Chapter 1

Introduction and Fundamental Theory

The general form of the differential equations we will study is

$$\frac{dx}{dt} = f(x,t), \tag{1.1}$$

where

$$x \in \mathbb{R}^n, \quad f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad f = \begin{pmatrix} f_1(x_1, x_2, \cdots x_n, t) \\ \vdots \\ f_n(x_1, x_2, \cdots x_n, t) \end{pmatrix}.$$

Differential equations like (1.1) arise frequently as models for processes that occur in Physics, Biology, and Economics, among other disciplines. In these applications, t usually represents time and x represent some properties of the system that evolve in time. x is referred to as the state of the system.

1.1 Preliminaries

We will use the following notation interchangeably:

$$\frac{dx}{dt}, \quad x', \quad \dot{x}$$

Equation (1.1) is called a *nonlinear system* if f is a nonlinear function of x, otherwise it is a *linear* system. If f depends explicitly on t, that is if t explicitly appears on the right hand side of (1.1), then equation (1.1) is called a *nonautonomous* system. Otherwise, it is called an *autonomous system* and it can be written as

$$x' = f(x) \tag{1.2}$$

where $\quad x \in \mathbb{R}^n, \quad f : \mathbb{R}^n \to \mathbb{R}.$

Recall: Any higher ordinary differential equation, $\dfrac{d^n y}{dt^n} = g(t, y, y^1, \cdot, y^{(n-1)})$ can be put into the form (1.1) by defining the variable

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y \\ \vdots \\ y_{(n-1)} \end{pmatrix}, \quad \dot{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Example 1.1:
Consider the model for a simple pendulum of mass m and length l:

$$m\ddot{\theta} + \frac{mg}{l} \sin\theta = 0 \tag{1.3}$$

where g is the acceleration due to gravity.

Defining $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}$, (1.3) can be written in the form (1.2) with

$$f = \begin{pmatrix} x_2 \\ -\dfrac{g}{l} \sin x_1 \end{pmatrix}.$$

This is a nonlinear, autonomous system.

Example 1.2:
Consider the system

$$\frac{dx_1}{dt} = \sin(t)x_1 + \cos(t)x_2 + e^{-t}$$
$$\frac{dx_2}{dt} = \cos(t)x_1 - \sin(t)x_2 + e^{t}$$

This can be written in the form (1.1) with $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and

$$f(x,t) = Ax + h(t), \quad \text{where} \quad A = \begin{pmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{pmatrix} \qquad h(t) = \begin{pmatrix} e^{-t} \\ e^{t} \end{pmatrix}.$$

This is a linear, nonautonomous system.

Note: The nonautonomous system (1.1) can be written as an autonomous system by introducing an extra variable, as follows. Append to the system (1.1) the (trivial) equation

$$t' = 1.$$

Let $y \in \mathbb{R}^{n+1}$ be defined by $y = (x_1, x_2, \ldots, x_n, t)^T$. Then y satisfies the autonomous DE

$$y' = g(y) \tag{1.4}$$

where $g : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is given by $g = (f_1(x), f_2(x), \ldots, f_n(x), 1)^T$.

Example 1.3:

Using the transformation above the linear, nonautomous system of Example 1.2 becomes an autonomous system of the form (1.4) with

$$g(y) = \left(\begin{array}{l} \sin(y_3)y_1 + \cos(y_3)y_2 + e^{-y_3} \\ \cos(y_3)y_1 - \sin(y_3)y_2 + e^{y_3} \\ 1 \end{array} \right).$$

Note that this system is nonlinear.

Because of the equivalence between autonomous and nonautonomous systems illustrated above, we will focus on autonomous systems, that is, equation (1.2).

Definition 1.4: A **solution** of (1.2) on the interval $\mathcal{I} \subset \mathbb{R}$ is a function $\psi : \mathbb{R} \to \mathbb{R}^n$ that is differentiable for $t \in \mathcal{I}$ and satisfies (1.2) on $\mathcal{I}$.

