A Mathematical and Numerical Study of Roll Waves

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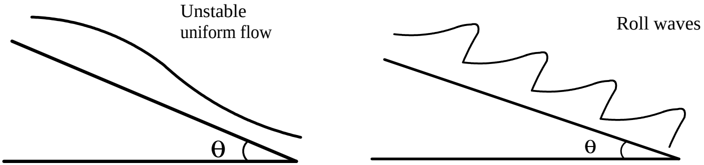
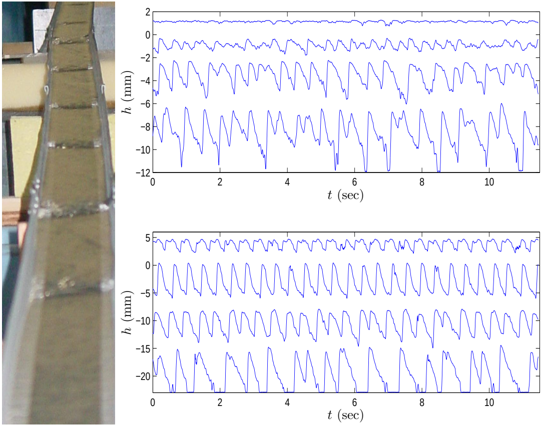
Outline

Unstable flow down an incline



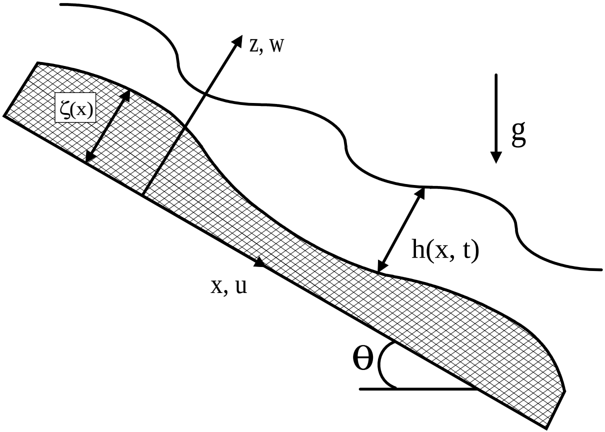
- Critical conditions for the onset of Instability.
- Structure of Roll Waves
- Investigate the effect of bottom topography

The spillway from the Llyn Brianne Dam in Wales



Experiment taken from Balmforth & Mandre (JFM, 2004)

Coordinate system



Equations of motion

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + g\rho \sin\theta + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} + g\cos\theta - \frac{\mu}{\rho}\frac{\partial^2 w}{\partial z^2} = 0$$

Assumed $Re \sim O(1)$ and neglected terms $O(\delta^2)$ and higher where $\delta = H/L$ is the aspect ratio

Interface conditions

Free surface conditions:

$$\left.\begin{aligned} &p - 2\mu\frac{\partial w}{\partial z} = 0 \\ &\mu\frac{\partial u}{\partial z} = 0 \\ &w = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + u\zeta'(x) \end{aligned}\right\} \text{ at } z = \zeta(x) + h(x,t)$$

Bottom boundary conditions:

$$u + \zeta'(x)w = 0 \ \text{ and } \ \zeta'(x)u - w = 0 \ \text{ at } \ z = \zeta(x)$$

$$\Rightarrow u = w = 0 \ \text{ at } \ z = \zeta(x)$$

Integral boundary layer (IBL) method

Depth-integrate equations and introduce flow variables

$$h(x,t) \quad \text{and} \quad q(x,t) = \int_{\zeta}^{\zeta+h} u dz$$

To convert terms $\displaystyle\int_{\zeta}^{\zeta+h} u^2 dz \ , \quad \frac{\mu}{\rho}\frac{\partial u}{\partial z}\bigg|_{z=\zeta}$

assume the parabolic velocity profile:

$$u(x,z,t) = \frac{3q}{2h^3}\left[2(h+\zeta)z - z^2 - (\zeta+2h)\zeta\right]$$

Dimensionless equations

In terms of h, q the dimensionless equations become

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0$$

$$\frac{\partial q}{\partial t} + \frac{6}{5}\frac{\partial}{\partial x}\left(\frac{q^2}{h}\right) = \frac{1}{Fr^2}\left(h - h\frac{\partial h}{\partial x} - \zeta'(x)h - \frac{q}{h^2}\right)$$

