ON WINDING DUCK SOLUTIONS IN R^4

KIYOYUKI TCHIZAWA* AND SUE ANN CAMPBELL**

*Dept of Maths, Musashi Institute of Technology
**Dept of Applied Maths, University of Waterloo

ABSTRACT. In this paper, we will prove the existence of duck solutions with winding in the coupled Fitzhugh-Nagumo equation. As the system is described by the slow -fast one in R^4, we will find the ducks in R^4.

Let consider the following slow-fast system:

$$
\begin{aligned}
\epsilon dx_1/dt &= h_1(x_1, x_2, y_1, y_2, u), \\
\epsilon dx_2/dt &= h_2(x_1, x_2, y_1, y_2, u), \\
dy_1/dt &= f(x_1, x_2, y_1, y_2, u), \\
dy_2/dt &= g(x_1, x_2, y_1, y_2, u),
\end{aligned}
\tag{1}
$$

where ϵ is infinitesimally small and u is a parameter, that is, we use nonstandard analysis. We assume that $H = (h_1, h_2)$ has $rank H = 2$ at almost every where. In this paper, we put

$$
\begin{aligned}
h_1 &= y_1 + x_1 - x_1^3/3 + \gamma(x_1 - x_2), \\
h_2 &= y_2 + x_2 - x_2^3/3 + \gamma(x_2 - x_1), \\
f &= -(x_1 - a + by_1)/c, \\
g &= -(x_2 - a + by_2)/c,
\end{aligned}
\tag{2}
$$

and for the simplicity, we put $a = 0$, $\gamma = -1$. So, the parameters are b and c but only b is essential. This slow-fast system is reduced from the coupled Fitzhugh-Nagumo equation proposed by S.A.Campbell [1], 2000.

When $\epsilon = 0$, the fast system gives a 2-dim differentiable manifold as a constrained surface. Because of satisfying $rank H = 2$ regarding especially x_1, x_2, the system (1) can be reduced to the slow-fast system projected in R^3:

1991 *Mathematics Subject Classification.* 34A34,34A47,34C35..

Key words and phrases. coupled Fitzhugh-Nagumo equation,singular perturbation, duck solution, winding number.

$$
\begin{aligned}
dy_1/dt &= -(x_1 + by_1)/c, \\
dy_2/dt &= -(x_1^3/3 - y_1 + by_2)/c, \\
\epsilon dx_1/dt &= y_2 - (x_1^3/3 - y_1)^3/3 + x_1,
\end{aligned}
\tag{3}
$$

under the condition, which $|dx_1/dt - dx_2/dt|$ is limited. On the constrained surface in the system (3), we can get the time scaled reduced system:

$$
\begin{aligned}
dy_1/dt &= -(x_1 + by_1)(1 - (x_1^3/3 - y_1)^2 x_1^2), \\
dy_2/dt &= -(x_1^3/3 - y_1 + by_2)(1 - (x_1^3/3 - y_1)^2 x_1^2), \\
dx_1/dt &= (x_1^3/3 - y_1)^2(x_1 + by_1) + x_1^3 - y_1 + by_2.
\end{aligned}
\tag{4}
$$

Then, the pseudo singular point, that is, the singular point of the system (4) is determined by

$$
\begin{aligned}
&(x_1^3/3 - y_1)^2(x_1 + by_1) + x_1^3/3 - y_1 + by_2 = 0, \\
&1 - (x_1^3/3 - y_1)^2 x_1^2 = 0.
\end{aligned}
\tag{5}
$$

Note that the second equation in (5) can be expressed as $x_1^3/3 - y_1 = +(-)1/x_1$. In the case $(-)$, there are 2 pseudo singular points $P_0 = (x_{1o}, y_{1o}, x_{2o}, y_{2o})$:
$(x_{1o}, y_{1o}, x_{2o}, y_{2o}) = (1, 4/3, -1, -4/3)$, and $(-1, -4/3, 1, 4/3)$.

These points do not depend on the parameter b, therefore they are structurally stable.

As the characteristic equation of the linearized system is

$$
\lambda(\lambda - (2 + 8b/3))(\lambda + 8b/3) = 0,
\tag{6}
$$

we can conclude that these will be node if $-3/4 < b < 0$. Then there are duck solutions at the pseudo-singular node. This fact implies they are winding. See[3], [4], [6].