Definition 1.5: Given $x_0 \in \mathbb{R}^n$, an **initial condition** for (1.2) is a specification of the solution at some particular time:

$$x(0) = x_0. \tag{1.5}$$

A differential equation together with an initial condition is called an **initial value problem**.

The following Lemma shows why, for autonomous systems, we need only consider initial value problems at $t = 0$.

Lemma 1.6. *$x(t) = \psi(t)$ is a solution on the interval $\mathcal{I}$ of the initial value problem consisting of* (1.2) *and* (1.5) *if and only if $\hat{x}(t) = \psi(t-t_0)$ is a solution on $\hat{\mathcal{I}} = \mathcal{I}+t_0$ of the initial value problem*

$$x' = f(x), \qquad x(t_0) = x_0. \tag{1.6}$$

Proof. Suppose $x(t) = \psi(t)$ is a solution of (1.2) and (1.5). Then $\psi'(t) = f(\psi(t))$ for all $t \in \mathcal{I}$ and $\psi(0) = x_0$. Thus we have

$$\begin{aligned} \hat{x}'(t) &= \psi'(t - t_0) \\ &= f(\psi(t - t_0)), \quad t - t_0 \in \mathcal{I} \\ &= f(\hat{x}(t)), \quad t \in \hat{\mathcal{I}} \end{aligned}$$

Further, $\hat{x}(t_0) = \psi(0) = x_0$. Thus $\hat{x}(t)$ satisfies (1.6). The proof the other direction is similar. □

Consequences of Nonlinear Models

If $f(x)$ in (1.2) is a nonlinear function of x, then we generally can't find explicit solutions for (1.2). However, we can use the theory of linear systems to give a qualitative description of the solutions. By "qualitative description", we mean answering questions like

- Are there constant (**equilibrium**) solutions? That is, solutions for which $x(t) = \bar{x}$, for $t \in \mathbb{R}$, where $\bar{x}$ is constant?
- Are there periodic solutions? That is, solutions for which there exists some $T \in \mathbb{R}$ for which $x(t + T) = x(t)$ for all $t \in \mathbb{R}$.
- Do solutions that start close to an equilibrium or periodic solution stay close to it? (Is the equilibrium solution or periodic solution **stable**?)
- What is the long term (**asymptotic**) behavior of solutions? ($\lim_{t\to\infty} x(t) =?$)

If $n = 2$ (or 3), we'll use the information gained from answering these questions to sketch the **phase portrait**: the set of all qualitatively different solution curves of (1.2) represented in the phase space $\mathbb{R}^2$ (or $\mathbb{R}^3$). These curves are called **trajectories** or **orbits**.

The following example show the phase portrait for a linear system.

Example 1.7:
Consider the linear, automous system

$$x' = Ax, \quad \text{where } A = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}, \tag{1.7}$$

subject to the initial condition $x(0) = x_0 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

The solution to this initial value problem is

$$x(t) = \begin{pmatrix} c_1 e^t \\ c_2 e^{-3t} \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \ t \in \mathbb{R}.$$

As seen in AMATH 250, the phase portrait may be determined by considering $(x_1(t), x_2(t)),\ t \in \mathbb{R}$, as parametric equations defining a curve in $\mathbb{R}^2$. Different value of c_1, c_2 will give rise to different curves. To see what these curves look like, one can eliminate t between the equations for $x_1(t)$ and $x_2(t)$. For $c_1 \neq 0$, this yields $x_2(t) = c_1^3 c_2 (x_1(t))^{-3}$. Considering positive, negative and zero values for c_1 and c_2 yields the phase portrait shown in Figure 1.1. There are 9 qualitatively different trajectories, including an equilibrium point at the origin.