$$+\frac{3Fr^2}{Re^2}\left[\frac{7}{2}\frac{\partial^2 q}{\partial x^2} - \frac{9}{h}\frac{\partial q}{\partial x}\frac{\partial h}{\partial x} + \frac{9q}{h^2}\left(\frac{\partial h}{\partial x}\right)^2 - \frac{9q}{2h}\frac{\partial^2 h}{\partial x^2}\right.$$

$$\left.-\frac{6\zeta'(x)}{h}\frac{\partial q}{\partial x} + \frac{6\zeta'(x)q}{h^2}\frac{\partial h}{\partial x} - \frac{3\zeta''(x)q}{h} - \frac{6\left(\zeta'(x)\right)^2 q}{h^2}\right]$$

where $Fr^2 = \dfrac{Re}{3\cot\theta}$, $Re = \dfrac{\rho Q}{\mu}$ and $\zeta(x) = a_b \cos(k_b x)$

Linear stability: $a_b = 0$ case

The steady-state flow is: $q_s = h_s = 1$
Imposing disturbances on this steady flow and linearizing yields the dispersion equation

$$Fr^2\sigma^2 + \left(\frac{21Fr^4}{2Re^2}k^2 + 1 + i\frac{12}{5}Fr^2k\right)\sigma + \left(1 - \frac{6}{5}Fr^2\right)k^2$$

$$+i\left(3k + \frac{27Fr^4}{2Re^2}k^3\right) = 0$$

where σ is the growth rate and k is the wavenumber of the disturbance

Linear stability results for $a_b = 0$

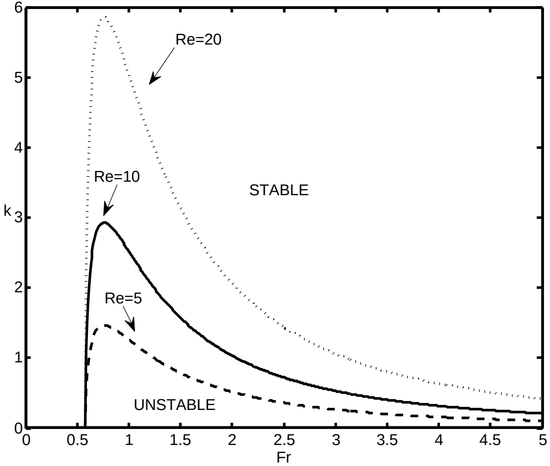
The flow is stable if $Fr < \frac{1}{\sqrt{3}}$ while for $Fr > \frac{1}{\sqrt{3}}$ instability occurs for wavenumbers $k < k_{max}$ where

$$k_{max} = \frac{10Re}{\sqrt{30}Fr^2}\sqrt{\frac{3Fr^2 - 1}{3Fr^2 + 35 + 12Fr\sqrt{6Fr^2 + 25}}}$$

For large Fr the asymptotic behaviour is

$$k_{max} \sim \frac{10Re}{Fr^2\sqrt{30(1 + 4\sqrt{6})}}$$

Neutral stability curves for $a_b = 0$



Neutral stability curves have a maximum at $Fr \approx 0.76286$ (independent of Re)

Linear stability: $a_b \neq 0$ case

The steady state solution is $q_s = 1$ and $h_s(x)$ satisfies

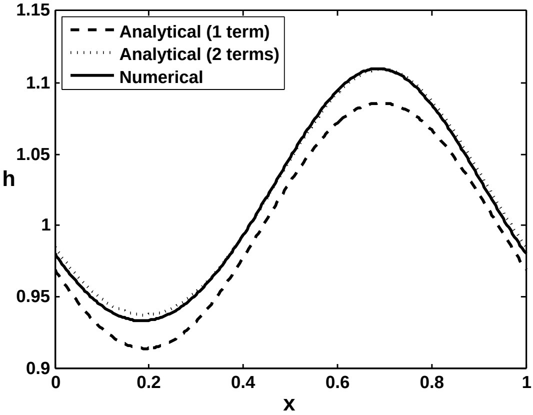
$$3\beta[h_s h_s'' - 2(h_s')^2] + (2\alpha h_s^3 - 4\beta\zeta' - \frac{12}{5})h_s'$$

$$+2\beta\zeta'' h_s - 2\alpha(1 - \zeta')h_s^3 = -2\alpha - 4\beta(\zeta')^2$$

where $\alpha = \dfrac{1}{Fr^2}$ and $\beta = 9\left(\dfrac{Fr}{Re}\right)^2$
An approximate solution can be constructed in the form