In the case $(+)$, there are 4 pseudo singular points which depend on the parameter b. The characteristic equation in this case is

$$
\lambda(A\lambda^2 + B\lambda + C)/(3 + D)^3 = 0,
\tag{7}
$$

where

$$
\begin{aligned}
&A = -D^3 - 27D + 36b^2 - 108 \\
&B = 2[(4b^2 - 9)D^3 + (16b^4 - 90b^2 + 243))D - 162b^2 + 486]/3b \\
&C - -4[405D^3 + (64b^6 - 720b^4 + 291b^2 - 3645)D + 576b^6 - 3024b^4 + 3888b^2]/9b^2 \\
&D = \sqrt{(3 - 2b)(3 + 2b)}.
\end{aligned}
$$

If $0 < b < 3/2$, there exist the pseudo singular points, as

$$x_1 = \pm\sqrt{\left(3/b \pm \sqrt{9/b^2 - 4}\right)/2}. \tag{8}$$

The eigen values of all four singular points are the same due to the symmetry. They arise in some sort of pitchfork bifurcation from the singular points in the $(-)$equation at $b = 3/2$. If $0.388 < b < 1.4489$, there will be the ducks at the pseudo-singular node and spirals if $0 < b < 0.388$ or $1.4489 < b < 3/2$.

Symmetric Solutions In our model (1)-(2) we have coupled two *identical* systems. One consequence of this and of the form of the coupling is that eqs. (1)-(2) are invariant under the transformation $(x_1, x_2, y_1, y_2) \leftrightarrow (x_2, x_1, y_2, y_1)$. This symmetry also results in the existence of a two dimensional invariant subspace for the equations: $I = \{(x_1, y_1, x_2, y_2) | x_1 = x_2, y_1 = y_2\}$. Note that any system of the form (1) will have this symmetry if $h_1(x_1, x_2, y_1, y_2, u) = h_2(x_2, x_1, y_2, y_1, u)$ and $f(x_1, x_2, y_1, y_2, u) = g(x_2, x_1, y_2, y_1, u)$.

On the invariant subspace, I, the solutions are given by $(x_1(t), x_2(t), y_1(t), y_2(t)) = (x(t), x(t), y(t), y(t))$ where $x(t)$, $y(t)$ a solutions of the two dimensional slow-fast system

$$\begin{aligned} \epsilon dx/dt &= y + x - x^3/3 \\ dy/dt &= -(x + by)/c. \end{aligned} \tag{9}$$

The slow manifold for these equations is $h(x, y) = y + x - x^3/3$, with critical points at $(\pm 1, \mp 2/3)$. Standard analysis, predicts that a duck limit cycle will exist for this system for $-3 < b < 3/2$. In fact, numerical bifurcation analysis of (9) shows that it exists and is stable for $-2.25 < b < 3/2$. Thus the identical solution should exist in the invariant subspace I of $(1) - (2)$ for the same parameter range; this has been verified numerically. Figure 2 shows the projection into the (x_1, y_1) of this limit cycle when $b = 1.485$. The projection in the (x_2, y_2) plane is identical. The critical points of the slow manifold are shown as circles on the figure.

The one thing that may change in the larger system is the stability of this cycle. Numerical bifurcation analysis shows that the cycle is stable for $-0.31 < b < 1.5$ and unstable for $-2.25 < b < -0.31$.