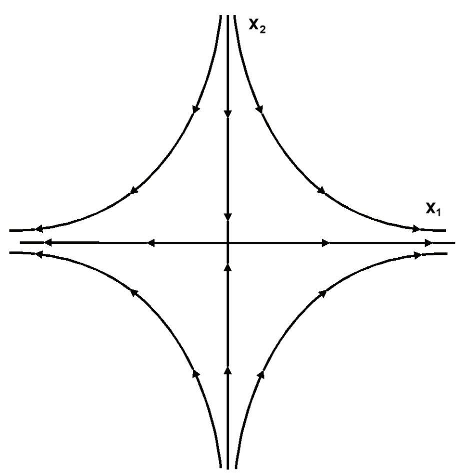

Figure 1.1: Phase portrait for the linear system (1.7).

1.2 Existence and Uniqueness Theory

Consider the initial value problem

$$x' = f(x), \qquad x(0) = x_0 \tag{1.8}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$.

The following lemma gives an equivalent representation of this initial value problem which is useful for proving results about the solutions.

Lemma 1.8. *$x(t)$ is a solution of (1.8) if and only if it is a solution of the integral equation*

$$x(t) = x_0 + \int_0^t f(x(s))ds. \tag{1.9}$$

Proof. Follows by integrating (1.8) and differentiating (1.9). □

We can now state our first theorem.

Theorem 1.9. *(Local Existence and Uniqueness Theorem)*
Let E be an open sunset of $\mathbb{R}^n$ containing x_0 and assume $f \in C^1(E)$. Then, there exists an $a > 0$ such that the initial value problem (1.8) has a unique solution on $[-a, a]$.

Proof. : An outline of the proof is as follows

- Since $f \in C^1(E)$ there is an $\epsilon > 0$ such that f is Lipschitz (with constant L) on $N_\epsilon(x_0) \subset E$.
- Define successive approximations, $u_k(t)$, to the solution $x(t)$ of (1.9) as follows:

$$\begin{aligned} u_0(t) &= x_0 \\ u_{k+1}(t) &= x_0 + \int_0^t f(u_k(s))ds \qquad k = 0, 1, 2, 3... \end{aligned}$$

- Let $b = \frac{\epsilon}{2}$. Show there exists $a > 0$ such that

$$\max_{t\in[-a,a]} |u_k(t) - x_0| \leq b, \quad k = 1, 2, \ldots$$

- Show that $u_k,\ k = 0, 1, \ldots$ are continuous on $[-a, a]$.
- Use the Lipschitz property of f to show

$$\max_{t\in[-a,a]} |u_k(t) - u_{k-1}(t)| \leq (La)^{k-1}b, \quad k = 1, 2, \ldots$$

- Show that $\{u_k\}$ is a Cauchy sequence, and thus it converges uniformly to a continuous function on $[-a, a]$. (Since the set of continuous functions on $[-a, a]$ is complete.)

- Show that $u(t) = \lim_{k\to\infty} u_k(t)$ satisfies (1.9) on $[-a, a]$.
- Show uniqueness by assuming there exist two solutions u, v of the initial value problem and showing $|u - v| = 0$

$\square$

Example 1.10:
Consider the following scalar, nonlinear initial value problem

$$x' = x^{2/3} \quad x(0) = x_0.$$

Here $f(x) = x^{2/3}$, thus $f \in C^1(E)$ where $E = (0, \infty)$ or $(-\infty, 0)$. Thus, for any $x_0 \neq 0$, Theorem (1.9) guarantees there exists a unique solution to the initial value problem on some interval $[-a, a]$.

If $x_0 \neq 0$ it is easy to solve the problem to find the unique solution is

$$x(t) = (t + \sqrt[3]{x_0})^3, \ t \in \mathbb{R}.$$

If $x_0 = 0$ the initial value problem has the solutions

$$\begin{aligned} x(t) &= t^3, \ t \in \mathbb{R} \\ x(t) &= 0, \ t \in \mathbb{R}. \end{aligned}$$

Thus the initial value problem has a solution, but it is not unique.

