_5(\hat{u}\hat{h})_{xx}$$

Linearizing the equations and assuming a wave-like solution (dropping the hats):

$$u(x,t) = u(\xi)\ ,\ h(x,t) = h(\xi) \text{ where } \xi = x - ct$$

yields the linearized speeds:

$$c_{\pm} = \left(\frac{1-2h_o}{1-h_o}\right) u_o \pm \sqrt{\frac{h_o}{1-h_o}}\sqrt{(1-h_o)^2 - u_o^2}$$

For $0 \leq u_o \leq 1$, the speeds are real in the triangular region $h_o \leq 1 - u_o$. Next introduce

$$\xi = x - c_- t\ ,\ \eta = x + c_- t\ ,\ T = \varepsilon t\ ,\ h = \varepsilon \tilde{h}\ ,\ u = \varepsilon \tilde{u}$$

and expand the variables in the following series

$$\tilde{h} = h^{(0)} + \varepsilon h^{(1)} + O(\varepsilon^2) \text{ and } \tilde{u} = u^{(0)} + \varepsilon u^{(1)} + O(\varepsilon^2)$$

The $O(1)$ Problem

The leading order equations

$$\alpha h_{\eta\eta}^{(0)} - \beta h_{\eta\xi}^{(0)} = 0$$

$$c_{-}(u_{\eta}^{(0)} - u_{\xi}^{(0)}) + \frac{u_o(1-3h_o)}{(1-h_o)}(u_{\eta}^{(0)} + u_{\xi}^{(0)}) =$$

$$-\frac{[(1-h_o)^3 - u_o^2]}{(1-h_o)^2}(h_{\eta}^{(0)} + h_{\xi}^{(0)})$$

have solutions of the form

$$h^{(0)} = \phi(\xi, T) + \psi(\eta + \frac{\alpha}{\beta}\xi, T)$$

$$u^{(0)} = c_1\phi(\xi, T) - c_2\psi(\eta + \frac{\alpha}{\beta}\xi, T)$$

The $O(\varepsilon)$ Problem

Carrying the analysis to the next order yields

$$\alpha h^{(1)}_{\eta\eta} - \beta h^{(1)}_{\eta\xi} = A(\xi, T) + B(\xi, \eta, T)$$

where

$$A(\xi, T) = c_3 \phi_{T\xi} + c_4 (\phi^2)_{\xi\xi}$$

Imposing the solvability condition $A = 0$ and integrating gives

$$\phi_T + b\phi\phi_\xi = 0$$

Letting $\phi(\xi, 0) = f(\xi)$ represent the initial condition, the solution to the above can be expressed implicitly in terms of the parameter τ as