$$h_s(x) = 1 + (a_b k_b)h_s^{(1)}(x) + (a_b k_b)^2 h_s^{(2)}(x) + \cdots$$

Periodic steady state solution



$Fr = 1$, $Re = 10$,
$a_b = 0.1$, $k_b = 2\pi$

Linear stability: $a_b \neq 0$ case

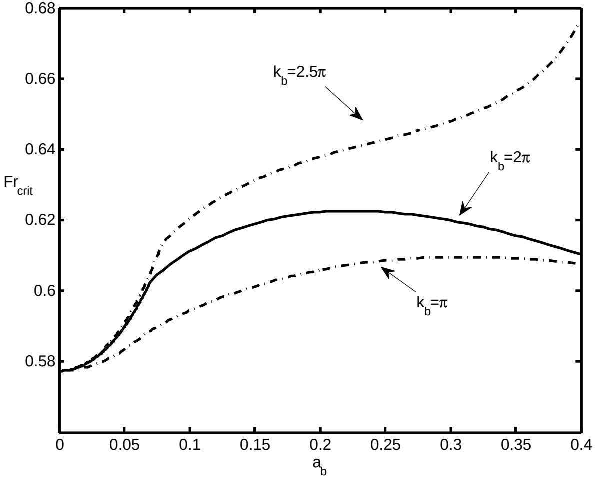
To study how small disturbances will evolve, introduce perturbations $\hat{h}$, $\hat{q}$ superimposed on the steady-state solution and linearize equations using

$$h = h_s(x) + \hat{h}\,, \;\; q = 1 + \hat{q}$$

For an uneven bottom, the coefficients in the linearized equations are periodic functions; hence apply Floquet-Bloch theory to conduct the stability analysis and represent the perturbations as Bloch-type functions having the form

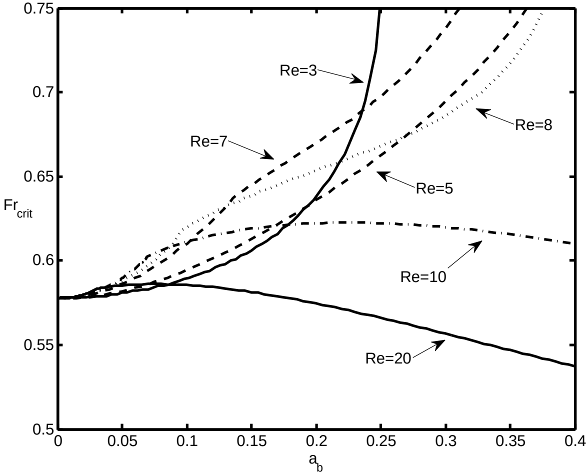
$$\hat{h} = e^{\sigma t} e^{iKx} \sum_{n=-\infty}^{\infty} \hat{h}_n e^{ink_b x}\,, \;\; \hat{q} = e^{\sigma t} e^{iKx} \sum_{n=-\infty}^{\infty} \hat{q}_n e^{ink_b x}$$

Numerical linear stability results for $a_b \neq 0$



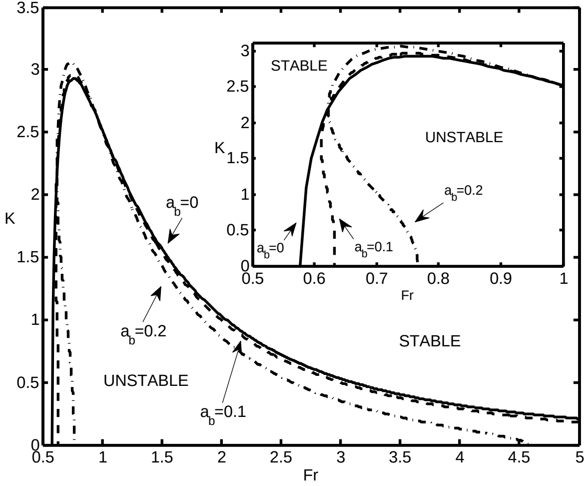
Critical Froude number as a function of bottom amplitude with $Re = 10$

Numerical linear stability results for $a_b \neq 0$



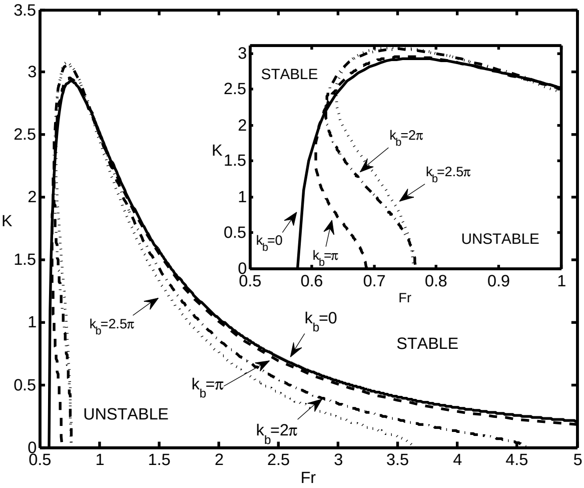
Critical Froude number as a function of bottom amplitude with $k_b = 2\pi$

