AMATH 250 — Assignment 9 — Due Wednesday March 22

Course Notes References: Sections 4.1, 4.2.1, 4.2.2

You may use the results in the table on p. 87 of the Course Notes.

1. (a) Use the definition to derive the Laplace Transform of $f(t) = t^{-1/2}$. (Hint: Make a change of variables $x = \sqrt{st}$ and use the fact that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.)

   (b) Use appropriate theorems/properties and the result of part (a) to find the Laplace Transform of $f(t) = t^{1/2}$.

2. Use appropriate theorems/properties to find the Laplace Transform of each of the following functions.

   (a) $f(t) = t^2 \sin(\pi t)$.

   (b) $t^n e^{at}$, where n is a positive integer.

   (c) $f(t) = 2t^2 + 3te^{-2t} + 5$.

3. Find the Inverse Laplace Transform of each of the following functions.

   (a) $F(s) = \dfrac{2s-3}{s^2+2s+10}$

   (b) $F(s) = \dfrac{1}{(s-5)(s+3)}$.

   (c) $F(s) = \dfrac{1}{(s^2+a^2)(s^2+b^2)}, \quad a \neq b$

4. Solve the following initial value problems using Laplace Transforms.

   (a) $y' + 3y = e^{5t}, \ y(0) = 2$.

   (b) $y'' - 3y' + 2y = e^t, \ y(0) = 0, \ y'(0) = 1$. You may use the fact that any term of the form $\dfrac{1}{(s-a)^2}$ may be expressed as $\dfrac{A}{s-a} + \dfrac{B}{(s-a)^2}$, where A and B are constants.

5. Consider the model for an undamped mass-spring system:

$$\frac{d^2u}{dt^2} + \omega^2 u = 0, \;\; u(0) = u_0, \;\; u'(0) = v_0,$$

where $\omega^2 = k/m$ (the ratio of the spring constant to the mass of the object) and u_0 and v_0 are the initial position and velocity respectively.

(a) Use Laplace Transforms to solve this initial value problem.

(b) A motor is now added to the system and the new model for the system is

$$\frac{d^2u}{dt^2} + \omega^2 u = e^{-t}, \;\; u(0) = u_0, \;\; u'(0) = v_0.$$

Solve this new initial value problem using Laplace Transforms. Identify the homogeneous part $u_h(t)$ and the particular part $u_p(t)$ of your solution.