$$\phi(\xi, T) = f(\tau) \text{ along } \xi = bTf(\tau) + \tau$$

Shock formation occurs when $|\phi_\xi| \to \infty$ where

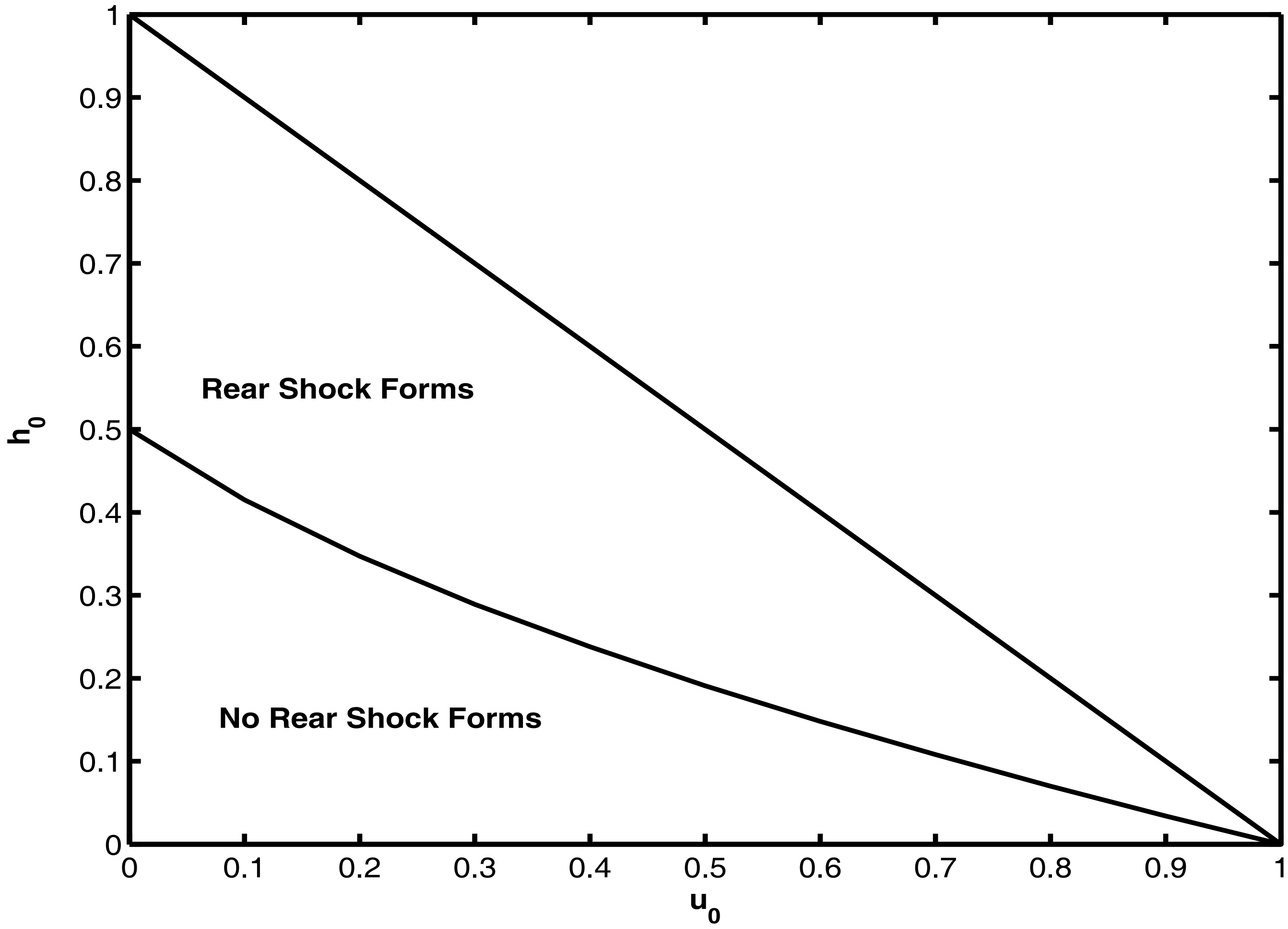
$$\phi_\xi = \frac{f'(\tau)}{1 + bTf'(\tau)}$$

which becomes infinite when $T = -1/bf'(\tau)$. Along the back side of a smooth curve $f(\tau)$, where $f'(\tau) > 0$, a shock will form if $b < 0$. In terms of the initial configuration specified by u_o and h_o this condition can be expressed as

$$2F_1F_2F_3 + F_1^2F_4 - F_2^2F_5 < 0$$

where F_1, F_2, F_3, F_4, F_5 are complicated functions of u_o and h_o. As a check, if we set $u_o = 0$ then the above condition collapses to $h_o > 1/2$ which is in full agreement with our previous result (Stud. Appl. Math. 96, 359-385, 1996).

Analytical Predictions



Numerical Solution Procedure

The **weak stratification** equations form a hyperbolic system of conservation laws. To numerically solve this system MacCormack's method was employed. This is a conservative second-order accurate finite difference scheme which correctly captures discontinuities and converges to the physical weak solution of the problem.