Theorem 1.11. *(Maximal Interval of Existence)*
Let E be an open subset of $\mathbb{R}^n$ and assume $f \in C^1(E)$. Then for each $x_0 \in E$, there is a maximal interval, J, on which the solution of the initial value problem (1.8) is unique. The maximal interval satisfies the following properties:

1) J is open, that is, $J = (\alpha, \beta)$

2) Either $\beta = \infty$ or $\lim_{t\to\beta^-} x(t) \in \partial E$

3) Either $\alpha = -\infty$ or $\lim_{t\to\beta^+} x(t) \in \partial E$

Example 1.12:
Consider the scalar, nonlinear initial value problem

$$x' = x^2, \quad x(0) = x_0. \tag{1.10}$$

Here $f(x) = x^2$, thus $f \in C^1(\mathbb{R})$ and Theorem 1.11 implies that, for any $x_0 \in \mathbb{R}$ the initial value problem has a unique solution on some maximal interval of existence.

Since the equation is separable, the initial value problem can be solved exactly to find:

$$x(t) = \frac{x_0}{1 - x_0 t}.$$

Thus one can determine that the maximal interval of existence is:

1. $\left(-\infty, \dfrac{1}{x_0}\right)$ if $x_0 > 0$
2. $(-\infty, \infty)$ if $x_0 = 0$
3. $\left(\dfrac{1}{x_0}, \infty\right)$ if $x_0 < 0$

Note that these satisfy the properties given in the theorem above. Some representative solutions are shown in Figure 1.2.

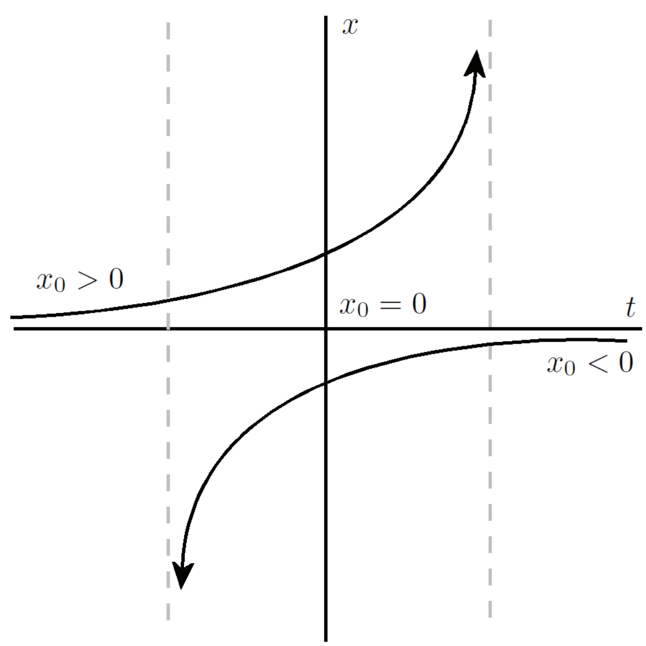

Figure 1.2: The solution to Example 1.12

The following corollary is useful for proving results about the global existence of solutions.

Corollary 1.13. *Let E be an open subset of $\mathbb{R}^n$ containing x_0. Let $f \in C^1(E)$, and let (α, β) be the maximal interval of existence of the unique solution of the initial* *value problem* (1.8)*. If there is a compact set* $K \subset E$ *such that the solution of* (1.8) *remains in* K *for at* $\in [0, \beta]$ *(respectively,* $t \in (\alpha, 0]$*) then* $\beta = \infty$ *(respectively,* $\alpha = -\infty$*).*

Theorem 1.14. *(Global Existence)*
Let $f \in C^1(\mathbb{R}^n)$ *and* f *be bounded on* $\mathbb{R}^n$*. Then for each* $x_0 \in \mathbb{R}^n$ *the initial value problem* (1.8) *has a unique solution defined for* $t \in \mathbb{R}$*.*