Numerical linear stability results for $a_b \neq 0$



Neutral stability curves for the case with $Re = 10$ and $k_b = 2\pi$

Numerical linear stability results for $a_b \neq 0$



Neutral stability curves for the case with $Re = 10$ and $a_b = 0.2$

Begin by expressing equations in the form

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0$$

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{6}{5}\frac{q^2}{h} + \frac{\alpha}{2}h^2\right) = \Psi(h,q) + \chi\left(x, h, q, \frac{\partial h}{\partial x}, \frac{\partial q}{\partial x}, \frac{\partial^2 h}{\partial x^2}, \frac{\partial^2 q}{\partial x^2}\right)$$

where $\Psi = \alpha\left(h - \frac{q}{h^2}\right)$

and $\chi = -\alpha\zeta' h - 2\beta\zeta'\left(\zeta' - \frac{\partial h}{\partial x}\right)\frac{q}{h^2} - \beta\zeta''\frac{q}{h} - 2\beta\frac{\zeta'}{h}\frac{\partial q}{\partial x}$

$$+\beta\left(\frac{7}{6}\frac{\partial^2 q}{\partial x^2} - \frac{3}{2}\frac{q}{h}\frac{\partial^2 h}{\partial x^2} - \frac{3}{h}\frac{\partial q}{\partial x}\frac{\partial h}{\partial x} + 3\frac{q}{h^2}\left(\frac{\partial h}{\partial x}\right)^2\right)$$

Fractional-step method (LeVeque, 2002)

Decouple the advective and diffusive components, first solve

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0$$

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{6}{5}\frac{q^2}{h} + \frac{\alpha}{2}h^2\right) = \Psi(h, q)$$

over a time step Δt, and then solve

$$\frac{\partial q}{\partial t} = \chi\left(x, h, q, \frac{\partial h}{\partial x}, \frac{\partial q}{\partial x}, \frac{\partial^2 h}{\partial x^2}, \frac{\partial^2 q}{\partial x^2}\right)$$

using the solution obtained from the first step as an initial condition for the second step; the second step returns the solution for q at the new time $t + \Delta t$

First step

This involves solving a nonlinear system of hyperbolic conservation laws; express in vector form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \mathbf{b}(\mathbf{U})$$

where $\mathbf{U} = \begin{bmatrix} h \\ q \end{bmatrix}$, $\mathbf{F}(\mathbf{U}) = \begin{bmatrix} q \\ \frac{6}{5}\frac{q^2}{h} + \frac{\alpha}{2}h^2 \end{bmatrix}$, $\mathbf{b}(\mathbf{U}) = \begin{bmatrix} 0 \\ \Psi \end{bmatrix}$

Utilize MacCormack's method to solve this system; this is a conservative second-order accurate finite difference scheme which correctly captures discontinuities and converges to the physical weak solution of the problem

First step

LeVeque & Yee (JCP, 1990) extended MacCormack's method to include source terms; this explicit predictor-corrector scheme takes the form

$$\mathbf{U}_j^* = \mathbf{U}_j^n - \frac{\Delta t}{\Delta x}\left[\mathbf{F}(\mathbf{U}_{j+1}^n) - \mathbf{F}(\mathbf{U}_j^n)\right] + \Delta t\, \mathbf{b}(\mathbf{U}_j^n)$$

$$\mathbf{U}_j^{n+1} = \frac{1}{2}\left(\mathbf{U}_j^n + \mathbf{U}_j^*\right) - \frac{\Delta t}{2\Delta x}\left[\mathbf{F}(\mathbf{U}_j^*) - \mathbf{F}(\mathbf{U}_{j-1}^*)\right] + \frac{\Delta t}{2}\mathbf{b}(\mathbf{U}_j^*)$$

where the notation $\mathbf{U}_j^n \equiv \mathbf{U}(x_j, t_n)$ was adopted, Δx is the grid spacing and Δt is the time step; second-order accuracy is achieved by first forward differencing and then backward differencing