A general system of conservation equations with a source term can be written compactly in vector form as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \mathbf{b}(\mathbf{U})$$

In our case $\mathbf{b}(\mathbf{U}) = \mathbf{0}$ and the vectors $\mathbf{U}$ and $\mathbf{F}(\mathbf{U})$ are given by

$$\mathbf{U} = \begin{bmatrix} u \\ h \end{bmatrix}, \; \mathbf{F}(\mathbf{U}) = \begin{bmatrix} \frac{1}{2}u^2 + \eta(u,h) \\ uh \end{bmatrix}, \; \eta(u,h) = -\frac{u^2 h}{1-h} - \frac{1}{2}h^2$$

LeVeque & Yee (JCP, 86, 187-210, 1990) extended MacCormack's method to include source terms. This explicit two step predictor-corrector scheme takes the form

$$\mathbf{U}_j^* = \mathbf{U}_j^n - \frac{\Delta t}{\Delta x}\left[\mathbf{F}(\mathbf{U}_{j+1}^n) - \mathbf{F}(\mathbf{U}_j^n)\right] + \Delta t\, \mathbf{b}(\mathbf{U}_j^n)$$

$$\mathbf{U}_j^{n+1} = \frac{1}{2}\left(\mathbf{U}_j^n + \mathbf{U}_j^*\right) - \frac{\Delta t}{2\Delta x}\left[\mathbf{F}(\mathbf{U}_j^*) - \mathbf{F}(\mathbf{U}_{j-1}^*)\right] + \frac{\Delta t}{2}\mathbf{b}(\mathbf{U}_j^*)$$

where the notation $\mathbf{U}_j^n \equiv \mathbf{U}(x_j, t_n)$ was adopted, Δx is the grid spacing and Δt is the time step.

To dampen spurious oscillations associated with second-order schemes artificial viscosity was introduced. Since adding artificial viscosity reduces the accuracy to first-order, Harten (Math. Comp., 32, 363-389, 1978) proposed an efficient strategy to deal with this. Harten's approach involves applying artificial viscosity in a solution dependent manner which adds significant artificial viscosity only around discontinuities. The resulting scheme then remains second-order accurate where the solution is smooth and is first-order accurate only near discontinuities.

This is achieved by replacing the approximation $\mathbf{U}_j^{n+1}$ by

$$\mathbf{U}_j^{n+1} + \frac{1}{8}\left[\theta_{j+1/2}^n(\mathbf{U}_{j+1}^n - \mathbf{U}_j^n) - \theta_{j-1/2}^n(\mathbf{U}_j^n - \mathbf{U}_{j-1}^n)\right]$$

where the scalar $\theta_{j+1/2}^n$ is solution dependent and is small if the solution is smooth and close to unity near discontinuities. Specifically,

$$\theta_{j+1/2} = \max(\hat{\theta}_j, \hat{\theta}_{j+1}) \text{ where}$$

$$\hat{\theta}_j = \begin{cases} \left|\dfrac{|\Delta_{j+1/2}h| - |\Delta_{j-1/2}h|}{|\Delta_{j+1/2}h| + |\Delta_{j-1/2}h|}\right| & \text{for } |\Delta_{j+1/2}h| + |\Delta_{j-1/2}h| > \epsilon \\ 0 & \text{for } |\Delta_{j+1/2}h| + |\Delta_{j-1/2}h| \leq \epsilon \end{cases}$$