Proof. Since $f \in C^1(\mathbb{R}^n)$, Theorem 1.11 implies that for any $x_0 \in \mathbb{R}^n$ the initial value problem (1.8) has a unique solution defined on some maximal interval of existence, (α, β) with $\alpha < 0 < \beta$. By Lemma 1.8 the solution satisfies

$$x(t) = x_0 + \int_0^t f(x(s))\, ds, \quad t \in (\alpha, \beta).$$

Since f is bounded on $\mathbb{R}^n$ there is $M > 0$ such that $|f(x)| < M$ for all $x \in \mathbb{R}^n$. It follows that, for $t \in [0, \beta)$

$$\begin{aligned} |x(t)| &\leq |x_0| + \int_0^t |f(x(s))|\, ds \\ &\leq |x_0| + Mt \end{aligned}$$

Suppose that $\beta < \infty$, then

$$|x(t)| \leq |x_0| + M\beta, \quad \text{for } t \in [0, \beta),$$

that is, the solution remains in a compact set for $t \in [0, \beta)$. So by Corollary 1.13, $\beta = \infty$, which contradicts are assumption that $\beta < \infty$. So we must have $\beta = \infty$.

The proof that $\alpha = -\infty$ is similar. □

Example 1.15:
Consider the nonlinear system

$$\begin{aligned} x_1' &= \tanh(x_2) \\ x_2' &= \tanh(x_1) \end{aligned}$$

subject to the initial condition, $x(0) = x_0$.

Clearly $f \in C^1(\mathbb{R}^2)$. Further, f is bounded on $\mathbb{R}^2$ since

$$|f(x)| = \sqrt{\tanh^2(x_2) + \tanh^2(x_1)} \leq \sqrt{2},$$

for all $x \in \mathbb{R}^2$.

Thus Theorem 1.14 shows that for each $x_0 \in \mathbb{R}^2$ the initial value problem above has a unique solution defined for $t \in \mathbb{R}$.

1.3 Properties of Solutions

The following Lemma is important for proving results about differential equations.

Lemma 1.16. *(Gronwall's Lemma) Suppose that $g(t)$ is a continuous, real-valued, nonnegative function and there exists $C > 0$ and $K > 0$ such that*

$$g(t) \leqslant C + K \int_0^t g(s)ds, \quad t \in [0, a].$$

Then $g(t) \leqslant Ce^{Kt}$ for all $t \in [0, a]$.

Theorem 1.17. *(Continuous Dependence on Initial Conditions)*
Let E be an open subset of $\mathbb{R}^n$ containing x_0 and assume $f \in C^1(E)$. Then there exist $a > 0$ and $\delta > 0$ such that, for all $y \in N_\delta(x_0)$, the initial value problem $x' = f(x)$, $x(0) = y$ has a unique solution $u \in C^1(G)$ where $G = [-a, a] \times N_\delta(x_0) \subset \mathbb{R}^{n+1}$. Furthermore, for each $y \in N_\delta(x_0)$, $u(t, y)$ is a twice continuously differentiable function of t on $[-a, a]$.

Proof. An outline of the proof is as follows:

- Since $f \in C^1(E)$ there is an $\epsilon > 0$ such that f is Lipschitz on $N_\epsilon(x_0) \subset E$.
- Set up the integral equation equivalent to the initial value problem

$$u(y, t) = y + \int_0^t f(u(y, s))\, ds. \tag{1.11}$$

- Define the successive approximations for (1.11)

$$\begin{aligned} u_0(t, y) &= y \\ u_{k+1}(t, y) &= y + \int_0^t f(u_k(y, s))\, ds \end{aligned}$$

- Let $\delta = \frac{\epsilon}{4}$. Show that $u_k(t, y)$ is continuous on G and $u_k \in N_{\frac{\epsilon}{2}}(x_0)$ for $(t, y) \in G$.
- Show that the sequence $u_k(t, y)$ converges to a function $u(t, y)$ which is continuous on G, satisfies (1.11) and $u \in N_{\frac{\epsilon}{2}}(x_0)$ for $(t, y) \in G$.
- Differentiate (1.11) with respect to t to find

$$\begin{aligned} u_t(t, y) &= f(u(t, y)) \\ u_{tt}(t, y) &= Df(u(t, y))\, u_t(t, y) \end{aligned}$$

  which are in $C(G)$ since $u(t, y) \in C^1(G)$, $f \in C^1(E)$ and $u \in N_{\frac{\delta}{2}}(x_0)$.