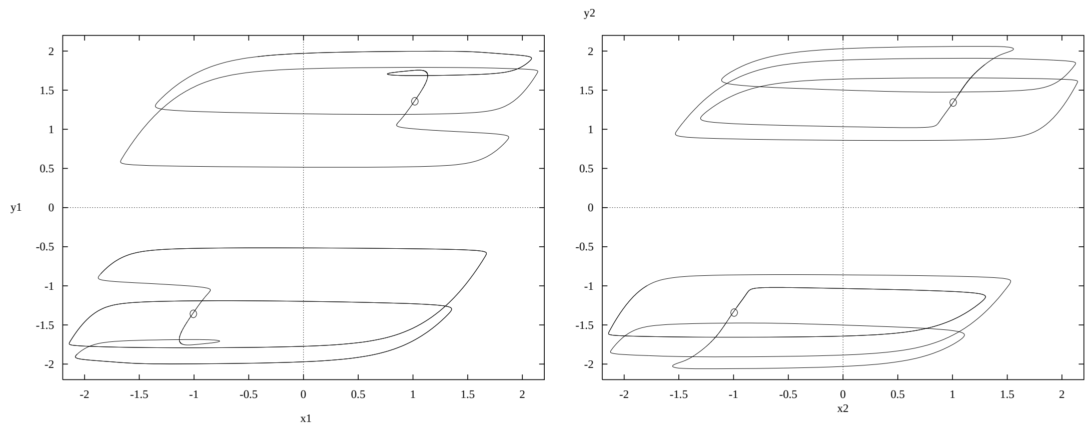

Figure 1: Limit cycles exhibiting winding. Parameter values : a=0, b=-0.6, c=70. (a) Projection onto the (x_1, y_1) plane. (b) Projection into (x_2, y_2) plane. Circles indicate the position of the pseudo singular points $(\pm 1, \pm 4/3, \mp 1, \mp 4/3)$

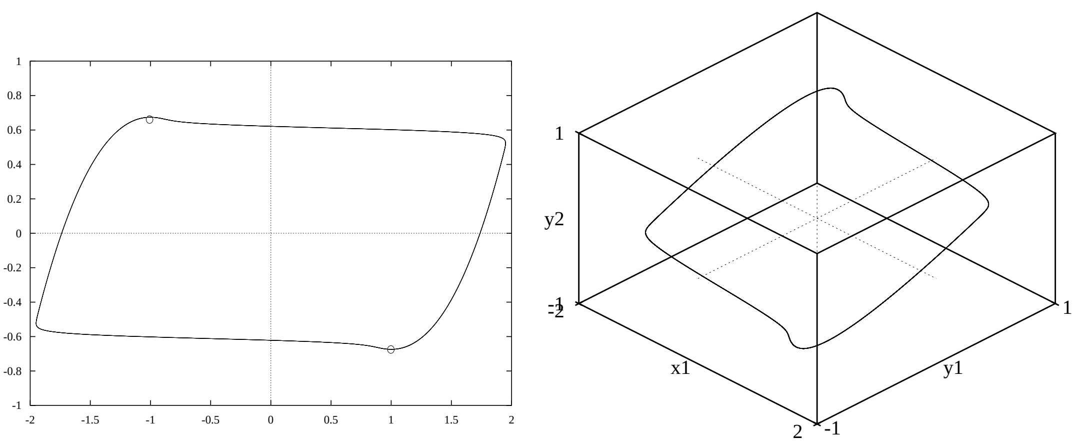

Figure 2: Limit cycle which has symmetry $x_1 = x_2$, $y_1 = y_2$. Parameter values : a=0, b=1.485, c=70. (a) Projection onto the (x_1, y_1) plane. Circles indicate the position of the pseudo singular points $(\pm 1, \mp 2/3, \pm 1, \mp 2/3)$. (b) Projection into (x_1, y_1, y_2) space.

At $b = 0.31$ the cycle undergoes what appears to be a pitchfork bifurcation of limit cycles with two stable cycles, which do not live in I, being born as the cycle in I loses stability. Some projections of these cycles when $b = -0.6$ are shown in Figure 1.

References

1. S.A. Campbell, M.Waite, *Multistability in Coupled Fitzhugh-Nagumo Oscillators*, Nonlinear Analysis **47** (2000), 1093–1104.
2. E. Benoit, *Systems lents-rapides dans R^3 et leurs canards*, Asterisque **109-110** (1983), 55–71.
3. E. Benoit, *Canards et enlacements*, Publ.Math.IHES **72** (1990), 63–91.
4. E. Benoit, *Existence of canards at a pseudo-singular node point*, Kyoto Univ RIMS Kokyuroku **1216** (2001), 90–98.
5. K. Tchizawa, *On an omega-incomplete duck and its application*, International J. of Applied mathematics **2 No.1** (2000), 25–38.
6. K. Tchizawa, *On a duck and its winding number in the minimal system*, Kyoto Univ RIMS Kokyuroku **1179** (2000), 131–142.
7. K. Tchizawa, P.Ashwin, *Delayed bursting oscillation with codimension* 3, preprint.

*1-28-1 Tamazutsumi Setagaya-Ku Tokyo, Japan
**Waterloo Ontario N2L 3G1, Canada