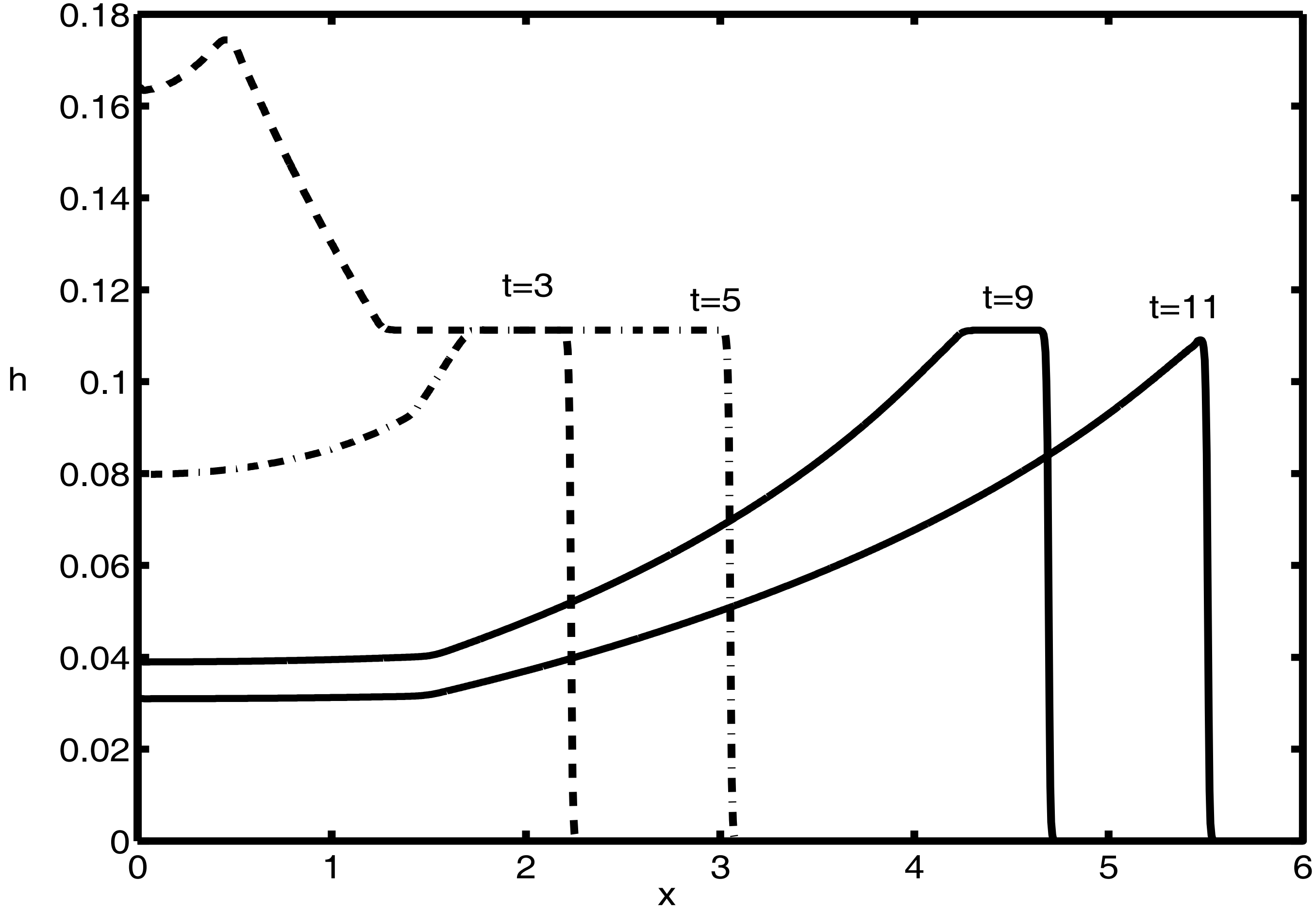
with $\Delta_{j+1/2}h = h_{j+1} - h_j$ and $\epsilon > 0$ is a specified tolerance.

