i.e. an exponential dominates a power as $b \to \infty$. $\Box$

The direct evaluation of Laplace transforms using the definition can be time-consuming, and so one seeks short cuts, for example

(i) by using complex functions,

(ii) by developing theorems to give new Laplace transforms from old, without extra calculation.

We first illustrate (i). The derivation leading to (4.5) is valid if α is complex, provided that we make one small change:

$$\lim_{r\to\infty} e^{-(s-\alpha)r} = 0 \quad \text{provided that} \quad s > Re(\alpha),$$

where $Re(\alpha)$ is the real part of α. We can now use Euler's formula

$$e^{ibt} = \cos bt + i \sin bt \tag{4.8}$$

and the linearity of $\mathcal{L}$ to calculate $\mathcal{L}[\cos bt]$ and $\mathcal{L}[\sin bt]$ directly from (4.5).

Example: Show that

$$\mathcal{L}[\cos bt] = \frac{s}{s^2+b^2}, \qquad \mathcal{L}[\sin bt] = \frac{b}{s^2+b^2}. \tag{4.9}$$

Solution: By linearity of $\mathcal{L}$ and (4.8):

$$\mathcal{L}[e^{ibt}] = \mathcal{L}[\cos bt] + i\mathcal{L}[\sin bt]. \tag{4.10}$$

The right hand side of (4.5), with $\alpha = ib$ is

$$\frac{1}{s-ib} = \frac{s+ib}{(s-ib)(s+ib)} = \frac{s+ib}{s^2+b^2}. \tag{4.11}$$

Choosing $\alpha = ib$ in (4.5) and substituting (4.10) and (4.11) gives

$$\mathcal{L}[\cos bt + i \sin bt] = \frac{s}{s^2+b^2} + i\frac{b}{s^2+b^2}.$$

Equating real and imaginary parts leads to (4.9). $\Box$

We now give a theorem which provides a quick way of calculating $\mathcal{L}[e^{ct}f(t)]$ if $\mathcal{L}[f(t)]$ is known.