Second step

This reduces to solving the generalized one-dimensional linear diffusion equation given by:

$$\frac{\partial q}{\partial t} = \frac{7\beta}{6}\frac{\partial^2 q}{\partial x^2} + S_1\frac{\partial q}{\partial x} + S_0 q + S$$

where $S = -\alpha\zeta' h$ *and*

$$S_0 = -\beta\frac{\zeta''}{h} - 2\beta\frac{\zeta'}{h^2}\left(\zeta' - \frac{\partial h}{\partial x}\right) - \frac{3}{2}\frac{\beta}{h}\frac{\partial^2 h}{\partial x^2} + 3\frac{\beta}{h^2}\left(\frac{\partial h}{\partial x}\right)^2$$

and $S_1 = -2\beta\frac{\zeta'}{h} - 3\frac{\beta}{h}\frac{\partial h}{\partial x}$

Computational parameters

The problem is completely specified by Fr, Re, a_b and k_b; typical computational parameters used were:

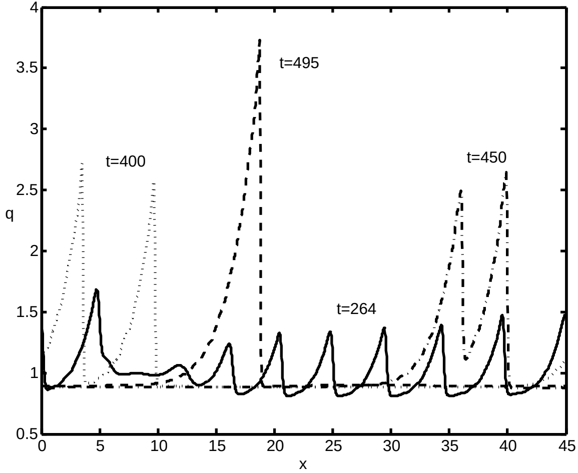
Computational Domain: $0 \leq x \leq L$

with $\lambda_b \leq L \leq 300\lambda_b$, $\lambda_b = \dfrac{2\pi}{k_b}$

Grid Spacing: $\Delta x = .01$

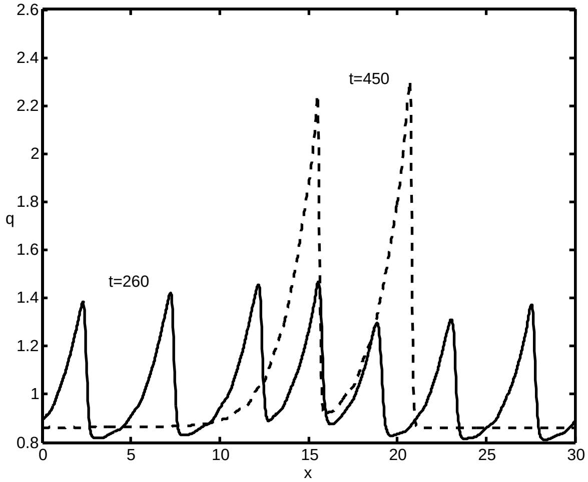
Time Step: $\Delta t = .002$

Evolution of flow rate



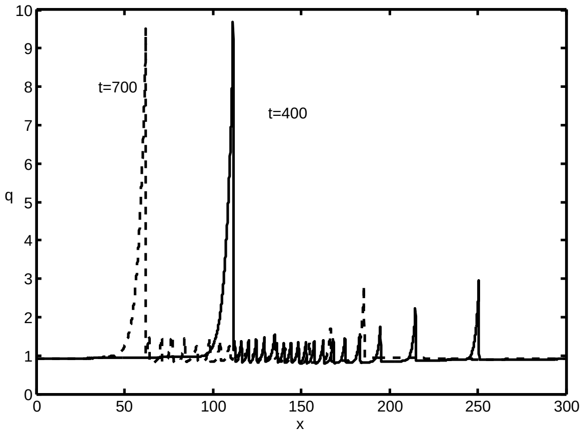
Parameters:
$a_b = 0.1, k_b = 2\pi,$
$Re = 10, Fr = 0.7$
A subharmonic instability known as wave coarsening occurs for $L = 45\lambda_b$

Evolution of flow rate



Parameters:
$a_b = 0.1, k_b = 2\pi,$
$Re = 10, Fr = 0.7$
Interruption in wave coarsening occurs for $L = 30\lambda_b$

Wave spawning



Parameters:
$a_b = 0.1, k_b = 2\pi,$
$Re = 10, Fr = 0.7$
A wave-spawning instability occurs for $L = 300\lambda_b$

Concluding remarks

- A mathematical model of roll waves along with a numerical method to solve the model were presented
- Investigated the effect of sinusoidal bottom topography on the formation of roll waves
- Bottom topography has a stabilizing effect on the flow for small to moderate waviness parameters
- Future work includes repeating the analysis for a porous wavy bottom and to include surface tension