Results & Comparisons

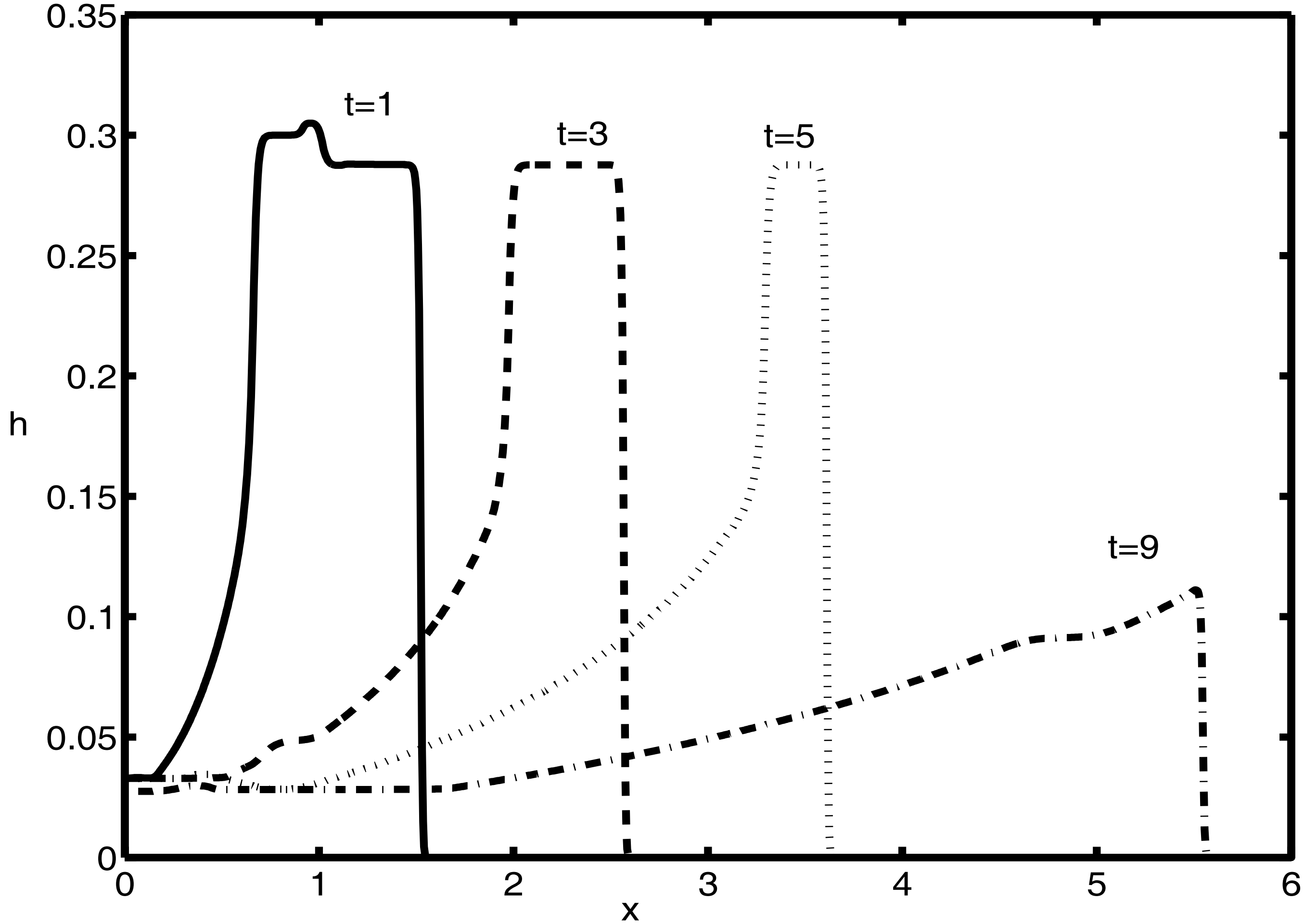
Computational Parameters

For numerical stability the following values were used:
Outer Boundary: $x_\infty = 6$
Grid Spacing Used: $\Delta x = .01$
Time Step Used: $\Delta t = .002$
Tolerance: $\epsilon = 10^{-5}$