- Use Gronwall's Lemma to show that, for all $t \in [-a, a]$ and $y_0 \in N_{\frac{\delta}{2}}(x_0)$,

$$\frac{\partial u}{\partial y}(t, y_0) = \Phi(t, y_0)$$

where $\Phi(t, y_0)$ satisfies

$$\begin{aligned} \dot{\Phi} &= A(t, y_0)\Phi \\ \Phi(0, y) &= I, \end{aligned} \tag{1.12}$$

with $A(t, y_0) = Df(u(t, y_0))$ and I is the $n \times n$ identity matrix.

- Show that $\Phi \in C(G)$ by applying successive approximations to (1.12).

□

Note: The continuity of u follows that of f. If $f \in C^r(E)$ with $r > 1$ then $u(t, y) \in C^r(G)$.

Example 1.18:
Recall that the initial value problem

$$x' = x^2, \quad x(0) = x_0.$$

Here $f \in C^\infty(\mathbb{R})$. Thus we should expect that the solution of the initial value problem will be $C^\infty(G)$ for some appropriate G.

Now the solution is

$$u(t, x_0) = \frac{x_0}{1 - x_0 t}.$$

It is easy to check that this is infinitely differentiable with respect to t and x_0 on the interval of existence of the solution. Some representative solutions of the initial value problem are shown in Figure 1.3. Note that the solution changes smoothly as x_0 changes.

We will now consider differential equations that depend on parameters, that is, equations of the form

$$x' = f(x, \mu) \tag{1.13}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Generally $\mu \in \mathbb{R}^m$ are physical constants that occur in the model.

Theorem 1.19. *(Continuous Dependence on Parameters)*
Let E be an open subset of $\mathbb{R}^{n+m}$ containing the point (x_0, μ_0) and assume $f \in C^1(E)$. Then, there exist $a > 0$ and $\delta > 0$ such that, for all $y \in N_g(x_0)$ and $\mu \in N_g(\mu_0)$, the initial value problem $x' = f(x, \mu)$, $x(0) = y$ has a unique solution $u(t, y, \mu)$ with $u \in C^1(G)$, where $G = [-a, a] \times N_\delta(x_0) \times N_\delta(\mu) \in \mathbb{R}^{n+m+1}$.

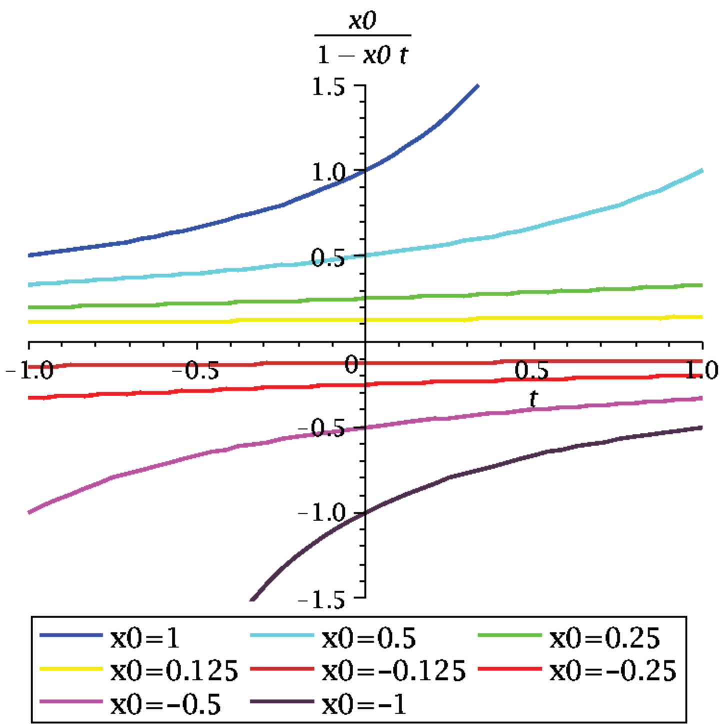

Figure 1.3: Plot of the solutions of Example 1.18

Proof. The proof is similar to that of Theorem 1.17. □

Example 1.20:
Consider the scalar initial value problem.