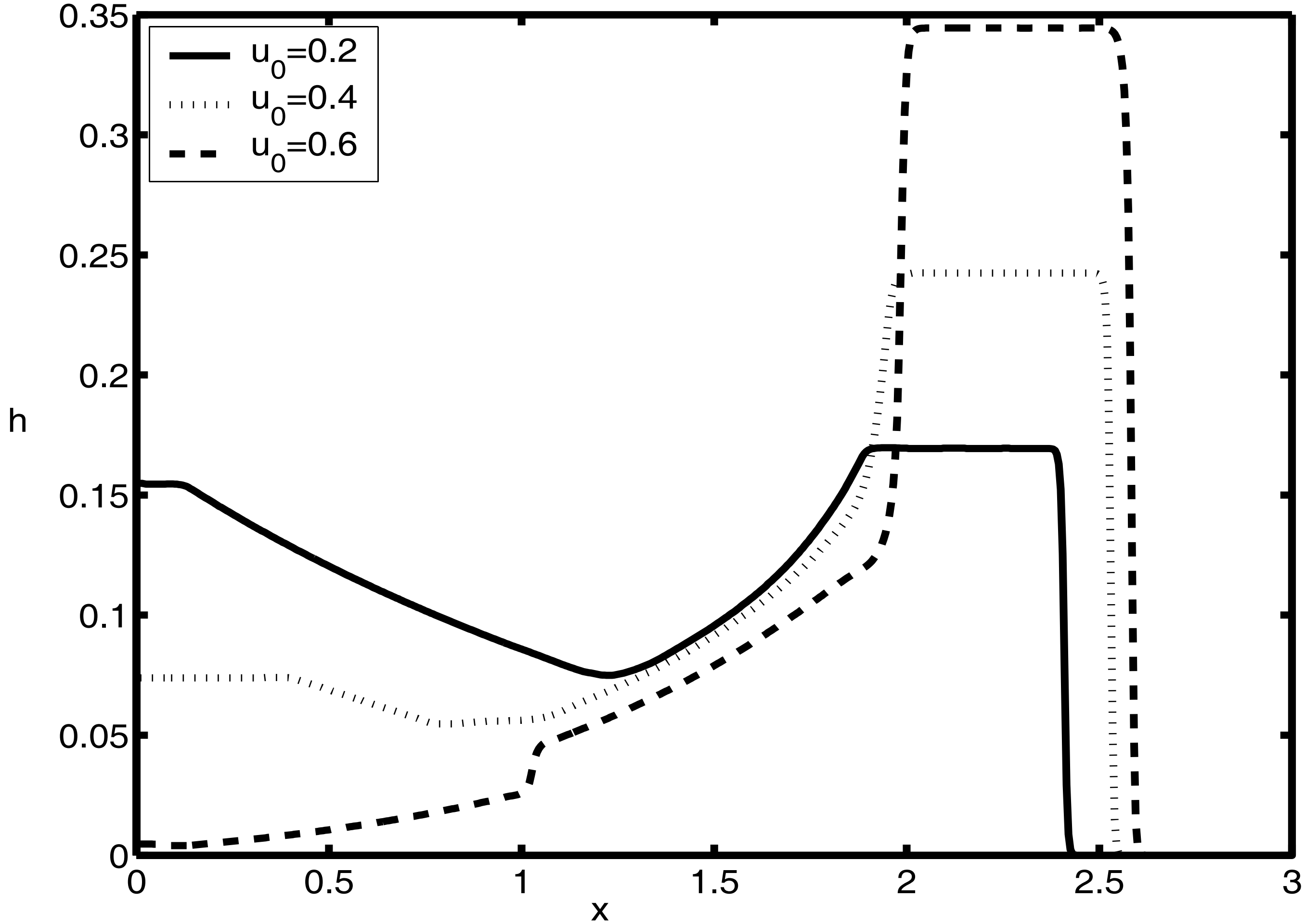
Case 1: $x_o = 1, h_o = 0.3, u_o = 0$



Case 2: $x_o = 1, h_o = 0.3, u_o = 0.5$



Case 3: $x_o = 1, h_o = 0.3, t = 3$



Conclusions

- Gravity currents flowing on a flat bottom of a rectangular channel were studied.

- For weak stratification the full 4×4 system can be reduced to a 2×2 system together with a set of 2 algebraic equations.

- A weakly nonlinear analysis applied to the weakly stratified equations was successful in predicting when a rear shock forms behind the head of the current. Full agreement with previous work is obtained for the special case of a gravity current initially at rest.

- Analytical predictions were further confirmed by extensive numerical experiments.

Thank You!

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