$$x' = \frac{1}{\mu}x, \;\; x(0) = x_0.$$

Here $f(x,\mu) = \dfrac{x}{\mu}$, thus $f \in C^1(E)$ where $E = \mathbb{R} \times (0,\infty)$ or $E = \mathbb{R} \times (-\infty, 0)$. Thus Theorem 1.19 predicts the solution will depend continously on the parameter μ in some neighbourhood of $\mu = \mu_0 > 0$ or $\mu = \mu_0 < 0$.

Now the actual solution of the initial value problem is $u(t, x_0, \mu) = x_0 e^{t/\mu}$. Thus we see that $x \in C^1(G)$ where $G = \mathbb{R} \times \mathbb{R} \times (0,\infty)$ or $G = \mathbb{R} \times \mathbb{R} \times (-\infty, 0)$. In particular, the solution is not continous on any set containing $\mu = 0$.

The solutions for $x_0 = 1$ and various values of μ are shown in Figure 1.3. It is clear that the solutions vary smoothly if μ is varied in $(0,\infty)$ or $(-\infty, 0)$, but if μ is varied through 0 there is drastic change in the solution.

Example 1.21:

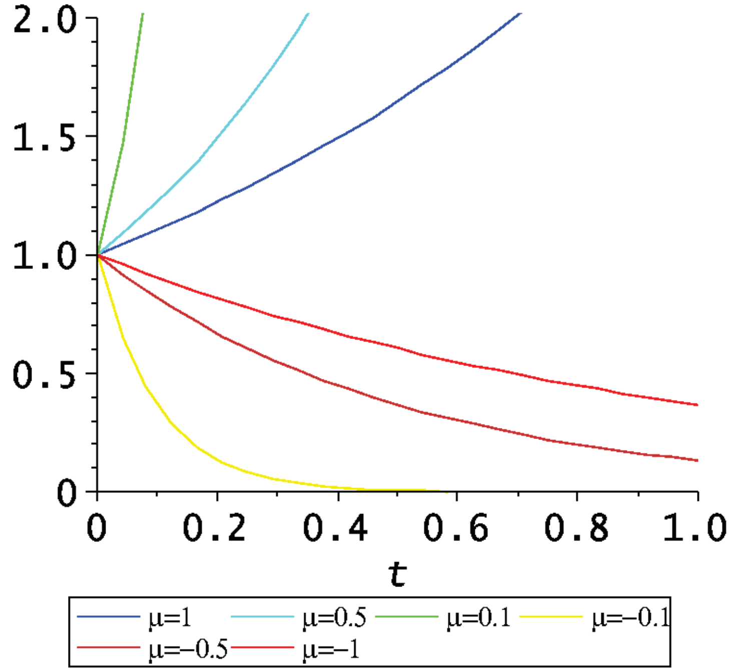

Figure 1.4: Plot of solutions for Example 1.20

Consider the model for the simple pendulum :

$$\begin{array}{rcl} x_1' & = & x_2 \\ x_2' & = & -\frac{g}{l}\sin(x_1) \end{array}, \quad \begin{array}{rcl} x_1(0) & = & x_{10} \\ x_2(0) & = & x_{20} \end{array}.$$

This can be written in the form (1.13) with $x = (x_1, x_2)$, $\mu = (g, l)$ and

$$f(x,\mu) = \begin{pmatrix} x_2 \\ -\frac{\mu_1}{\mu_2}\sin(x_1) \end{pmatrix}.$$

Thus $f \in C^1(E)$ where $E = \mathbb{R}^2 \times \mathbb{R} \times (0,\infty)$, and we expect the solution $u(t, x_0, \mu) \in C^1(G)$ for some appropriate G. Note that we cannot say what will happen to solutions if l varies through 0, but this is not important from a physical standpoint.